Treatise on the theory of invariants

Oliver E. Glenn

Title: Treatise on the Theory of Invariants

Author: Oliver E. Glenn

Release Date: February, 2006 [EBook #9933]
[Yes, we are more than one year ahead of schedule]
[This file was first posted on November 1, 2003]
Edition: 10
Language: English
Character set encoding: TeX

A TREATISE ON THE THEORY OF
INVARIANTS
OLIVER E. GLENN, PH.D.
PROFESSOR OF MATHEMATICS IN THE UNIVERSITY OF PENNSYLVANIA

PREFACE
The object of this book is, first, to present in a volume of medium size the fundamental principles and processes and a few of the multitudinous applications of invariant theory, with emphasis upon both the nonsymbolical and the symbolical method. Secondly, opportunity has been taken to emphasize a logical development of this theory as a whole, and to amalgamate methods of English mathematicians of the latter part of the nineteenth century–Boole, Cayley, Sylvester, and their contemporaries–and methods of the continental school, associated with the names of Aronhold, Clebsch, Gordan, and Hermite.
The original memoirs on the subject, comprising an exceedingly large and classical division of pure mathematics, have been consulted extensively. I have deemed it expedient, however, to give only a few references in the text. The student in the subject is fortunate in having at his command two large and

meritorious bibliographical reports which give historical references with much greater completeness than would be possible in footnotes in a book. These are the article “Invariantentheorie” in the “EnzyklopÂ¨adie der mathematischen Wissenschaften” (I B 2), andW. Fr. Meyer’s “Bericht Â¨uber den gegenwÂ¨artigen Stand der Invarianten-theorie” in the “Jahresbericht der deutschen Mathematiker-Vereinigung” for 1890-1891.

The first draft of the manuscript of the book was in the form of notes for a course of lectures on the theory of invariants, which I have given for several years in the Graduate School of the University of Pennsylvania.

The book contains several constructive simplifications of standard proofs and, in connection with invariants of finite groups of transformations and the algebraical theory of ternariants, formulations of fundamental algorithms which may, it is hoped, be of aid to investigators.

While writing I have had at hand and have frequently consulted the following texts:

- CLEBSCH, Theorie der binÂ¨aren Formen (1872).
- CLEBSCH, LINDEMANN, Vorlesungen uher Geometrie (1875).
- DICKSON, Algebraic Invariants (1914).
- DICKSON, Madison Colloquium Lectures on Mathematics (1913). I. Invariants and the Theory of lumbers.
- ELLIOTT, Algebra of Quantics (1895).
- FA`A DI BRUNO, Theorie des formes binaires (1876).
- GORDAN, Vorlesungen Â¨uber Invariantentheorie (1887).
- GRACE and YOUNG, Algebra of Invariants (1903).
- W. FR. MEYER, Allgemeine Formen und Invariantentheorie (1909).

- W. FR. MEYER, ApolaritÂ¨at und rationale Curven (1883).
- SALMON, Lessons Introductory to Modern Higher Algebra (1859; 4th ed., 1885).
- STUDY, Methoden zur Theorie der temaren Formen (1889).

O. E. GLENN
PHILADELPHIA, PA.

Contents

Chapter 1

THE PRINCIPLES OF INVARIANT THEORY

1.1 The nature of an invariant. Illustrations

We consider a definite entity or system of elements, as the totality of points in a plane, and suppose that the system is subjected to a definite kind of a transformation, like the transformation of the points in a plane by a linear transformation of their coÂ¨ordinates. Invariant theory treats of the properties of the system which persist, or its elements which remain unaltered, during the changes which are imposed upon the system by the transformation.

By means of particular illustrations we can bring into clear relief several defining properties of an invariant.

1.1.1 An invariant area.

Given a triangle ABC drawn in the Cartesian plane with a vertex at the origin. Suppose that the coordinates of A are (x_1, y_1); those of B (x_2, y_2). Then the area is

$$= \frac{1}{2}(x_1y_2 - x_2y_1),$$

or, in a convenient notation,

$$= \frac{1}{2}(xy).$$

Let us transform the system, consisting of all points in the plane, by the substitutions

$$x = 1x_0 + \mu_1y_0, \quad y = 2x_0 + \mu_2y_0.$$

The area of the triangle into which is then carried will be

$0 =$

1

2
$(x01y01 - x02y01) =$
1
2
$(x0y0)$,
and by applying the transformations directly to ,
 $= (1\mu2 - 2\mu1)0$. (1)
If we assume that the determinant of the transformation is unity,
$D = (\mu) = 1$,
then
$0 = .$
Thus the area of the triangle ABC remains unchanged under a transformation of determinant unity and is an invariant of the transformation. The triangle itself is not an invariant, but is carried into abC. The area is called an absolute invariant if $D = 1$. If D 6= I, all triangles having a vertex at the origin will have their areas multiplied by the same number $D-1$ under the transformation. In such a case is said to be a relative invariant. The adjoining figure illustrates the transformation of $A(5, 6)$, $B(4, 6)$, $C(0, 0)$ by means of
$x = x0 + y0$, $y = x0 + 2y0$.

1.1.2 An invariant ratio.
In I the points (elements) of the transformed system are located by means of two lines of reference, and consist of the totality of points in a plane. For a second illustration we consider the system of all points on a line EF.
We locate a point C on this line by referring it to two fixed points of reference P,Q. Thus C will divide the segment PQ in a definite ratio. This ratio,
PC/CQ,
is unique, being positive for points C of internal division and negative for points of external division. The point C is said to have for coordinates any pair of numbers $(x1, x2)$ such that

x1
x2
= PC
CQ
, (2)
where is a multiplier which is constant for a given pair of reference points P,Q. Let the segment PC be positive and equal to μ. Suppose that the point C is represented by the particular pair $(p1, p2)$, and let $D(q1, q2)$ be any other point. Then we can find a formula for the length of CD. For,
CQ
p2
= PC
p1
= PQ
p1 + p2
= μ
p1 + p2
,
and
DQ
q2
= μ
q1 + q2
.
Consequently
$CD = CQ = DQ = \mu(qp)$
$(q1 + q2)(p1 + p2)$. (3)
Theorem. The anharmonic ratio {CDEF} of four points $C(p2, p2)$, $D(q1, q2)$,

E(r1, r2), F(1, 2), defined by
{CDEF} = CD Â· EF
CF Â· ED
.
is an invariant under the general linear transformation
T : x1 = 1x01 + µ1x02, x2 = 2x01 + µ2x02, (µ) 6= 0. (31)

In proof we have from (3)
{CDEF} =
(qp)(r)
(p)(qr) .
But under the transformation (cf. (1)),
(qp) = (µ)(q0p0), (4)
and so on. Also, C,D,E, F are transformed into the points
C0(p01, p02),D0(q01, q02),E0(r01 , r02), F0(s01, s02),
respectively. Hence
{CDEF} =
(qp)(sr)
(sp)(qr)
=
(q0p0)(s0r0)
(s0p0)(q0r0)
= {C0D0E0F0},
and therefore the anharmonic ratio is an absolute invariant.

1.1.3 An invariant discriminant.

A homogeneous quadratic polynomial,
f = a0x21
+ 2a1x1x2 + a2x22
,
when equated to zero, is an equation having two roots which are values of the ratio x1/x2. According to II we may represent these two ratios by two points C(p1, p2),D(q1, q2) on the line EF. Thus we may speak of the roots (p1, p2), (q1, q2) of f.
These two points coincide if the discriminant of f vanishes, and conversely; that is if
D = 4(a0a2 − a21) = 0
If f be transformed by T, the result is a quadratic polynomial in x01, x02, or
f0 = a00x02
1 + 2a01x01x02 + a02x02
2 ,
Now if the points C,D coincide, then the two transformed points C0,D0 also coincide. For if CD = 0, (3) gives (qp) = 0. Then (4) gives (q0p0) = 0, since by hypothesis (µ) 6= 0. Hence, as stated, C0D0 = 0.
It follows that the discriminant D0 of f0 must vanish as a consequence of the vanishing of D. Hence
D0 = KD.
The constant K may be determined by selecting in place of f the particular quadratic f1 = 2x1x2 for which D = −4. Transforming f1 by T we have
f01 = 212x21
+ 2(1µ2 + 2µ1)x1x2 + 2µ1µ2x22
;

and the discriminant of f01 is D0 = −4(µ)2. Then the substitution of these particular discriminants gives
−4(µ)2 = −4K,
K = (µ)2.
We may also determine K by applying the transformation T to f and computing the explicit form of f0. We obtain
a00 = a021

+ 2a112 + a222
,
a01 = a01µ1 + a1(1µ2 + 2µ1) + a22µ2, (5)
a02 = a0µ21
+ 2a1µ1µ2 + a2µ22
,
and hence by actual computation,
4(a00a02 − a02
1) = 4(µ)2(a0a2 − a21
),
or, as above,
D0 = (µ)2D.
Therefore the discriminant of f is a relative invariant of T (Lagrange 1773); and, in fact, the discriminant of f0 is always equal to the discriminant of f multiplied by the square of the determinant of the transformation..
Preliminary Geometrical Definition.
If there is associated with a geometric figure a quantity which is left unchanged by a set of transformations of the figure, then this quantity is called an absolute invariant of the set (Halphen). In I the set of transformations consists of all linear transformations for which (µ) = 1. In II and III the set consists of all for which (µ) 6= 0.
1.1.4 An Invariant Geometrical Relation.
Let the roots of the quadratic polynomial f be represented by the points (p1, p2), (r1, r2), and let be a second polynomial,
 = b0x21
+ 2b1x1x2 + b2x22
,
whose roots are represented by (q1, q2), (s1, s2), or, in a briefer notation, by (q), (s). Assume that the anharmonic ratio of the four points (p), (q), (r), (s), equals minus one,
(qp)(sr)
(sp)(qr)
= −1 (6)

The point pairs f = 0, = 0 are then said to be harmonic conjugates. We have from (6)
2h 2p2r2s1q1 + 2p1r1s2q2 − (p1r2 + p2r1)(q1s2 + q2s1) = 0.
But
f = (x1p2 − x2p1)(x1r2 − x2r1),
 = (x1q2 − x2q1)(x1s2 − x2s1).
Hence
a0 = p2, 2a1 = −(p2r1 + p1r2), a2 = p1r1,
b0 = q2s2, 2b1 = −(q2s1 + q1s2), b2 = q1s1,
and by substitution in (2h) we obtain
h a0b2 − 2a1b1 + a2b0 = 0. (7)
That h is a relative invariant under T is evident from (6): for under the transformation f, become, respectively,
f0 = (x01p02 − x02p01)(x01r02 − x02r01),
0 = (x01q02 − x02q01)(x01s02 − x02s01),
where
p01 = µ2p1 − µ1p2, p02 = −2p1 + 1p2,
r01 = µ2r1 − µ1r2, r02 = −2r1 + 1r2,
Hence
(q0p0)(s0r0) + (s0p0)(q0r0) = (µ)2[(qp)(sr) + (sp)(qr)].
That is,
h0 = (µ)2h.
Therefore the bilinear function h of the coefficients of two quadratic polynomials, representing the condition that their root pairs be harmonic conjugates, is a relative invariant of the transformation T. It is sometimes called a joint

invariant, or simultaneous invariant of the two polynomials under the transformation.

1.1.5 An invariant polynomial.
To the pair of polynomials f, , let a third quadratic polynomial be adjoined,
 = c0x21
+ 2c1x2x2 + c2x22
= (x1u2 − x2u1)(x1v2 − x2v1).
Let the points (u1, u2) (v1, v2), be harmonic conjugate to the pair (p), (r); and also to the pair (q), (s). Then
c0a2 − 2c1a1 − c2a0 =0,
c0b2 − 2c1b1 − c2b0 =0,
c0x21
+ 2c1x1x2 + c2x22
=0.
Elimination of the c coefficients gives
C =
a0 a1 a2
b0 b1 b2
x22
−x1x2 x21

= 0 (8)
This polynomial,
C = (a0b1 − a1b0)x21
+ (a0b2 − a2b0)x1x2 + (a1b2 − a2b1)x22
,
is the one existent quadratic polynomial whose roots form a common harmonic conjugate pair, to each of the pairs f, .
We can prove readily that C is an invariant of the transformation T. For we have in addition to the equations (5),
b00 = b021
+ 2b112 + b222
,
b01 = b01μ1 + b1(1μ2 + 2μ1) + b22μ2, (9)
b02 = b0μ21
+ 2b1μ1μ2 + b2μ22
.
Also if we solve the transformation equations T for x01, x02 in terms of x1, x2 we obtain
x01 = (μ)−1(μ2x1 − μ1x2),
x02 = (μ)−1(−2x1 + 1x2), (10)
Hence when f, are transformed by T, C becomes
C0 =
{(a021
+ 2a112 + a222
)[b01μ1 + b1(1μ2 + 2μ1) + b22μ2]
−(b021
+ 2b112 + b222
[a01μ1 + a1(1μ2 + 2μ1) + a22μ2]}
Ã—(μ)−2(μ2x1 − μ1x2)2 + Â· Â· Â· .
(11)

When this expression is multiplied out and rearranged as a polynomial in x1, x2, it is found to be (μ)C. That is,
C0 = (μ)C
and therefore C is an invariant.
It is customary to employ the term invariant to signify a function of the coefficients of a polynomial, which is left unchanged, save possibly for a numerical multiple, when the polynomial is transformed by T. If the invariant function

involves the variables also, it is ordinarily called a covariant. Thus D in III is a relative invariant, whereas C is a relative covariant.

The Inverse of a Linear Transformation.

The process (11) of proving by direct computation the invariancy of a function we shall call verifying the invariant or covariant. The set of transformations (10) used in such a verification is called the inverse of T and is denoted by T^{-1}.

1.1.6 An invariant of three lines.

Instead of the Cartesian coÂ¨ordinates employed in I we may introduce homogeneous variables (x_1, x_2, x_3) to represent a point P in a plane. These variables may be regarded as the respective distances of N from the three sides of a triangle of reference. Then the equations of three lines in the plane may be written

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = 0,$
$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = 0,$
$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = 0.$

The eliminant of these,

D =
a_{11} a_{12} a_{13}
a_{21} a_{22} a_{23}
a_{31} a_{32} a_{33}

,

evidently represents the condition that the lines be concurrent. For the lines are concurrent if D = 0. Hence we infer from the geometry that D is an invariant, inasmuch as the transformed lines of three concurrent lines by the following transformations, S, are concurrent:

S :
$x_1 = 1x_{01} + \mu_1 x_{02} + 1x_{03},$
$x_2 = 2x_{01} + \mu_2 x_{02} + 2x_{03},$
$x_1 = 3x_{01} + \mu_3 x_{02} + 3x_{03}.$
$(\mu) \neq 0.$ (12)

To verify algebraically that D is an invariant we note that the transformed of

$a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3$ $(i = 1, 2, 3),$

by S is

$(a_{i1}1 + a_{i2}2 + a_{i3}3)x_{01} + (a_{i1}\mu_1 + a_{i2}\mu_2 + a_{i3}\mu_3)x_{02} + (a_{i1}v_1$
$+ a_{i2}v_2 + a_{i3}v_3)x_{03}$ $(i = 1, 2, 3).$ (13)

Thus the transformed of D is

D0 =
$a_{11}1 + a_{12}2 + a_{13}3$ $a_{11}\mu_1 + a_{12}\mu_2 + a_{13}\mu_3$ $a_{11}1 + a_{12}2 + a_{13}3$
$a_{21}1 + a_{22}2 + a_{23}3$ $a_{21}\mu_1 + a_{22}\mu_2 + a_{23}\mu_3$ $a_{21}1 + a_{22}2 + a_{23}3$
$a_{31}1 + a_{32}2 + a_{33}3$ $a_{31}\mu_1 + a_{32}\mu_2 + a_{33}\mu_3$ $a_{31}1 + a_{32}2 + a_{33}3$

$= (\mu)D.$ (14)

The latter equality holds by virtue of the ordinary law of the product of two determinants of the third order. Hence D is an invariant.

1.1.7 A Differential Invariant.

In previous illustrations the transformations introduced have been of the linear homogeneous type. Let us next consider a type of transformation which is not linear, and an invariant which represents the differential of the arc of a plane curve or simply the distance between two consecutive points (x, y) and (x + dx, y + dy) in the (x, y) plane.

We assume the transformation to be given by

$x0 = X(x, y, a),\ y0 = Y(x, y, a),$

where the functions X, Y are two independent continuous functions of x, y and the parameter a. We assume (a) that the partial derivatives of these functions exist, and (b) that these are continuous. Also (c) we define X, Y to be such that when $a = a0$

$X(x, y, a0) = x,\ Y(x.y, a0) = y.$

Then let an increment a be added to a0 and expand each function as a power series in a by Taylor's theorem. This gives

x0 = X(x, y, a0) + @X(x, y, a0)
@a0
a + . . . ,
y0 = Y (x, y, a0) + @Y (x, y, a0)
@a0
a + ... (15)

Since it may happen that some of the partial derivatives of X, Y may vanish for a = a0, assume that the lowest power of a in (15) which has a non-vanishing coefficient is (a)k, and write (a)k = t. Then the transformation, which is infinitesimal, becomes

I : x0 = x + t,
y0 = y + t.

where , are continuous functions of x, y. The effect of operating I upon the coÂ¨ordinates of a point P is to add infinitesimal increments to those coÂ¨ordinates, viz.

x = t,
y = t. (16)

Repeated operations with I produce a continuous motion of the point P along a definite path in the plane. Such a motion may be called a stationary streaming in the plane (Lie).

Let us now determine the functions , , so that

= dx2 + dy2

shall be an invariant under I.

By means of I, receives an infinitesimal increment . In order that may be an absolute invariant, we must have

1
2 = dxdx + dydy = 0,

or, since differential and variation symbols are permutable,

dxdx + dydy = dxd + dyd = 0.

Hence

(xdx + ydy)dx + (xdx + ydy)dy = 0.

Thus since dx and dy are independent differentials

x = y = 0, y + x = 0.

That is, is free from x and from y. Moreover

xy = xx = yy = 0.

Hence is linear in y, and is linear in x; and also from

y = −x,
= y + , = −x + . (17)

Thus the most general infinitesimal transformation leaving invariant is

I : x0 = x + (y +)t, y0 = y + (−x +)t. (18)

Now there is one point in the plane which is left invariant, viz.

x = /, y = −/.

The only exception to this is when = 0. But the transformation is then completely defined by

x0 = x + t, y0 = y + t,

and is an infinitesimal translation parallel to the coÂ¨ordinate axes. Assuming then that 6= 0, we transform coÂ¨ordinate axes so that the origin is moved to the invariant point. This transformation,

x = x + /, y = y − /,

leaves unaltered, and I becomes

x0 = x + yt, y0 = y − xt. (19)

But (19) is simply an infinitesimal rotation around the origin. We may add that the case = 0 does not require to be treated as an exception since an infinitesimal translation may be regarded as a rotation around the point at infinity. Thus,

Theorem. The most general infinitesimal transformation which leaves =
dx2+dy2 invariant is an infinitesimal rotation around a definite invariant point
in the plane.
We may readily interpret this theorem geometrically by noting that if is
invariant the motion is that of a rigid figure. As is well known, any infinitesimal
motion of a plane rigid figure in a plane is equivalent to a rotation around a
unique point in the plane, called the instantaneous center. The invariant point
of I is therefore the instantaneous center of the infinitesimal rotation.
The adjoining figure shows the invariant point (C) when the moving figure
is a rigid rod R one end of which slides on a circle S, and the other along a
straight line L. This point is the intersection of the radius produced through
one end of the rod with the perpendicular to L at the other end.
1.1.8 An Arithmetical Invariant.
Finally let us introduce a transformation of the linear type like

T : xl = 1x01 + μ1x02, x2 = 2x01 + μ2x02,
but one in which the coefficients , μ are positive integral residues of a prime
number p. Call this transformation Tp. We note first that Tp may be generated
by combining the following three particular transformations:
(a) x1 = x01 + tx02, x2 = x02,
(b) x1 = x01, x2 = x02, (20)
(c) x1 = x02, x2 = −x01,
where t, are any integers reduced modulo p. For (a) repeated gives
x1 = (x00 1 + tx00 2) + tx00 2 = x00 1 + 2tx00 2 , x2 = x00 2 .
Repeated r times (a) gives, when rt u (mod p),
(d) x1 = x01 + ux02, x2 = x02.
Then (c) combined with (d) becomes
(e) x1 = −ux01 + x02, x2 = −x01.
Proceeding in this way Tp may be built up.
Let
f = a0x21
+ 2a1x1x2 + a2x22
,
where the coefficients are arbitrary variables; and
g = a0x41
+ a1(x31
x2 + x1x32
) + a2x42
, (21)
and assume p = 3. Then we can prove that g is an arithmetical covariant; in
other words a covariant modulo 3. This is accomplished by showing that if f be
transformed by T3 then g0 will be identically congruent to g modulo 3. When f
is transformed by (c) we have
f0 = a2x02
1 − 2a1x01x02 + a0x02
2 .
That is,
a00 = a2, a01 = −a1, a02 = a0.
The inverse of (c) is x02 = x1, x01 = −x2. Hence
g0 = a2x42
+ a1(x1x32
+ x31
x2) + a0x41
= g,
and g is invariant, under (c).
Next we may transform f by (a); and we obtain
a00 = a0, a01 = a0t + a1, a02 = a0t2 + 2a1t + a2.

The inverse of (a) is

$x'_2 = x_2, \; x'_1 = x_1 - tx_2.$

Therefore we must have

$$g' = a_0(x_1 - tx_2)^4 + (a_0t + a_1)\left((x_1 - tx_2)^3x_2 + (x_1 - tx_2)x_2^3\right) + (a_0t^2 + 2a_1t + a_2)x_2^4$$

$$\equiv a_0x_1^4 + a_1(x_1^3x_2 + x_1x_2^3) + a_2x_2^4 \pmod 3 \qquad (22)$$

But this congruence follows immediately from the following case of Fermat's theorem:

$$t^3 \equiv t \pmod 3).$$

Likewise g is invariant with reference to (b). Hence g is a formal modular covariant of f under T_3.

1.2 Terminology and Definitions. Transformations

We proceed to formulate some definitions upon which immediate developments depend.

1.2.1 An invariant.

Suppose that a function of n variables, f, is subjected to a definite set of transformations upon those variables. Let there be associated with f some definite quantity ϕ such that when the corresponding quantity ϕ' is constructed for the transformed function f' the equality

$$\phi' = M\phi$$

holds. Suppose that M depends only upon the transformations, that is, is free from any relationship with f. Then ϕ is called an invariant of f under the transformations of the set.

The most extensive subdivision of the theory of invariants in its present state of development is the theory of invariants of algebraical polynomials under linear transformations. Other important fields are differential invariants and number-theoretic invariant theories. In this book we treat, for the most part, the algebraical invariants.

1.2.2 Quantics or forms.

A homogeneous polynomial in n variables $x_1, x_2, \ldots, x_n$ of order m in those variables is called a quantic, or form, of order m. Illustrations are

$$f(x_1, x_2) = a_0x_1^3 + 3a_1x_1^2x_2 + 3a_2x_1x_2^2 + a_3x_2^3,$$

$$f(x_1, x_2, x_3) = a_{200}x_1^2 + 2a_{110}x_1x_2 + a_{020}x_2^2 + 2a_{101}x_1x_3 + 2a_{011}x_2x_3 + a_{002}x_3^2$$

With reference to the number of variables in a quantic it is called binary, ternary; and if there are n variables, n-ary. Thus $f(x_1, x_2)$ is a binary cubic form; $f(x_1, x_2, x_3)$ a ternary quadratic form. In algebraic invariant theories of binary forms it is usually most convenient to introduce with each coefficient a_i the binomial multiplier $\binom{m}{i}$ as in $f(x_1, x_2)$. When these multipliers are present, a common notation for a binary form of order m is (Cayley)

$$f(x_1, x_2) = (a_0, a_1, \cdots, a_m \between x_1, x_2)^m = a_0x_1^m + ma_1x_1^{m-1}x_2 + \cdots.$$

If the coefficients are written without the binomial numbers, we abbreviate

$$f(x_1, x_2) = (a_0, a_1, \cdots, a_m \between x_1, x_2)^m = a_0x_1^m + a_1x_1^{m-1}x_2 + \cdots.$$

The most common notation for a ternary form of order m is the generalized form of $f(x_1, x_2, x_3)$ above. This is

$$f(x_1, x_2, x_3) = \sum_{p,q,r=0}^{m} \frac{m!}{p!q!r!} a_{pqr} x_1^p x_2^q x_3^r,$$

where p, q, r take all positive integral values for which $p + q + r = m$. It will be observed that the multipliers associated with the coefficients are in this case multinomial numbers. Unless the contrary is stated, we shall in all cases consider the coefficients a of a form to be arbitrary variables. As to coordinate representations we may assume (x_1, x_2, x_3), in a ternary form for instance, to be homogenous coÂ¨ordinates of a point in a plane, and its coefficients a_{pqr} to be homogenous coordinates of planes in M-space, where $M + 1$ is the number of the a's. Thus the ternary form is represented by a point in M dimensional space and by a curve in a plane.

1.2.3 Linear Transformations.

The transformations to which the variables in an n-ary form will ordinarily be subjected are the following linear transformations called collineations:

$$x_1 = {}_1x'_1 + \mu_1 x'_2 + \dots + {}_1x'_n$$
$$x_2 = {}_2x'_1 + \mu_2 x'_2 + \dots + {}_2x'_n \quad (23)$$
$$\dots$$
$$x_n = {}_nx'_1 + \mu_n x'_2 + \dots + {}_nx'_n.$$

In algebraical theories the only restriction to which these transformations will be subjected is that the inverse transformation shall exist. That is, that it be possible to solve for the primed variables in terms of the un-primed variables (cf. (10)). We have seen in Section 1, V (11), and VIII (22) that the verification of a covariant and indeed the very existence of a covariant depends upon the existence of this inverse transformation.

Theorem. A necessary and sufficient condition in order that the inverse of (23) may exist is that the determinant or modulus of the transformation,

$$M = (\mu \dots) = \begin{vmatrix} {}_1, & \mu_1, & {}_1, & \dots & {}_1 \\ {}_2, & \mu_2, & {}_2, & \dots & {}_2 \\ \dots & \dots & \dots & \dots & \dots \\ {}_n, & \mu_n, & {}_n, & \dots & {}_n \end{vmatrix},$$

shall be different from zero.

In proof of this theorem we observe that the minor of any element, as of μ_i, of M equals $\frac{\partial M}{\partial \mu_i}$. Hence, solving for a variable as x'_2, we obtain

$$x'_2 = M^{-1}\left(x_1 \frac{\partial M}{\partial \mu_1} + x_2 \frac{\partial M}{\partial \mu_2} + \dots + x_n \frac{\partial M}{\partial \mu_n}\right),$$

and this is a defined result in all instances except when $M = 0$, when it is undefined. Hence we must have $M \neq 0$.

1.2.4 A theorem on the transformed polynomial.
Let f be a polynomial in x1, x2 of order m,
f(x1, x2) = a0xm1
+ ma1xm−1
1 x2 + m
2 a2xm−2
1 x22
+ . . . + amxm2
.
Let f be transformed into f0 by T (cf. 31)
f0 = a00x0m
1 + ma01x0m−1
1 x02 + Â· Â· Â· + m
r a0rx0m−r
1 x0r
2 + Â· Â· Â· + a0mx0m
2 .
We now prove a theorem which gives a short method of constructing the coefficients a0r in terms of the coefficients a0, . . . , am.
Theorem. The coefficients a0r of the transformed form f0 are given by the formulas
a0r =
(m − r)!
m! µ1
@
@1
+ µ2
@
@2r
f(1, 2) (r = 0, . . . ,m). (231)
In proof of this theorem we note that one form of f0 is f(1x01+µ1x02, 2x01+ µ2x02). But since f0 is homogeneous this may be written
f0 = x0m
1 f(1 + µ1x02/x01, 2 + µ2x02/x01).
We now expand the right-hand member of this equality by Taylor's theorem, regarding x02/x01 as a parameter,

f0 = x0m
1 f(1, 2) + x02
x01 µ
@
@f(1, 2)
+
1
2! x02
x012 µ
@
@2
f(1, 2) + Â· Â· Â·
+
1
r! x02
x01r µ
@
@r
f(1, 2) + Â· Â· Â· ,
where
µ
@

@= µ1
@
@1
+ µ2
@
@2,
f0 = f(1, 2)x0m
1 + Â· Â· Â· +
1
r! µ
@
@r
f(1, 2)x0m−r
1 x0r
2 + Â· Â· Â·
+
1
m! µ
@
@m
f(1, 2)x0m
2 .
Comparison of this result with the above form of f0 involving the coefficients a0r
gives (231).
An illustration of this result may be obtained from (5). Here m = 2, and
a00 = a021
+ 2a112 + a222
= f(1, 2) = f0,
a01 = a01µ1 + a1(1µ2 + 2µ1) + a22µ2 =
1
2 µ
@
@f(1, 2), (24)
a02 = a0µ21
+ 2a11µ2 + a2µ22
=
1
2 µ
@
@2
f(1, 2).
1.2.5 A group of transformations.
If we combine two transformations, as T and
T0 : x01 = 1x00 1 + 1x00 2 ,
x02 = 2x00 2 + 2x00 2 ,
there results
TT0 : x1 = (11 + µ12)x00 1 + (11 + µ12)x00 2 ,
x2 = (21 + µ22)x00 1 + (21 + µ22)x00 2 ,
This is again a linear transformation and is called the product of T and T0.
If now we consider 1, 2, µ1, µ2 in T to be independent continuous variables
assuming, say, all real values, then the number of linear transformations is
infinite, i.e. they form an infinite set, but such that the product of any two
transformations of the set is a third transformation of the set. Such a set of

transformations is said to form a group. The complete abstract definition of a
group is the following:
Given any set of distinct operations T, T00, T000, Â· Â· Â· , finite or infinite in number
and such that:
() The result of performing successively any two operations of the set is

another definite operation of the set which depends only upon the component operations and the sequence in which they are carried out:
() The inverse of every operation T exists in the set; that is, another operation T−I such that TT−I is the identity or an operation which produces no effect.
This set of operations then forms a group.
The set described above therefore forms an infinite group. If the transformations of this set have only integral coefficients consisting of the positive residues of a prime number p, it will consist of only a finite number of operations and so will form a finite group.
1.2.6 The induced group.
The equalities (24) constitute a set of linear transformations on the variables a0, a1, a2. Likewise in the case of formulas (231). These transformations are said to be induced by the transformations T. If T carries f into f0 and T00 carries f0 into f00, then
a00 r =
(m − r)!
m!
@
@r
f0(1, 2)
=
(m − r)!
m!
@
@ m Xs=0
1
s! µ
@
@s
f(1, 2)m−2
1 s
2. (241)
(r = 0, 1, Â· Â· Â· ,m).
This is a set of linear transformations connecting the a00 r directly with a0, Â· Â· Â· , am. The transformations are induced by applying T, T00 in succession to f. Now the induced transformations (231) form a group; for the transformations induced by applying T and T0 in succession is identical with the transformation induced by the product TT0. This is capable of formal proof. For by (281) the result of transforming f by TT0 is
a00 r =
(m − r)!
m!
rf(11 + µ12, 21 + µ22),
where
 = (11 + µ12) @
@(11 + µ12)
+ (21 + µ22) @
@(21 + µ22)

But
(11 + µ12) @
@(11 + µ12)
= 1
@
(11 + µ12)
@(11 + µ12)
@1
+ 2

@
(11 + µ12)
@(11 + µ12)
@2
= 1
@
@1
+ 2
@
@2
.
Hence
=
@
@+
@
@
and by the method of (IV) combined with this value of
a00 r =
(m − r)!
m!
@
@r m X0
1
s! µ
@
@s
f(1, 2)m−s
1 s
2.
But this is identical with (241). Hence the induced transformations form a group, as stated. This group will be called the induced group.
Definition.
A quantic or form, as for instance a binary cubic f, is a function of two distinct sets of variables, e.g. the variables x1, x2, and the coefficients a0, . . . , a3. It is thus quaternary in the coefficients and binary in the variables x1, x2. We call it a quaternary-binary function. In general, if a function F is homogeneous and of degree i in one set of variables and of order ! in a second set, and if the first set contains m variables and the second set n, then F is said to be an m-ary-n-ary function of degree-order (i, !). If the first set of variables is a0, . . . , am, and the second and the second x1, . . . , xn, we frequently employ the notation
F = (a0, . . . , am)i(x1, . . . , xn)!.
1.2.7 Cogrediency.
In many invariant theory problems two sets of variables are brought under consideration simultaneously. If these sets (x1, x2, Â· Â· Â· , xn), (y1, y2, Â· Â· Â· , yn) are subject to the same scheme of transformations, as (23), they are said to be cogredient sets of variables.
As an illustration of cogredient sets we first take the modular binary transformations,
Tp : x1 = 1x01 + µ1x02, x2 = 2x01 + µ2x02,
where the coefficients , µ are integers reduced modulo p as in Section 1, VIII.
We can prove that with reference to Tp the quantities xp
1, xp
2, are cogredient to

x1, x2. For all binomial numbers p
i, where p is a prime, are divisible by p
except p
0 and p
p. Hence, raising the equations of Tp to the pth power, we

have

$$x_1^p \equiv \lambda_1^p x_1'^p + \mu_1^p x_2'^p, \quad x_2^p \equiv \lambda_2^p x_1'^p + \mu_2^p x_2'^p \pmod p.$$

But by Fermat's theorem,

$$\lambda_i^p \equiv \lambda_i, \quad \mu_i^p \equiv \mu_i \pmod p \quad (i = 1, 2).$$

Therefore

$$x_1^p = \lambda_1 x_1'^p + \mu_1 x_2'^p, \quad x_2^p = \lambda_2 x_1'^p + \mu_2 x_2'^p,$$

and the cogrediency of x_1^p, x_2^p with x_1, x_2 under T_p is proved.

1.2.8 Theorem on the roots of a polynomial.

Theorem. The roots $(r_1^{(1)}, r_2^{(1)}), (r_1^{(2)}, r_2^{(2)}), \cdots, (r_1^{(m)}, r_2^{(m)})$ of a binary form

$$f = a_0x_1^m + ma_1x_1^{m-1}x_2 + \ldots + a_mx_2^m,$$

are cogredient to the variables.

To prove this we write

$$f = (r_2^{(1)} x_1 - r_1^{(1)} x_2)(r_2^{(2)} x_1 - r_1^{(2)} x_2) \cdots (r_2^{(m)} x_1 - r_1^{(m)} x_2),$$

and transform f by T. There results

$$f' = \prod_{i=1}^{m} \left[(r_2^{(i)}\lambda_1 - r_1^{(i)}\lambda_2)x_1' + (r_2^{(i)}\mu_1 - r_1^{(i)}\mu_2)x_2'\right].$$

Therefore

$$r_2'^{(i)} = r_2^{(i)}\lambda_1 - r_1^{(i)}$$

$r'^{(i)}_1 = -(r^{(i)}_2 \mu_1 - r^{(i)}_1 \mu_2)$.

Solving these we have

$$(\mu) r^{(i)}_1 = \lambda_1 r'^{(i)}_1 + \mu_1 r'^{(i)}_2,$$

$$(\mu) r^{(i)}_2 = \lambda_2 r'^{(i)}_1 + \mu_2 r'^{(i)}_2.$$

Thus the r's undergo the same transformation as the x's (save for a common multiplier (μ)), and hence are cogredient to x_1, x_2 as stated.

1.2.9 Fundamental postulate.

We may state as a fundamental postulate of the invariant theory of quantics subject to linear transformations the following: Any covariant of a quantic or system of quantics, i.e. any invariant formation containing the variables $x_1, x_2, \ldots$ will keep its invariant property unaffected when the set of elements $x_1, x_2, \ldots$ is replaced by any cogredient set.

This postulate asserts, in effect, that the notation for the variables may be changed in an invariant formation provided the elements introduced in place of the old variables are subject to the same transformation as the old variables.

Since invariants may often be regarded as special cases of covariants, it is desirable to have a term which includes both types of invariant formations. We shall employ the word concomitant in this connection.

BINARY CONCOMITANTS

Since many chapters of this book treat mainly the concomitants of binary forms, we now introduce several definitions which appertain in the first instance to the binary case.

1.2.10 Empirical definition.

Let

$$f = a_0 x_1^m + m a_1 x_1^{m-1} x_2 + \frac{1}{2} m(m-1) a_2 x_1^{m-2} x_2^2 + \cdots + a_m x_2^m,$$

be a binary form of order m. Suppose f is transformed by T into

$$f' = a'_0 x'^m_1 + m a'_1 x'^{m-1}_1 x'_2 + \cdots + a'_m x'^m_2.$$

We construct a polynomial ϕ in the variables and coefficients of f. If this function ϕ is such that it needs at most to be multiplied by a power of the determinant or modulus of the transformation (μ), to be made equal to the same function of the variables and coefficients of f', then ϕ is a concomitant of f under T. If the order of ϕ in the variables x_1, x_2 is zero, ϕ is an invariant. Otherwise it is a covariant. An example is the discriminant of the binary quadratic, in Paragraph III of Section 1.

If ϕ is a similar invariant formation of the coefficients of two or more binary forms and of the variables x_1, x_2, it is called a simultaneous concomitant. Illustrations are h in Paragraph IV of Section 1, and the simultaneous covariant C in Paragraph V of Section 1.

We may express the fact of the invariancy of ϕ in all these cases by an

equation
0 = (μ)k,
in which 0 is understood to mean the same function of the coefficients a00, a01, . . ., and of x01, x02 that is of a0, a1, . . ., and x1, x2. Or we may write more explicitly
(a00, a01, . . . ; x01, x02) = (μ)k(a0, a1, . . . ; x1, x2). (25)
We need only to replace T by (23) and (μ) by M = (μ Â· Â· Â·) in the above to obtain an empirical definition of a concomitant of an n-ary form f under (23). The corresponding equation showing the concomitant relation is
(a0; x01, x02, . . . , x0n) = Mk(a; x1, x2, . . . , xn). (26)

An equation such as (25) will be called the invariant relation corresponding to the invariant .
1.2.11 Analytical definition.
1 We shall give a proof in Chapter II that no essential particularization of the above definition of an invariant of a binary form f is imposed by assuming that is homogeneous both in the a's and in the x's. Assuming this, we define a concomitant of f as follows:
(1) Let be a function of the coefficients and variables of f, and 0 the same function of the coefficients and variables of f0. Assume that it is a function such that
μ1
@0
@1
+ μ2
@0
@2
= 0, 1
@0
@μ1
+ 2
@0
@μ2
= 0. (27)
(2) Assume that 0 is homogeneous in the sets 1, 2; μ1, μ2 and of order k in each.
Then is called a concomitant of f.
We proceed to prove that this definition is equivalent to the empirical definition above.
Since 0 is homogeneous in the way stated, we have by Euler's theorem and (1) above
1
@0
@1
+ 2
@0
@2
= k0, μ
@
@0 = 0, (28)
where k is the order of 0 in 1, 2. Solving these,
@0
@1
= kμ20(μ)−1,
@0
@2
= −kμ10(μ)−1.
Hence
d0 = @0

@1
d1 + @0
@2
d2 = (µ)−1k0(µ2d1 − µ1d2).
Separating the variables and integrating we have
d0
0
= k
d(µ)
(µ) , 0 = C(µ)k,
where C is the constant of integration. To determine C, let T be particularized
to
x1 = x01, x2 = x02.
Then a0i = ai(i = 0, 1, 2, Â· Â· Â· ,m), and 0 . Also (µ) = 1. Hence by
substitution
0 = (µ)k,
1The idea of an analytical definition of invariants is due to Cayley. Introductory Memoir
upon Quantics. Works, Vol. II.

and this is the same as (25). If we proceed from

@
@µ0 = 0, µ
@
@µ0 = k0,
we arrive at the same result. Hence the two definitions are equivalent.
1.2.12 Annihilators.
We shall now need to refer back to Paragraph IV (231) and Section 1 (10) and
observe that
µ
@
@a0r = (m − r)a0r+1,µ
@
@x01 = 0,µ
@
@x02 = −x01. (29)
Hence the operator µ @
@ applied to 0, regarded as a function of 1, 2, µ1, µ2,
has precisely the same effect as some other linear differential operator involving
only a0i(i = 0, Â· Â· Â· ,m) and x01, x02, which would have the effect (29) when applied
to 0 regarded as a function of a0i, x01, x02 alone. Such an operator exists. In fact
we can see by empirical considerations that
O0 − x01
@
@x02 ma01
@
@a00
+ (m − 1)a02
@
@a01
+ Â· Â· Â· + a0m
@
@a0m−1 − x01
@
@x02
(291)
is such an operator. We can also derive this operator by an easy analytical
procedure. For,
µ

$$\frac{\partial \phi'}{\partial \mu} = \frac{\partial \phi'}{\partial a'_0}\frac{\partial a'_0}{\partial \mu} + \frac{\partial \phi'}{\partial a'_1}\frac{\partial a'_1}{\partial \mu} + \cdots + \frac{\partial \phi'}{\partial a'_m}\frac{\partial a'_m}{\partial \mu} + \frac{\partial \phi'}{\partial x'_2}\frac{\partial x'_2}{\partial \mu} = 0,$$

or, by (29) $O'\phi' - x'_1 \frac{\partial \phi'}{\partial x'_2} = 0.$ (291)

In the same manner we can derive from $\left(\frac{\partial \phi'}{\partial \mu}\right)_0 = 0,$

$$\Omega'\phi' - x'_2 \frac{\partial \phi'}{\partial x'_1} \equiv a'_0 \frac{\partial \phi'}{\partial a'_1} + 2a'_1 \frac{\partial \phi'}{\partial a'_2} + \cdots + m a'_{m-1} \frac{\partial \phi'}{\partial a'_m} - x'_2 \frac{\partial \phi'}{\partial x'_1} = 0. \qquad (292)$$

The operators (291), (292) are called annihilators (Sylvester). Since ϕ is the same function of a_i, $x_1 x_2$, that ϕ' is of a'_i, $x'_1 x'_2$ we have, by dropping primes, the result:

Theorem. A set of necessary and sufficient conditions that a homogeneous function, ϕ, of the coefficients and variables of a binary form f should be a concomitant is

$$O\phi - x_1 \frac{\partial \phi}{\partial x_2} = 0, \quad \Omega\phi - x_2 \frac{\partial \phi}{\partial x_1} = 0.$$

In the case of invariants these conditions reduce to $O\phi = 0$, $\Omega\phi = 0$. These operators are here written again, for reference, and in the un-primed variables:

$$O = m a_1 \frac{\partial}{\partial a_0} + (m-1) a_2 \frac{\partial}{\partial a_1} + \cdots + a_m \frac{\partial}{\partial a_{m-1}},$$

$$\Delta = a_0\frac{\partial}{\partial a_1} + 2a_1\frac{\partial}{\partial a_2} + \cdots + ma_{m-1}\frac{\partial}{\partial a_m}.$$

A simple illustration is obtainable in connection with the invariant

$$D_1 = a_0a_2 - a_1^2$$

(Section 1, III).

Here $m = 2$:

$$\Delta = a_0\frac{\partial}{\partial a_1} + 2a_1\frac{\partial}{\partial a_2},\quad O = 2a_1\frac{\partial}{\partial a_0} + a_2\frac{\partial}{\partial a_1}.$$

$$\Delta D_1 = -2a_0a_1 + 2a_0a_1 \equiv 0,\quad OD_1 = 2a_1a_2 - 2a_1a_2 \equiv 0.$$

It will be noted that this method furnishes a convenient means of checking the work of computing any invariant.

1.3 Special Invariant Formations

We now prove the invariancy of certain types of functions of frequent occurrence in the algebraic theory of quantics.

1.3.1 Jacobians.

Let $f_1, f_2, \cdots, f_n$ be n homogeneous forms in n variables $x_1, x_2, \cdots, x_n$. The determinant,

$$J = \begin{vmatrix} f_{1x_1}, & f_{1x_2}, & \cdots, & f_{1x_n} \\ f_{2x_1}, & f_{2x_2}, & \cdots, & f_{2x_n} \\ \cdot & \cdot & \cdot & \cdot \\ f_{nx_1}, & f_{nx_2}, & \cdots, & f_{nx_n} \end{vmatrix} \qquad (30)$$

in which $f_{1x_1} = \frac{\partial f_1}{\partial x_1}$, etc., is the functional determinant, or Jacobian of the n forms. We prove that J is invariant when the forms f_i are transformed by (23), i.e. by

$$x_i = \lambda_i x'_1 + \mu_i x'_2 + \cdots + \nu_i x'_n \quad (i = 1, 2, \cdots, n). \qquad (31)$$

To do this we construct the Jacobian J' of the transformed quantic f'_j. We have from (31),

$$\frac{\partial f'_j}{\partial x'_2} = \frac{\partial f'_j}{\partial x_1}\frac{\partial x_1}{\partial x'_2}$$

+
@f0 j
@x2
@x2
@x02
+ Â· Â· Â· +
@f0 j
@xn
@xn
@x02
.
But by virtue of the transformations (31) we have in all cases, identically,
f0 j = fj(j = 1, 2, Â· Â· Â· , n). (32)

Hence
@f0 j
@x02
= µ1
@fj
@x1
+ µ2
@fj
@x2
+ Â· Â· Â· + µn
@fj
@xn
, (33)
and we obtain similar formulas for the derivatives of f0 j with respect to the other variables. Therefore
J0 =
1f1x1 + 2f1x2 + Â· Â· Â· + nf1xn, µ1f1x1 + µ2f1x2 + Â· Â· Â· + µnf1xn, Â· Â· Â· ....
1fnx1 + 2fnx2 + Â· Â· Â· + nfnxn, µ1fnx1 + µ2fnx2 + Â· Â· Â· + µnfnxn, Â· Â· Â·

.
(1.1)
But this form of J0 corresponds exactly with the formula for the product of two nth order determinants, one of which is J and the other the modulus M.
Hence
J0 = (µ Â· Â· Â·)J,
and J is a concomitant. It will be observed that the covariant C in Paragraph V of Section 1 is the Jacobian of f and .
1.3.2 Hessians.
If f is an n-ary form, the determinant
H =
fx1x1 , fx1x2 , . . . , fx1xn
fx2x1 , fx2x2 , . . . , fx2xn ...
...
. . .
...
fxnx1 , fxnx2 , . . . , fxnxn

(34)
is called the Hessian of f. That H possesses the invariant property we may prove as follows: Multiply H by M = (µ Â· Â· Â·), and make use of (33). This gives
MH

1 µ1 . . . 1
2 µ2 . . . 2

...
...
. . .
...
n µn . . . n

H =
@
@x01
@f
@x1
, @
@x02
@f
@x1
, . . . , @
@x0n
@f
@x1
@
@x01
@f
@x2
, @
@x02
@f
@x2
, . . . , @
@x0n
@f
@x2
...
...
. . .
...
@
@x01
@f
@xn
, @
@x02
@f
@xn
, . . . , @
@x0n
@f
@xn

.
Replacing f by f0 as in (32) and writing
@
@x01
@f
@x1
= @
@x1
@f0
@x01
, etc.,

we have, after multiplying again by M,

$$M^2H = \begin{vmatrix} f_{x'_1x'_1}, & f_{x'_2x'_1}, & \dots, & f_{x'_nx'_1} \\ f_{x'_1x'_2}, & f_{x'_2x'_1}, & \dots, & f_{x'_nx'_2} \\ \vdots & \vdots & \cdots & \vdots \\ f_{x'_1x'_n}, & f_{x'_2x'_n}, & \dots, & f_{x'_nx'_n} \end{vmatrix},$$

that is to say,

$$H' = (\mu \cdots)^2 H,$$

and H is a concomitant of f.

It is customary, and avoids extraneous numerical factors, to define the Hessian as the above determinant divided by $\frac{1}{2}m^n \times (m-1)^n$. Thus the Hessian covariant of the binary cubic form

$$f = a_0x_1^3 + 3a_1x_1^2x_2 + 3a_2x_1x_2^2 + a_3x_2^3,$$

is

$$= 2\begin{vmatrix} a_0x_1 + a_1x_2, & a_1x_1 + a_2x_2 \\ a_1x_1 + a_2x_2, & a_2x_1 + a_3x_2 \end{vmatrix},$$

$$= 2(a_0a_2 - a_1^2)x_1^2 + 2(a_0a_3 - a_1a_2)x_1x_2 + 2(a_1a_3 - a_2^2)x_2^2. \quad (35)$$

1.3.3 Binary resultants.

Let f, ϕ be two binary forms of respective orders, m, n;

$$f = a_0x_1^m + ma_1x_1^{m-1}x_2 + \cdots + a_mx_2^m = \prod_{i=1}^{m}(r_2^{(i)}x_1 - r_1^{(i)}x_2),$$

$$\phi = b_0x_1^n + nb_1x_1^{n-1}x_2 + \cdots + b_nx_2^n = \prod_{j=1}^{n}(s_2^{(j)}x_1 - s_1^{(j)}x_2).$$

It will be well known to students of the higher algebra that the following symmetric function of the roots $(r_1^{(i)}, r_2^{(i)}$

$_2$), $(s_1^{(j)}, s_2^{(j)})$, $R(f,)$ is called the resultant of f and . Its vanishing is a necessary and sufficient condition in order that f and should have a common root.

$$R(f,) = \prod_{j=1}^{n} \prod_{i=1}^{m} (r_1^{(i)} s_2^{(j)} - r_2^{(i)} s_1^{(j)}). \quad (36)$$

To prove that R is a simultaneous invariant of f and it will be sufficient to recall that the roots (r_1, r_2), (s_1, s_2) are cogredient to x_1, x_2. Hence when f, are each transformed by T, R undergoes the transformation

$$(\mu)r_1^{(i)} = {}_1 r_1'^{(i)} + \mu_1 r_2'^{(i)}, \quad (\mu)r_2^{(i)} = {}_2 r_1'^{(i)} + \mu_2 r_2'^{(i)},$$

$$(\mu)s_1^{(j)} = {}_1 s_1'^{(j)} + \mu_1 s_2'^{(j)}, \quad (\mu)s_2^{(j)} = {}_2 s_1'^{(j)} + \mu_2 s_2'^{(j)},$$

in which, owing to homogeneity the factors (μ) on the left may be disregarded. But under these substitutions,

$$r_1^{(i)} s_2^{(j)} - r_2^{(i)} s_1^{(j)} = (\mu)^{-1}(r_1'^{(i)} s_1'^{(j)} - r_2'^{(t)} s_1'^{(j)}).$$

2Throughout this book the notation for particular algebraical concomitants is that of Clebsch.

Hence

$$R'(f', ') = (\mu)^{mn} R(f,),$$

which proves the invariancy of the resultant.

The most familiar and elegant method of expressing the resultant of two forms f, in terms of the coefficients of the forms is by Sylvester's dialytic method of elimination. We multiply f by the n quantities $x_1^{n-1}, x_1^{n-2}x_2, \ldots, x_2^{n-1}$ in succession, and obtain

$$a_0 x_1^{m+n-1} + m a_1 x_1^{m+n-2} x_2 + \ldots + a_m x_1^{n-1} x_2^m,$$

$$a_0 x^{m+n-2}$$

1 x2 + ... + mam−1xn−2
1 xm+1
2 + amxn−1
1 xm2
, (37)
....
a0xm1
xn−1
2 + ... + mam−1x1xm+n−2
2 + amxm+n−1
2 .
Likewise if we multiply by the succession xm−1
1 , xm−2
1 x2, ..., xm−1
2 , we have
the array
b0xm+n−1
1 + nb1xm+n−2
1 x2 + ... + bnxm−1
1 xn2
,
....
b0xn1
xm−1
2 + ... + nbn−1x1xm+n−2
2 + bnxm+n−1
2 . (38)
The eliminant of these two arrays is the resultant of f and , viz.
R(f,) =

n rows8>><>>:
a0 ma1 . . . . . . am 0 0 . . . 0
0 a0 ma1 . . . mam−1 am 0 . . . 0
....
0 0 0 . . . . am
m rows8>><>>:
b0 nb1 . . . .
0 b0 nb1 . . . .
....
0 0 0 . . . . bn

.
A particular case of a resultant is shown in the next paragraph. The degree of
R(f,) in the coefficients of the two forms is evidently m + n.
1.3.4 Discriminant of a binary form.
The discriminant D of a binary form f is that function of its coefficients which
when equated to zero furnishes a necessary and sufficient condition in order that
f = 0 may have a double root. Let
f = f(x1, x2) = a0xm1
+ ma1xm−1
1 x2 + Â· Â· Â· + amxm2
,
and let fx1 (x1, x2) = @f
@x1
, fx2 (x1, x2) = @f
@x2
. Then, as is well known, a common
root of f = 0, @f
@x1

$= 0$ is a double root of $f = 0$ and conversely. Also

$$x_2^{-1}\left(mf - x_1\frac{\partial f}{\partial x_1}\right) = \frac{\partial f}{\partial x_2};$$

hence a double root of $f = 0$ is a common root of $f = 0$, $\frac{\partial f}{\partial x_1} = 0$, $\frac{\partial f}{\partial x_2} = 0$, and conversely; or D is equal either to the eliminant of f and $\frac{\partial f}{\partial x_1}$, or to that of f and $\frac{\partial f}{\partial x_2}$. Let the roots of $f_{x_1}(x_1, x_2) = 0$ be $(s_1^{(i)}, s_2^{(i)})$ $(i = 1, \cdots, m - 1)$, those of $f_x(x_1, x_2) = 0$, $(t_1^{(i)}, t_2^{(i)})$ $(i = 1, \cdots, m - 1)$, and those of $f = 0$ be $(r_1^{(j)}, r_2^{(j)})$ $(j = 1, 2, \cdots, m)$. Then

$$a_0 D = f(s_1^{(1)}, s_2^{(1)}) f(s_1^{(2)}, s_2^{(2)}) \cdots f(s_1^{(m-1)}, s_2^{(m-1)}),$$

$$a_m D = f(t_1^{(1)}, t_2^{(1)}) f(t_1^{(2)}, t_2^{(2)}) \cdots f(t_1^{(m-1)}, t_2^{(m-1)}).$$

Now $Of(x_1, x_2) = x_1 \frac{\partial f}{\partial x_2}$, $f(x_1, x_2) = x_2 \frac{\partial f}{\partial x_1}$, where 0 and are the annihilators of Section 2, XII. Hence

$$OD = \sum t_1^{(1)} f_{x_2}(t_1^{(1)}, t_2^{(1)}) f(t_1^{(2)}, t_2^{(2)}) \cdots f(t_1^{(m-1)}, t_2^{(m-1)})\ 0,$$

$$D = \sum s_2^{(1)} f_{x_1}(s_1^{(1)}, s_2^{(1)}) f(s_1^{(2)}, s_2^{(2)}) \cdots f(s_1^{(m-1)}, s_2^{(m-1)}) \equiv 0,$$

Thus the discriminant satisfies the two differential equations $OD = 0$, $D = 0$ and is an invariant. Its degree is $2(m - 1)$.

An example of a discriminant is the following for the binary cubic f, taken as the resultant of $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}$:

$$-\frac{1}{2}R = \begin{vmatrix} a_0 & 2a_1 & a_2 & 0 \\ 0 & a_0 & 2a_1 & a_2 \\ a_1 & 2a_2 & a_3 & 0 \\ 0 & a_1 & 2a_2 & a_3 \end{vmatrix} \quad (39)$$

$$= (a_0a_3 - a_1a_2)^2 - 4(a_0a_2 - a_1^2)(a_1a_3 - a_2^2).$$

1.3.5 Universal covariants.

Corresponding to a given group of linear transformations there is a class of invariant formations which involve the variables only. These are called universal covariants of the group. If the group is the infinite group generated by the transformations T in the binary case, a universal covariant is

$$d = (xy) = x_1y_2 - x_2y_1,$$

where (y) is cogredient to (x). This follows from

$$d = \begin{vmatrix} \lambda_1x'_1 + \mu_1x'_2, & \lambda_2x'_1 + \mu_2x'_2 \\ \lambda_1y'_1 + \mu_1y'_2, & \lambda_2y'_1 + \mu_2y'_2 \end{vmatrix} = (\lambda\mu)(x'y'). \quad (40)$$

If the group is the finite group modulo p, given by the transformations T_p, then since x_1^p, x_2^p are cogredient to x_1, x_2, we have immediately, from the above result for d, the fact that

$$L = x_1^p x_2 - x_1 x_2^p \quad (41)$$

is a universal covariant of this modular group.[3]

Another group of linear transformations, which is of consequence in geometry, is given by the well-known transformations of coördinate axes from a pair inclined at an angle ω to a pair inclined at an angle $\omega' = \beta - \alpha$, viz.

$$x_1 = \frac{\sin(\omega - \alpha)}{\sin\omega} x'_1 + \frac{\sin(\omega - \beta)}{\sin\omega} x'_2,$$

$$x_2 = \frac{\sin}{\sin\omega} x'_1 + \frac{\sin}{\sin\omega} x'_2. \quad (42)$$

Under this group the quadratic,

$$x_1^2 + 2x_1x_2\cos\omega + x_2^2 \quad (43)$$

is a universal covariant.[4]

[3] Dickson, Transactions Amer. Math. Society, vol. 12 (1911)

[4] Study, Leipz. Ber. vol. 40 (1897).

Chapter 2

PROPERTIES OF INVARIANTS

2.1 Homogeneity of a Binary Concomitant

2.1.1 Homogeneity.

A binary form of order m

$$f = a_0x_1^m + ma_1x_1^{m-1}x_2 + \cdots + a_mx_2^m,$$

is an $(m+1)$-ary-binary function, of degree-order $(1,m)$. A concomitant of f is an $(m + 1)$-ary-binary function of degree-order (i, ω). Thus the Hessian of the binary cubic (Chap. I, §3, II),

$$2(a_0a_2 - a_1^2)x_1^2 + 2(a_0a_3 - a_1a_2)x_1x_2 + 2(a_1a_3 - a_2^2)x_2^2, \quad (44)$$

is a quaternary-binary function of degree-order (2, 2). Likewise $f +$ is quaternary-binary of degree-order (2, 3), but non-homogeneous.

An invariant function of degree-order $(i, 0)$ is an invariant of f. If the degreeorder is $(0, \omega)$, the function is a universal covariant (Chap. I, §3, V). Thus $a_2a_2 - a_1^2$ of degree-order (2, 0) is an invariant of the binary quadratic under T, whereas $x_1^px_2 - x_1x_2^p$ of degree-order $(0, p + 1)$ is a universal modular covariant of T_p.

Theorem. If $C \equiv (a_0, a_1, \cdots, a_m)^i(x_1, x_2)^\omega$ is a concomitant of $f = (a_0, \cdots, a_m)(x_1, x_2)^m$, its theory as an invariant function loses no generality if we assume that it is homogeneous both as regards the variables x_1, x_2 and the variables $a_0, \cdots, a_m$.

Assume for instance that it is non-homogeneous as to x_1, x_2. Then it must equal a sum of functions which are separately homogeneous in x_1, x_2. Suppose

$$C = C_1 + C_2 + \cdots + C_s,$$

where $C_j = (a_0, a_1, \cdots, a_m)^{i_0}(x_1, x_2)^{\omega_j}$ $(j = 1, 2, \cdots, s)$ $i_0 \leq i$. Suppose now that we wish to verify the covariancy of C, directly. We will have

$$C' = (a'_0, a'_1, \cdots, a'_m)^i(x'_1, x'_2)^\omega = (\mu)^kC, \quad (45)$$

in which relation we have an identity if a'_i is expressed as the appropriate linear expression in $a_0, \cdots, a_m$ and the x'_i as the linear expression in x_1, x_2, of Chapter I, Section 1 (10). But we can have

$$\sum_{j=1}^{s} C'_j = (\mu)^k \sum_{j=1}^{s} C_j,$$

identically in x_1, x_2 only provided

C0j = (µ)kCj (j = 1, Â· Â· Â· , s).
Hence Cj is itself a concomitant, and since it is homogeneous as to x1, x2, no generality will be lost by assuming all invariant functions C homogeneous in x1, x2.
Next assume C to be homogeneous in x1, x2 but not in the variables a0, a1, Â· Â· Â· , am.
Then
C = 1 + 2 + Â· Â· Â· + ,
where i is homogeneous both in the a's and in the x's. Then the above process of verification leads to the fact that
0j = (µ)k j ,
and hence C may be assumed homogeneous both as to the a's and the x's; which was to be proved. The proof applies equally well to the cases of invariants, covariants, and universal covariants.

2.2 Index, Order, Degree, Weight

In a covariant relation such as (45) above, k, the power of the modulus in the relation, shall be called the index of the concomitant. The numbers i, ! are respectively the degree and the order of C.

2.2.1 Definition.

Let = ap
0aq
1ar
2 Â· Â· Â· av
mxµ
1x!−µ
2 be any monomial expression in the coefficients and variables of a binary m-ic f. The degree of is of course i = p+q +r +Â· Â· Â·+v.
The number
w = q + 2r + 3s + Â· Â· Â· + mv + µ (46)
is called the weight of . It equals the sum of all of the subscripts of the letters forming factors of excluding the factors x2. Thus a3 is of weight 3; a0a21
a4 of
weight 6; a31
a4x21
x32
of weight 9. Any polynomial whose terms are of the type

and all of the same weight is said to be an isobaric polynomial. We can, by a method now to be described, prove a series of facts concerning the numbers !, i, k,w.
Consider the form f and a corresponding concomitant relation
C0 = (a00, a01, Â· Â· Â· , a0m)i(x01, x02)! = (µ)k(a0, a1, Â· Â· Â· , am)i(x1, x2)!. (47)
This relation holds true when f is transformed by any linear transformation
T : x1 = 1x01 + µ1x02,
x2 = 2x01 + µ2x02.
It will, therefore, certainly hold true when f is transformed by any particular case of T. It is by means of such particular transformations that a number of facts will now be proved.

2.2.2 Theorem on the index.

Theorem. The index k, order !, and degree i of C satisfy the relation
k =
1
2
(im − !). (48)
And this relation is true of invariants, i.e. (48) holds true when ! = 0.
To prove this we transform
f = a0xm1
+ ma1xm−1
1 x2 + Â· Â· Â· + amxm2
,

by the following special case of T:

$x_1 = x'_1$, $x_2 = x'_2$

The modulus is now 2, and $a'_j = ma_j (j = 0, \cdots, m)$. Hence from (47),

$(ma_0, ma_1, \cdots, ma_m)^i(-1x_1, -1x_2)^! = 2^k(a_0, a_1, \cdots, a_m)^i(x_1, x_2)^!$. (49)

But the concomitant C is homogeneous. Hence, since the degree-order is (i, !),

$^{im-!}(a_0, \cdots, a_m)^i(x_1, x_2)^! = 2^k(a_0, \cdots, a_m)^i(x_1, x_2)^!$

Hence

$2K = im - !$.

2.2.3 Theorem on weight.

Theorem. Every concomitant C of f is isobaric and the weight is given by

$$w = \frac{1}{2}(im + !),\quad (50)$$

where (i, !) is the degree-order of C, and m the order of f. The relation is true for invariants, i.e. if ! = 0.

In proof we transform f by the special transformation

$x_1 = x'_1$, $x_2 = x'_2$. (51)

Then the modulus is , and $a'_j = ia_i (j = 0, 2, \cdots, m)$. Let

$$ = a_0^p a_1^q a_2^r \cdots x_1^\mu x_2^{!-\mu}$$

be any term of C and $'$ the corresponding term of C', the transformed of C by (51). Then by (47),

$$' = ^{q+2r+\cdots+\mu-!} a_0^p a_1^q a_2^r \cdots x_1^\mu x_2^{!-\mu} = ^k .$$

Thus

$$w - ! = k = \frac{1}{2}(im - !),$$

or

$$w = \frac{1}{2}(im + !).$$

COROLLARY 1. The weight of an invariant equals its index,

$$w = k = \frac{1}{2}im.$$

COROLLARY 2. The degree-order (i, !) of a concomitant C cannot consist of an even number and an odd number except when m is even. Then i may be odd and ! even. But if m is even ! cannot be odd.

These corollaries follow directly from (48), (50).

As an illustration, if C is the Hessian of a cubic, (44), we have

$i = 2, ! = 2, m = 3$,

$$w = \frac{1}{2}(2 \cdot 3 + 2) = 4,$$

$$k = \frac{1}{2}(2 \cdot 3 - 2) = 2.$$

These facts are otherwise evident (cf. (44), and Chap. I, Section 3, II).

COROLLARY 3. The index k of any concomitant of f is a positive integer.

For we have

$$w - \omega = k,$$

and evidently the integer w is positive and $\omega \leqq w$.

2.3 Simultaneous Concomitants

We have verified the invariancy of two simultaneous concomitants. These are the bilinear invariants of two quadratics (Chap. I, Section 1, IV),

$$\phi = a_0x_1^2 + 2a_1x_1x_2 + a_2x_2^2,$$
$$\psi = b_0x_1^2 + 2b_1x_1x_2 + b_2x_2^2,$$

viz. $h = a_0b_2 - 2a_1b_1 + a_2b_0$,

and the Jacobian C of ϕ and ψ (cf. (8)). For another illustration we may introduce the Jacobian of ϕ and the Hessian, Δ, of a binary cubic f. This is (cf. (44))

$$J_{\phi,\Delta} = [b_0(a_0a_3 - a_1a_2) - 2b_1(a_0a_2 - a_1^2)]x_1^2 + 2[b_0(a_1a_3 - a_2^2) - b_2(a_0a_2 - a_1^2)]x_1x_2 + [2b_1(a_1a_3 - a_2^2) - b_2(a_0a_3 - a_1a_2)]x_2^2,$$

and it may be verified as a concomitant of ϕ and

$$f = a_0x_1^3 + \ldots.$$

The degree-order of J is (3, 2). This might be written (1 + 2, 2), where by the sum 1 + 2 we indicate that J is of partial degree 1 in the coefficients of the first form ϕ and of partial degree 2 in the coefficients of the second form f.

2.3.1 Theorem on index and weight.

Theorem. Let $f, \phi, \psi, \cdots$ be a set of binary forms of respective orders $m_1, m_2, m_3, \cdots$. Let C be a simultaneous concomitant of these forms of degree-order

$$(i_1 + i_2 + i_3 + \cdots, \omega)$$

Then the index and the weight of C are connected with the numbers m, i, ω by the relations

$$k = \frac{1}{2}\left(\sum i_1m_1 - \omega\right), w = \frac{1}{2}\left(\sum i_1m_1 + \omega\right), \quad (52)$$

and these relations hold true for invariants (i.e. when $\omega = 0$).

The method of proof is similar to that employed in the proofs of the theorems in Section 2. We shall prove in detail the second formula only. Let

$$f = a_0x_1^{m_1} + \cdots, \phi = b_0x_1^{m_2} + \cdots, \psi = c_0x_1^{m_3} + \cdots, \cdots.$$

Then a term of C will be of the form

$$\tau = a_0^{q_1} a_1^{r_1}$$

1 as1
2 Â· Â· Â· bq2
0 br2
1 bs2
2 Â· Â· Â· xµ
1x!−µ
2 .
Let the forms be transformed by x1 = x01, x2 = x02. Then a0j = jaj , b0j =
jbj , Â· Â· Â· (j = 0, Â· Â· Â· ,mi), and if 0 is the term corresponding to in the transformed
of C by this particular transformation, we have
 0 = r1+2s1+Â·Â·Â·+r2+2s2+Â·Â·Â·+µ−! = k.
Hence
w − ! = k =
1
2 Xi1m1 − !,

which proves the theorem.
We have for the three simultaneous concomitants mentioned above; from
formulas (52)
h C J
k = 2 k = I k = 3
w = 2 w = 3 w = 5

2.4 Symmetry. Fundamental Existence Theorem

We have shown that the binary cubic form has an invariant, its discriminant, of
degree 4, and weight 6. This is (cf. (39))
1
2R = −(a0a3 − a1a2)2 + 4(a0a2 − a21
)(a1a3 − a22
).

2.4.1 Symmetry.

We may note concerning it that it is unaltered by the substitution (a0a1)(a1a2).
This fact is a case of a general property of concomitants of a binary form of order
m. Let f = a0xm1
+Â· Â· Â· ; and let C be a concomitant, the invariant relation being
C0 = (a00, a01, . . . , a0m)i(x01, x02)! = (µ)k(a0, . . . , am)i(x1, x2)!.
Let the transformation T of f be particularized to
x1 = x02, x2 = x01.
The modulus is −1. Then a0j = am−j , and
C0 = (am, am−1, . . . , a0)i(x2, x1)! = (−1)k(a0, . . . , am)i(x1, x2)!. (53)
That is; any concomitant of even index is unchanged when the interchanges
(a0am)(a1am−1) Â· Â· Â· (x1x2) are made, and if the index be odd, the concomitant
changes only in sign. On account of this property a concomitant of odd index is
called a skew concomitant. There exist no skew invariants for forms of the first
four orders 1, 2, 3, 4. Indeed the simplest skew invariant of the quintic is quite
complicated, it being of degree 18 and weight 45 1 (Hermite). The simplest
skew covariant of a lower form is the covariant T of a quartic of (125) (Chap.
IV, Â§1).
We shall now close this chapter by proving a theorem that shows that the
number of concomitants of a form is infinite. We state this fundamental existence
theorem of the subject as follows:
Theorem. Every concomitant K of a covariant C of a binary form f is a
concomitant of f.
1Fa`a di Bruno, Walter. Theorie der BinÂ¨aren Formen, p. 320.

That this theorem establishes the existence of an infinite number of concomitants
of f is clear. In fact if f is a binary quartic, its Hessian covariant
H (Chap. I, Â§3) is also a quartic. The Hessian of H is again a quartic, and is
a concomitant of f by the present theorem. Thus, simply by taking successive
Hessians we can obtain an infinite number of covariants of f, all of the fourth

order. Similar considerations hold true for other forms.
In proof of the theorem we have
f = a0xm1
+ Â· Â· Â· ,
C = (a0, Â· Â· Â· , am)i(x1, x2)! = c0x!
1 + !c1x!−1
1 x2 + Â· Â· Â· ,
where ci is of degree i in a0, Â· Â· Â· , am.
Now let f be transformed by T. Then we can show that this operation
induces a linear transformation of C, and precisely T. In other words when f
is transformed, then C is transformed by the same transformation. For when f
is transformed into f0, C goes into
C0 (µ)k(c0x!
1 + !c1x!−1
1 x2 + Â· Â· Â·).
But when C is transformed directly by T, it goes into a form which equals C
itself by virtue of the equations of transformation. Hence the form C, induced by
transforming f, is identical with that obtained by transforming C by T directly,
save for the factor (µ)k. Thus by transformation of either f or C,
c00x0!
1 + !c01x0!−1
1 x02 + Â· Â· Â· = (µ)kc0x!
1 + !(µ)kc1x!−1
1 x2 + Â· Â· Â· (54)
is an equality holding true by virtue of the equations of transformation. Now
an invariant relation for K is formed by forming an invariant function from
the coefficients and variables of the left-hand side of (54) and placing it equal
to (µ)k times the same function of the coefficients and the variables of the
right-hand side,
K0 = (c00, . . . , c0!)(x01, x02)
= (µ)((µ)kc0, . . . , (µ)kc!)(x1, x2).
But K0 is homogeneous and of degree-order (,). Hence
K0 = (c00, . . . , c0!)(x01, x02) = (µ)k+(c0, . . . , c!)(x1, x2) (55)
= (µ)k+K.
Now c0j is the same function of the a00, . . . , a0m that cj is of a0, . . . , am. When
the c0's and c's in (55) are replaced by their values in terms of the a's, we have
K0 = [a00, . . . , a0m]i(x01, x02) = (µ)k+[a0, . . . , am]i(x1, x2) (56)
= (µ)k+K.
where, of course, [a0, . . . , am]i(x1, x2) considered as a function, is different
from (a0, . . . , am)i(x1, x2). But (56) is a covariant relation for a covariant of
f. This proves the theorem.

The proof holds true mutatis mutandis for concomitants of an n-ary form
and for simultaneous concomitants.
The index of K is
 = Â·
1
2
(im − !) +
1
2
(! −)
=
1
2
(im −),
and its weight,
w =
1

2
(im +).
Illustration. If f is a binary cubic,
$f = a_0x_1^3$
$+ 3a_1x_1^2$
$x_2 + 3a_2x_1x_2^2$
$+ a_2x_2^3$
,
then its Hessian,
 = 2[($a_0a_2 - a_1^2$
)x_1^2
+ ($a_0a_3 - a_1a_2$)x_1x_2 + ($a_1a_3 - a_2^2$
)x_2^2
],
is a covariant of f. The Hessian 2R of is the discriminant of , and it is also twice the discriminant of f,
2R = 4[−($a_0a_3 - a_1a_2$)2 + 4($a_0a_2 - a_1^2$
)($a_1a_2 - a_2^2$
)].

Chapter 3
THE PROCESSES OF INVARIANT THEORY

3.1 Invariant Operators

We have proved in Chapter II that the system of invariants and covariants of a form or set of forms is infinite. But up to the present we have demonstrated no methods whereby the members of such a system may be found. The only methods of this nature which we have established are those given in Section 3 of Chapter I on special invariant formations, and these are of very limited application. We shall treat in this chapter the standard known processes for finding the most important concomitants of a system of quantics.

3.1.1 Polars.

In Section 2 of Chapter I some use was made of the operations 1
@
@µ1
+2
@
@µ2
, µ1
@
@1
+
µ2
@
@2
. Such operators may be extensively employed in the construction of invariant formations. They are called polar operators.

Theorem. Let f = $a_0x_1^m$
+ Â· Â· Â· be an n-ary quantic in the variables x1, Â· Â· Â· , xn
and a concomitant of f, the corresponding invariant relation being
0 = (a00, Â· Â· Â·)i(x01, Â· Â· Â· , x0n)!
= (µ Â· Â· Â·)k(a0, Â· Â· Â·)i(x1, Â· Â· Â· , xn)! = Mk. (57)
Then if y1, y2, Â· Â· Â· , yn are cogredient to x1, x2, Â· Â· Â· , xn, the function
y
@
@x y1
@
@x1
+ y2
@

$$\frac{\partial}{\partial x_2} + \cdots + y_n \frac{\partial}{\partial x_n}$$

is a concomitant of f.

It will be sufficient to prove that

$$y' \frac{\partial}{\partial x'} = M y \frac{\partial}{\partial x}; \quad (58)$$

the theorem will then follow directly by the definition of a covariant. On account of cogrediency we have

$$x_i = \lambda_i x'_1 + \mu_i x'_2 + \cdots + \sigma_i x'_n,$$
$$y_i = \lambda_i y'_1 + \mu_i y'_2 + \cdots + \sigma_i y'_n \quad (i = 1, \cdots, n). \quad (59)$$

Hence

$$\frac{\partial}{\partial x'_1} = \frac{\partial}{\partial x_1}\frac{\partial x_1}{\partial x'_1} + \frac{\partial}{\partial x_2}\frac{\partial x_2}{\partial x'_1} + \cdots + \frac{\partial}{\partial x_n}\frac{\partial x_n}{\partial x'_1},$$

$$\frac{\partial}{\partial x'_1} = \lambda_1 \frac{\partial}{\partial x_1} + \lambda_2 \frac{\partial}{\partial x_2} + \cdots + \lambda_n \frac{\partial}{\partial x_n},$$

$$\frac{\partial}{\partial x'_2} = \mu_1 \frac{\partial}{\partial x_1} + \mu_2 \frac{\partial}{\partial x_2} + \cdots + \mu_n \frac{\partial}{\partial x_n},$$

....

$$\frac{\partial}{\partial x'_n}$$

= 1
@
@x1
+ 2
@
@x2
+ Â· Â· Â· + n
@
@xn
.
Therefore
y01
@
@x01
+ Â· Â· Â· + y0n
@
@x0n
= (1y01 + µ1y02 + Â· Â· Â· + 1y0n) @
@xn
+ Â· Â· Â· + (ny01 + µny02 + Â· Â· Â· + ny0n) @
@xn
= y1
@
@x1
+ Â· Â· Â· + yn
@
@xn
.
Hence (58) follows immediately when we operate upon (57) by
y0 @
@x0= y
@
@x. (60)
The function (y @
@x) is called the first polar covariant of , or simply the first
polar of . It is convenient, however, and avoids adventitious numerical factors,
to define as the polar of the expression (y @
@x) times a numerical factor. We
give this more explicit definition in connection with polars of f itself without
loss of generality. Let f be of order m. Then
|m − r
|m y1
@
@x1
+ Â· Â· Â· + yn
@
@xnr
f fyr , (61)
the right-hand side being merely an abbreviation of the left-hand side, is called
the rth y-polar of f. It is an absolute covariant of f by (60).

For illustration, the first polars of
f = a0x31
+ 3a1x21
x2 + 3a2x1x22
+ a3x32
,
g = a200x21
+ 2a110x1x2 + a020x22

$$+ 2a_{101}x_1x_3 + 2a_{011}x_2x_3 + a_{002}x_3^2,$$

are, respectively,

$$f_y = (a_0x_1^2 + 2a_1x_1x_2 + a_2x_2^2)y_1 + (a_1x_1^2 + 2a_2x_1x_2 + a_3x_2^2)y_2,$$

$$g_y = (a_{200}x_1 + a_{110}x_2 + a_{101}x_3)y_1 + (a_{110}x_1 + a_{020}x_2 + a_{011}x_3)y_2 + (a_{101}x_1 + a_{011}x_2 + a_{002}x_3)y_3.$$

Also,

$$f_{y^2} = (a_0y_1^2 + 2a_1y_1y_2 + a_2y_2^2)x_1 + (a_1y_1^2 + 2a_2y_1y_2 + a_3y_2^2)x_2.$$

If $g = 0$ is the conic C of the adjoining figure, and $(y) = (y_1, y_2, y_3)$ is the point P, then $g_y = 0$ is the chord of contact AB, and is called the polar line of P and the conic. If P is within the conic, $g_y = 0$ joins the imaginary points of contact of the tangents to C from P.

We now restrict the discussion in the remainder of this chapter to binary forms.

We note that if the variables (y) be replaced by the variables (x) in any polar of a form f the result is f itself, i.e. the original polarized form. This follows by Euler's theorem on homogeneous functions, since

$$\left(y\frac{\partial}{\partial x}\right)f\Big]_{y=x} = \left(x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}\right)f = mf. \quad (62)$$

In connection with the theorem on the transformed form of Chapter I, Section 2, we may observe that the coefficients of the transformed form are given by the polar formulas

$$a_0' = f(\lambda_1, \lambda_2) = f_0, a_1' = f_{0\mu}, a_2' = f_{0\mu^2}, \ldots, a_m' = f_{0\mu^m}. \quad (63)$$

The rth y-polar of f is a doubly binary form in the sets (y_1, y_2), (x_1, x_2) of degree-order $(r, m - r)$. We may however polarize a number of times as to (y) and then a number of times as to another cogredient set (z);

$$f_{y^r}\Big]_{z^s} = \frac{(m-r)!(m-r-s)!}{m!(m-r)!}\left(y\frac{\partial}{\partial x}\right)^r\left(z\frac{\partial}{\partial x}\right)^s f. \quad (64)$$

This result is a function of three cogredient sets (x), (y), (z).

Since the polar operator is a linear differential operator, it is evident that the polar of a sum of a number of forms equals the sum of the polars of these forms,

$$(f + \phi + \cdots)_{y^r} = f_{y^r} + \phi_{y^r} + \cdots.$$

3.1.2 The polar of a product.

We now develop a very important formula giving the polar of the product of two binary forms in terms of polars of the two forms.

If $F(x_1, x_2)$ is any binary form in x_1, x_2 of order M and (y) is cogredient to (x), we have by Taylor's theorem, k being any parameter,

$$F(x_1 + ky_1, x_2 + ky_2) = F(x_1, x_2) + k\left(y\frac{\partial}{\partial x}\right)F + k^2\frac{\left(y\frac{\partial}{\partial x}\right)^2 F}{2!} + \cdots + k^r\frac{\left(y\frac{\partial}{\partial x}\right)^r F}{r!} + \cdots$$
$$= F + \binom{M}{1}F_y k + \binom{M}{2}F_{y^2}k^2 + \cdots + \binom{M}{r}F_{y^r}k^r + \cdots. \quad (65)$$

Let $F = f(x_1, x_2)\phi(x_1, x_2)$, the product of two binary forms of respective orders m, n. Then the rth polar of this product will be the coefficient of k^r in the expansion of

$$f(x_1 + ky_1, x_2 + ky_2) \times \phi(x_1 + ky_1, x_2 + ky_2),$$

divided by $\binom{m+n}{r}$, by (65). But this expansion equals

$$\left(f + \binom{m}{1}f_y k + \binom{m}{2}f_{y^2}k^2 + \ldots + \binom{m}{r}f_{y^r}k^r + \ldots\right)\left(\phi + \binom{n}{1}\phi_y k + \binom{n}{2}\phi_{y^2}k^2 + \ldots + \binom{n}{r}\phi_{y^r}k^r + \ldots\right).$$

Hence by direct multiplication,

$$(f\phi)_{y^r} = \frac{1}{\binom{m+n}{r}}\left[\binom{m}{0}\binom{n}{r}f\phi_{y^r} + \binom{m}{1}\binom{n}{r-1}f_y\phi_{y^{r-1}} + \ldots + \binom{m}{r}\binom{n}{0}f_{y^r}\phi\right],$$

or

$$(f\phi)_{y^r} = \frac{1}{\binom{m+n}{r}}\sum_{s=0}^{r}\binom{m}{s}\binom{n}{r-s}f_{y^s}\phi_{y^{r-s}} \quad (66)$$

This is the required formula.

The sum of the coefficients in the polar of a product is unity. This follows from the fact (cf. (62)) that if (y) goes into (x) in the polar of a product it becomes the original polarized form.

An illustration of formula (66) is the following:

Let $f = a_0x_1^4 + \ldots$, $\phi = b_0x_1^2$

+ . . .. Then

$$f_{y^3} = \frac{1}{20}\left[\binom{4}{3}\binom{2}{0} f_{y^3} + \binom{4}{2}\binom{2}{1} f_{y^2 y} + \binom{4}{1}\binom{2}{2} f_{y y^2}\right] = \frac{1}{5} f_{y^3} + \frac{3}{5} f_{y^2 y} + \frac{1}{5} f_{y y^3}.$$

3.1.3 Aronhold's polars.

The coefficients of the transformed binary form are given by

$$a'_r = f_{\mu^r}(1, 2) \quad (r = 0, \ldots, m).$$

These are the linear transformations of the induced group (Chap. I, §2). Let be a second binary form of the same order as f,

$$= b_0 x_1^m + m b_1 x_1^{m-1} x_2 + \ldots.$$

Let be transformed by T into 0. Then

$$b'_r = u_r(1, 2).$$

Hence the set $b_0, b_1, \cdots, b_m$ is cogredient to the set $a_0, a_1, \cdots, a_m$, under the induced group. It follows immediately by the theory of Paragraph I that

$$b'_0 \frac{\partial}{\partial a'_0} + \cdots + b'_m \frac{\partial}{\partial a'_m} = b_0 \frac{\partial}{\partial a_0} + \cdots + b_m \frac{\partial}{\partial a_m} \equiv \left(b \frac{\partial}{\partial a}\right). \quad (67)$$

That is, $\left(b \frac{\partial}{\partial a}\right)$ is an invariant operation. It is called the Aronhold operator but was first discovered by Boole in 1841. Operated upon any concomitant of f it gives a simultaneous concomitant of f and . If m = 2, let

$$I = a_0 a_2 - a_1^2.$$

Then

$$\left(b \frac{\partial}{\partial a}\right) I = b_0 \frac{\partial}{\partial a_0} + b_1 \partial$$

@a1
+ b2
@
@a2I = a0b2 − 2a1b1 + a2b0.
This is h (Chap. I, Â§1). Also
2b
@
@a2
I = 4(b0b2 − b21
),

the discriminant of . In general, if is any concomitant of f,
0 = (a00, Â· Â· Â· , a0m)i(x01, x02)! = (µ)(a0, Â· Â· Â· , am)i(x1, x2)!,
then b0 @
@a0r
0 = (µ)b
@
@ar
(r = 0, 1, Â· Â· Â· , i) (68)
are concomitants of f and . When r = i, the concomitant is
x = (b0, Â· Â· Â· , bm)i(x1, x2)!.
The other concomitants of the series, which we call a series of Aronhold's polars of , are said to be intermediate to and , and of the same type as . The theory of types will be referred to in the sequel.
All concomitants of a series of Aronhold's polars have the same index k.
Thus the following series has the index k = 2, as may be verified by applying (52) of Section 3, Chapter II to each form (f = a0x31
+ Â· Â· Â· ; = b0x31
+ Â· Â· Â·):
H = (a0a2 − a21
)x21
+ (a0a3 − a1a2)x1x2 + (a1a3 − a22
)x22
,
b
@
@aH = (a0b2 − 2a1b1 + a2b0)x21
+ (a0b3 + a3b0 − a1b2 − a2b1)x1x2
+ (a1b3 − 2a2b2 + a3b1)x22
,
1
2 b
@
@a2
H = (b0b2 − b21
)x21
+ (b0b3 − b1b2)x1x2 + (b1b3 − b22
)x22
.
3.1.4 Modular polars.
Under the group Tp, we have shown, xp
1, xp
2 are cogredient to x1, x2. Hence the
polar operation
p = xp
1
@
@x1
+ xp

2 $\frac{\partial}{\partial x_2}$, (69)

applied to any algebraic form f, or covariant of f, gives a formal modular concomitant of f. Thus if

$$f = a_0x_1^2 + 2a_1x_1x_2 + a_2x_2^2,$$

then,

$$\frac{1}{2}3f = a_0x_1^4 + a_1(x_1^3x_2 + x_1x_2^3) + a_2x_2^4.$$

This is a covariant of f modulo 3, as has been verified in Chapter I, Section 1. Under the induced modular group $a_0^p, x_1^p, \cdots, a_m^p$ will be cogredient to $a_0, a_1, \cdots, a_m$. Hence we have the modular Aronhold operator

$$d_p = a_0^p \frac{\partial}{\partial a_0} + \cdots + a_m^p \frac{\partial}{\partial a_m}.$$

If m = 2, and

$$D = a_0a_2 - a_1^2,$$

then

$$d_pD \equiv a_0^pa_2 - 2a_1^{p+1} + a_0a_2^p \pmod p.$$

This is a formal modular invariant modulo p. It is not an algebraic invariant; that is, not invariantive under the group generated by the transformations T.

We may note in addition that the line

$$l = a_0x_1 + a_1x_2 + a_2x_3$$

has among its covariants modulo 2, the line and the conic

$$d_2l = a_0^2x_1 + a_1^2x_2 + a_2^2x_3,$$

$$2l = a_0x_1^2 + a_1x_2^2 + a_2x_3^2.$$

3.1.5 Operators derived from the fundamental postulate.

The fundamental postulate on cogrediency (Chap. I, §2) enables us to replace the variables in a concomitant by any set of elements cogredient to the variables, without disturbing the property of invariance.

Theorem. Under the binary transformations T the differential operators $\frac{\partial}{\partial x_2}$

,− @
@x1 are cogredient to the variables.
From T we have
@
@x01
= 1
@
@x1
+ 2
@
@x2
,
@
@x02
= μ1
@
@x1
+ μ2
@
@x2
.
Hence
(μ) @
@x2
= 1
@
@x02
+ μ1 −
@
@x01,
−(μ) @
@x1
= 2
@
@x02
+ μ2 −
@
@x01.
This proves the theorem.
It follows that if = (a0, . . . , am)i(x1, x2)! is any invariant function, i.e. a concomitant of a binary form f, then
@ = (a0, . . . , am)i @
@x2
,−
@
@x1!
(70)
is an invariant operator (Boole). If this operator is operated upon any covariant of f, it gives a concomitant of f, and if operated upon a covariant of any set

of forms g, h, . . ., it gives a simultaneous concomitant of f and the set. This process is a remarkably prolific one and enables us to construct a great variety of invariants and covariants of a form or a set of forms. We shall illustrate it by means of several examples.
Let f be the binary quartic and let be the form f itself. Then
@ = @f = a0
@4
@x42
− 4a1

$$\frac{\partial^4}{\partial x_2^3 \partial x_1} + 6a_2 \frac{\partial^4}{\partial x_2^2 \partial x_1^2} - 4a_3 \frac{\partial^4}{\partial x_2 \partial x_1^3} + a_4 \frac{\partial^4}{\partial x_1^4},$$

and

$$\frac{1}{24} f \cdot f = 2(a_0a_4 - 4a_1a_3 + 3a_2^2) = i.$$

This second degree invariant i represents the condition that the four roots of the quartic form a self-apolar range. If this process is applied in the case of a form of odd order, the result vanishes identically.

If H is the Hessian of the quartic, then

$$\partial H = (a_0a_2 - a_1^2) \frac{\partial^4}{\partial x_2^4} - 2(a_0a_3 - a_1a_2) \frac{\partial^4}{\partial x_2^3 \partial x_1} + (a_0a_4 + 2a_1a_3 - 3a_2^2) \frac{\partial^4}{\partial x_2^2 \partial x_1^2} - 2(a_1a_4 - a_2a_3) \frac{\partial^4}{\partial x_2 \partial x_1^3} + (a_2a_4 - a_3^2) \frac{\partial^4}{\partial x_1^4}.$$

And

$$\frac{1}{12} \partial H \cdot f = 6(a_0a_2a_4 + 2a_1a_2a_3 - a_0a_3^2 - a_2^3) = J. \quad (701)$$

This third-degree invariant equated to zero gives the condition that the roots of the quartic form a harmonic range.

If H is the Hessian of the binary cubic f and $g = b_0x_1^3 + \dots,$ then

$$\frac{1}{6} \partial H \cdot g = [b_0(a_1a_3 - a_2^2) + b_1(a_1a_2 - a_0a_3) + b_2(a_0a_2 - a_1^2)]x_1 + [b_1(a_1a_3 - a_2^2) + b_2(a_1a_2 - a_0a_3) + b_3(a_0a_2 - a_1^2)]x_2;$$

a linear covariant of the two cubics.

Bilinear Invariants

If $f = a_0x_1^m + \cdots$ is a binary form of order m and $g = b_0x_1^m + \cdots$ another of the same order, then

1
m!@f Â· g = a0bm − m
1 a1bm−1 + Â· Â· Â· + (−1)rm
r arbm−r + Â· Â· Â· + (−1)mamb0.
(71)

This, the bilinear invariant of f and g, is the simplest joint invariant of the two forms. If it is equated to zero, it gives the condition that the two forms be apolar. If m = 2, the apolarity condition is the same as the condition that the two quadratics be harmonic conjugates (Chap. I, Â§1, IV).

3.1.6 The fundamental operation called transvection.

The most fundamental process of binary invariant theory is a differential operation called transvection. In fact it will subsequently appear that all invariants and covariants of a form or a set of forms can be derived by this process. We proceed to explain the nature of the process. We first prove that the following operator is an invariant:

 =
@
@x1
, @
@x2
@
@y1
, @
@y2
(72)

where (y) is cogredient to (x). In fact by (70),

0 =
1
@
@x1
+ 2
@
@x2
, µ1
@
@x1
+ µ2
@
@x2
1
@
@y1
+ 2
@
@y2
, µ1
@
@y1
+ µ2
@
@y2
= (µ)
,

which proves the statement.

Evidently, to produce any result, must be applied to a doubly binary function. One such type of function is a y-polar of a binary form. But Theorem. The result of operating upon any y-polar of a binary form f is zero.

For, if $f = a_0x_1^m + \cdots$,

$$P = \frac{m!}{(m-r)!} f_{r}\, y = \left(y \frac{\partial}{\partial x}\right)^r f = y_1^r \frac{\partial^r f}{\partial x_1^r} + r y_1^{r-1} y_2 \frac{\partial^r f}{\partial x_1^{r-1}\partial x_2} + \cdots + y_2^r \frac{\partial^r f}{\partial x_2^r}.$$

Hence

$$P = r y_1^{r-1} \frac{\partial^{r+1} f}{\partial x_1^r \partial x_2} + \cdots + r y_2^{r-1} \frac{\partial^{r+1} f}{\partial x_1 \partial x_2^r} - r y_1^{r-1} \frac{\partial^{r+1} f}{\partial x_1^r \partial x_2} - \cdots - r y_2^{r-1} \frac{\partial^{r+1} f}{\partial x_1 \partial x_2^r},$$

and this vanishes by cancellation.

If is operated upon another type of doubly binary form, not a polar, as for instance upon fg, where f is a binary form in x_1, x_2 and g a binary form in y_1, y_2, the result will generally be a doubly binary invariant formation, not zero.

DEFINITION.

If $f(x) = a_0x_1^m + \cdots$ is a binary form in (x) of order m, and $g(y) = b_0y_1^n + \cdots$ a binary form in (y) of order n, then if y_1, y_2 be changed to x_1, x_2 respectively in

$$\frac{(m-r)!(n-r)!}{m!n!} \, {}^{r} f(x)g(y), \qquad (73)$$

after the differentiations have been performed, the result is called the rth transvectant (Cayley, 1846) of $f(x)$ and $g(x)$. This will be abbreviated $(f, g)^r$, following a well-established notation. We evidently have for a general formula

$$(f, g)^r = \frac{(m-r)!(n-r)!}{m!n!} \sum_{s=0}^{r} (-1)^s \binom{r}{s} \frac{\partial^r f(x)}{\partial x_1^{r-s} \partial x_2^{s}} \cdot \frac{\partial^r g(x)}{\partial x_1^{s} \partial x_2^{r-s}}. \qquad (74)$$

We give at present only a few illustrations. We note that the Jacobian of two binary forms is their first transvectant. Also the Hessian of a form f is its second transvectant. For

$$H = \frac{2}{m^2(m-1)^2}\left(f_{x_1x_1}f_{x_2x_2} - f^2_{x_1x_2}\right) = \frac{(bm-2)^2}{(bm)^2}\left(f_{x_1x_1}f_{x_2x_2} - 2f^2_{x_1x_1} + f_{x_2x_2}f_{x_1x_1}\right) = (f, f)^2.$$

As an example of multiple transvection we may write the following covariant of the cubic f:

$$Q = (f, (f, f)^2)^1 = (a_0^2 a_3 - 3a_0a_1a_2 + 2a_1^3)x_1^3 + 3(a_0a_1a_3 - 2a_0a_2^2 + a_1^2 a_2)x_1^2 x_2 - 3(a_0a_2a_3 - 2a_1^2 a_3 + a_1a_2^2)x_1x_2^2 - (a_0a_3^2 - 3a_1a_2a_3 + 2a_2^3)x_2^3 \qquad (74_1)$$

If f and g are two forms of the same order m, then $(f, g)^m$ is their bilinear invariant. By forming multiple transvections, as was done to obtain Q, we can evidently obtain an unlimited number of concomitants of a single form or of a set.

3.2 The Aronhold Symbolism. Symbolical Invariant Processes

3.2.1 Symbolical Representation.

A binary form f, written in the notation of which

f = a0x31
+ 3a1x21
x2 + 3a2x1x22
+ a3x32
is a particular case, bears a close formal resemblance to a power of linear form, here the third power. This resemblance becomes the more noteworthy when we observe that the derivative @f
@x1
bears the same formal resemblance to the derivative of the third power of a linear form:
@f
@x1
= 3(a0x21
+ 2a1x1x2 + a2x22
).
That is, it resembles three times the square of the linear form. When we study the question of how far this formal resemblance may be extended we are led to a completely new and strikingly concise formulation of the fundamental processes of binary invariant theory. Although f = a0xm + Â· Â· Â· is not an exact power, we assume the privilege of placing it equal to the mth power of a purely symbolical linear form 1x1 + 2x2 which we abbreviate x.
f = (1x1 + 2x2)m = m
x = a0xm + Â· Â· Â· .
This may be done provided we assume that the only defined combinations of the symbols 1, 2, that is, the only combinations which have any definite meaning, are the monomials of degree m in 1, 2;
m
1 = a0, m−1
1 2 = a0, Â· Â· Â· , m
2 = am,
and linear combinations of these. Thus m
1 + 2m−1
1 2
2 means a0 + 2a2. But of
m−2
1 2 is meaningless; an umbral expression (Sylvester). An expression of the second degree like a0a8 cannot then be represented in terms of ’s alone, since
m
1 Â·m−3
1 3
2 = 2m−3
1 3
2 is undefined. To avoid this difficulty we give f a series of symbolical representations,
f = m
x = m
x = m
x Â· Â· Â· ,
wherein the symbols (1, 2), (1, 2), (1, 2), ... are said to be equivalent symbols as appertaining to the same form f. Then
m
1 = m
1 = m
1 Â· Â· Â· = a0, m−1
1 2 = m−1
1 2 = m−1
1 2 Â· Â· Â· = a1, Â· Â· Â· .
Now a0a3 becomes (m
1 m−3

1 3
2) and this is a defined combination of symbols.

In general an expression of degree i in the a's will be represented by means of i equivalent symbol sets, the symbols of each set entering the symbolical expressions only to the mth degree; moreover there will be a series of (equivalent) symbolical representations of the same expression, as

a0a3 = m
1 m−3
1 3
2 = m
1 m−3
1 3
2 = m
1 m−3
1 3
2 = Â· Â· Â· .

Thus the discriminant of
f = 2x
= 2x
= . . . = a0x21
+ 2a1x1x2 + a2x22

is
D = 4(0a2 − a21
) = 4(2
12
2 − 1212)
= 2(2
12
2 − 21212 + 2
22
1),
or
D = 2()2,
a very concise representation of this invariant.

Conversely, if we wish to know what invariant a given symbolical expression represents, we proceed thus. Let f be the quadratic above, and
g = 2
x = 2x
= . . . = b0x21
+ 2b1x1x2 + b2x32
,
where is not equivalent to . Then to find what J = ()xx, which evidently contains the symbols in defined combinations only, represents in terms of the actual coefficients of the forms, we multiply out and find
J = (12 − 21)(1x1 + 2x2)(1x1 + 2x2)
= (2
112 − 1221
)x21
+ (2
122
− 2
221
)x1x2 + (1222
− 2
212)x22
,
= (a0b1 − a1b0)x21
+ (a0b2 − a2b0)x1x2 + 9a1b2 − a2b1)x22

.
This is the Jacobian of f and g. Note the simple symbolical form
J = ()xx.

3.2.2 Symbolical polars.

We shall now investigate the forms which the standard invariant processes take when expressed in terms of the above symbolism (Aronhold, 1858).
For polars we have, when f = m
x = m
x = . . .,
fy =
1
m y
@
@xf = m−1
x y
@
@x(1x1 + 2x2) = m−1
x y.
Hence
fyr = m−r
x ry
. (75)
The transformed form of f under T will be
f0 = [1(1x01 + μ1x02) + 2(2x01 + μ2x02)]m
= [(11 + 22)x01 + 1μ1 + 2μ2)x02]m,
or
f0 = (ax01 + aμx02)m
= am x0m
1 + . . . + m
r am−r
 ar
μx0m−r
1 x0r
2 + . . . + amμ
x0m
2 . (76)

In view of (75) we have here not only the symbolical representation of the transformed form but a very concise proof of the fact, proved elsewhere (Chap. I, (29)), that the transformed coefficients are polars of a00 = f(1, 2) = am .
The formula (66) for the polar of a product becomes
m
x n
x iyr
=
1
m+n
r
r Xs=0 m
s n
r − sm−s
x sy
n−r+s
x r−s
y , (77)
where the symbols , are not as a rule equivalent.

3.2.3 Symbolical transvectants.

If f = m
x = amx

= . . . , g = n
x = bnx
= . . ., then
(f, g)1 =
1
mn

m
x n
y iy=x
= m−1
x n−1
y @2
@x1@y2 −
@2
@x2@y1xyiy=x
= ()m−1
x n−1
x .
Hence the symbolical form for the rth transvectant is
(f, g)r = ()rm−r
x n−r
x . (78)
Several properties of transvectants follow easily from this. Suppose that g = f so that and are equivalent symbols. Then obviously we can interchange and
 in any symbolical expression without changing the value of that expression.
Also we should remember that () is a determinant of the second order, and formally
() = −().
Suppose now that r is odd, r = 2k + 1. Then
(f, f)2k+1 = ()2k+1am−2k−1
x n−2k−1 = −()2k+1am−2k−1
x n−2k−1
x
Hence this transvectant, being equal to its own negative, vanishes. Every odd transvectant of a form with itself vanishes.
If the symbols are not equivalent, evidently
(f, g)r = (−1)r(g, f)r. (79)
Also if C is a constant,
(Cf, g)r = C(f, g)r; (80)
(c1f1 + c2f2 + . . . , d1g1 + d2g2 + . . .)r = c1d1(f1, g1)r + c1d2(f1, g2)r + ...
(81)

3.2.4 Standard method of transvection.
We may derive transvectants from polars by a simple application of the fundamental postulate. For, as shown in section 1, if f = a0xm1
+ . . . = amx
,
fyr =
(m − r)!
m! @f
@xr
1
yr
1 + r
1 @rf
@xr−1
1 @x2
yr−1

$$\dots \frac{\partial^r f}{\partial x_1^{r-1}\partial x_2} y_1^{r-1} y_2 + \dots + \frac{\partial^r f}{\partial x_2^r} y_2^r\Big]. \quad (82)$$

Now (y) is cogredient to (x). Hence $\frac{\partial}{\partial x_2}, -\frac{\partial}{\partial x_1}$ are cogredient to y_1, y_2. If we replace the y's by these derivative symbols and operate the result, which we abbreviate as ∂f_y^r, upon a second form $g = b_x^n$, we obtain

$$\frac{(n-r)!}{n!}\partial f_y^r = \left[a_1^r b_2^r - r a_1^{r-1} a_2 b_2^{r-1} b_1 + \dots + (-1)^r a_2^r b_1^r\right] a_x^{m-r} b_x^{n-r} = (ab)^r a_x^{m-r} b_x^{n-r} = (f, g)^r. \quad (83)$$

When we compare the square bracket in (82) with a_x^{m-r} times the square bracket in (83), we see that they differ precisely in that y_1, y_2 has been replaced by $b_2, -b_1$. Hence we enunciate the following standard method of transvection. Let f be any symbolical form. It may be simple like f in this paragraph, or more complicated like (78), or howsoever complicated. To obtain the rth transvectant of f and $\phi = b_x^n$ we polarize f r times, change y_1, y_2 into $b_2, -b_1$ respectively in the result and multiply by b_x^{n-r}. In view of the formula (77) for the polar of a product this is the most desirable method of finding transvectants.

For illustration, let F be a quartic, $F = a_x^4 = b_x^4$, and f its Hessian,

$$f = (ab)^2 a_x^2 b_x^2.$$

Let

$$g = \alpha_x^3.$$

Then

$$(f, g)^2 = \left[(ab)^2 a_x^2 b_x^2\right]_{y^2, y=\alpha} \times \alpha_x$$

$$= \frac{(ab)^2}{6}\Big[\binom{2}{0}^2 a_x^2 b_y^2 + \binom{2}{1}^2 \dots$$

1axaybxby + 2
22
0a2
yb2
xy=a Ã— x
(84)
=
1
6
(ab)2(b)2a2
xx +
2
3
(ab)2(a)(b)axbxx +
1
6
(ab)2(a)2b2
xx.

Since the symbols a, b are equivalent, this may be simplified by interchanging a, b in the last term, which is then identical with the first,

(f, g)2 =
1
3
(ab)2(b)2a2
xx +
2
3
(ab)2(a)(b)axbxx.

By the fundamental existence theorem this is a joint covariant of F and g.

Let f be as above and g = ()2x
x, where and are not equivalent. To find (f, g)2, say, in a case of this kind we first let

g = ()2x
x = 3x
,

introducing a new symbolism for the cubic g. Then we apply the method just given, obtaining

(f, g)2 =
1
3
(ab)2(b)2a2
xx +
2
3
(ab)2(a)(b)axbxx.

We now examine this result term by term. We note that the first term could have been obtained by polarizing g twice changing y1, y2 into b2,−b1 and multiplying the result by (ab)2a2
x. Thus

1
3
(ab)2(b)2a2
xx =
1
3
()2x
xiy2;y=b
(ab)2a2

x. (85)
Consider next the second term. It could have been obtained by polarizing g once with regard to y, and then the result once with regard to z; then changing y1, y2 into a2,−a1, and z1, z2 into b2,−b1, and multiplying this result by (ab)2axbx;
2
3
(ab)2(a)(b)axbxx
=
2
3
()2x
xy;y=a#z;z=b
Ã— (ab)2axbx. (86)
From (85),
1
9
() 2
11
1xyy + 2
21
02y
xy=b
(ab)2a2
x
=
2
9
(ab)2()(ab)(b)a2
xx +
1
9
(ab)2()(ab)2a2
xx.
From (86),
2
9
() 2
01
12x
y + 2
11
0xyxy=a#z;z=b
(ab)2axbx
=
2
92x
(a) +
4
9xx(a)z;z=b Ã— (ab)2()axbx
=
2
9
(ab)2()(b)(a)xaxbx +
2
9
(ab)2()(a)(b)xaxbx
+
2
9

(ab)2()(b)(a)xaxbx.

Hence we have in this case
(f, g)2 =
2
9
(ab)2()(b)(b)a2
xax +
1
9
(ab)2()(ab)2a2
xx
+
4
9
(ab)2()(b)(a)xaxbx +
2
9
(ab)2()(a)(b)xaxbx. (87)

3.2.5 Formula for the rth transvectant.

The most general formulas for f, g respectively are
f = (1)
x (2)
x Â· Â· Â· (m)
x , g = (1)
y (2)
y Â· Â· Â· (n)
y ,
in which
a(i)
x = a(i)
1 x1 + a(i)
x x2, (i)
x = (i)
1 x1 + (i)
2 x2.
We can obtain a formula of complete generality for the transvectant (f, g)r by
applying the operator directly to the product fg. We have
@2
@x1@y2
fg =X(q)
1 (r)
2
fg
(q)
x (r)
y
,
@2
@x2@y1
fg =X(q)
2 (r)
1
fg
(q)
x (r)
y
.
Subtracting these we obtain

(f, g)1 =
(m − 1)!(n − 1)!
m!n! X((q)(r)) fg
(q)
x (r)
x
.
Repetitions of this process, made as follows:
(f, g)2 =
(m − 2)!(n − 2)!
m!n! X((q)(r)) " fg
(q)
x (r)
y #y=x
, (88)
lead to the conclusion that the rth transvectant of f and g, as well as the mere result of applying the operator to fg r times, is a sum of terms each one of which contains the product of r determinant factors (),m−r factors x, and n − r factors x. We can however write (f, g)r in a very simple explicit form.
Consider the special case
f = (1)
x (2)
x (3)
x , g = (1)
y (2)
y .

Here, by the rule of (88),
(f, g)2 = {((1)(1))((2)(2))(3)
x + ((1)(1))((3)(2))(2)
x
+ ((1)(2))((2)(1))(3)
x + ((1)(2))((3)(1))(2)
x
+ ((2)(1))((3)(2))(1)
x + ((2)(1))((1)(2))(3)
x (89)
+ ((2)(2))((3)(1))(1)
x + ((2)(2))((1)(1))(3)
x
+ ((3)(1))((1)(2))(2)
x + ((3)(1))((2)(2))(1)
x
+ ((3)(2))((1)(1))(2)
x + ((3)(2))((2)(1))(1)
x Ã· 2!3!,
in which occur only six distinct terms, there being a repetition of each term. Now consider the general case, and the rth transvectant. In the first transvectant one term contains t1 = ((1)(1))(2)
x Â· Â· Â· (m)
x (2)
y Â· Â· Â· (n)
y . In the second transvectant
there will be a term u1 = ((1)(1))((2)(2))(3)
x Â· Â· Â· (3)
y Â· Â· Â· arising from

t1, and another term u1 arising from
t2, where t2 = ((2)(2))(1)

x (3)
x Â· Â· Â· (m)
x (1)
y (3)
y Â· Â· Â· (n)
y .
Thus u1(y = x) and likewise any selected term occurs just twice in (f, g)2. Again the term v1 = ((1)(1))((2)(2))((3)(3))(4)
x Â· Â· Â· (4)
x Â· Â· Â· will occur in (f, g)3
as many times as there are ways of permuting the three superscripts 1, 2, 3 or 3! times. Finally in (f, g)r, written by (88) in the form (89), each term will be repeated r! times. We may therefore write (f, g)r as the following summation, in which all terms are distinct and equal in number to m
r n
rr!:
(f,)r =
1
m
r n
rr!X"((1)(1))((2)(2)) . . . ((r)(r))
(1)
x (2)
x . . . (m)
x (1)
y (2)
y . . . (n)
y
fg#y=x
. (90)
3.2.6 Special cases of operation by upon a doubly binary
form, not a product.
In a subsequent chapter Gordan's series will be developed. This series has to do with operation by upon a doubly binary form which is neither a polar nor a simple product. In this paragraph we consider a few very special cases of such a doubly binary form and in connection therewith some results of very frequent application.
We can establish the following formula:

r(xy)r = constant = (r + 1)(r!)2. (91)
In proof (74),

r =
r Xi=0
(−1)ir
i @r
@xr−i
1 @xi
2
@r
@yi1
@yr−i
2
,
and
(xy)r =
r Xi=0
(−1)ir
ixr−i

1 xi
2yi1
yr−i
2 .

Hence it follows immediately that

r(xy)r =
r Xi=0 r
i2
((r − i)!)2(i!)2
=
r Xi=0
(r!)2 = (r + 1)(r!)2.
A similar doubly binary form is
F = (xy)jm−j
x n−j
y .
If the second factor of this is a polar of m+n−2j
x , we may make use of the fact,
proved before, that on a polar is zero. An easy differentiation gives

F = j(m + n − j + 1)(xy)j−1m−j
x n−j
y ,
and repetitions of this formula give

iF = j!
(j − 1)!
(m + n − j + 1)!
(m + n − j − i + 1)!
(xy)j−im−j
x n−j
y If i 5 j;
= 0 if i > j (911)
This formula holds true if m = n = j, that is, for
i(xy)j .
3.2.7 Fundamental theorem of symbolical theory.
Theorem. Every monomial expression which consists entirely of symbolical factors of two types, e.g. determinants of type () and linear factors of the type x and which is a defined expression in terms of the coefficients and variables of a set of forms f, g, . . . is a concomitant of those forms. Conversely, every concomitant of the set is a linear combination of such monomials.
Examples of this theorem are given in (78), (84), (87).
In proof of the first part, let
 = ()p()q . . .
x
x
x . . . ,
where f = m
x ; and , , . . . may or may not be equivalent to , depending upon whether or not appertains to a single form f or to a set f, g, ... Transform the form f, that is, the set, by T. The transformed of f is (76)
f0 = (x01 + μx02)m.
Hence on account of the equations of transformation,
0 = (μ − μ)p(μ − μ)q . . .
x
x . . .
But

μ − μ = (μ)(). (92)

Hence
0 = (μ)p+q+...,
which proves the invariancy of . Of course if all factors of the second type, x,
are missing in , the latter is an invariant.
To prove the converse of the theorem let be a concomitant of the set f, g, . . .
and let the corresponding invariant relation be written
(a00, a01, . . . ; x01, x02) = (μ)k(a0, a1, . . . ; x1, x2). (93)
Now a0j = m−j
 jμ
(j = 0, 1, . . .m). Hence if we substitute these symbolical
forms of the transformed coefficients, the left-hand side of (93) becomes a summation
of the type
XPQx01
!1x02
!1 = (μ)k(a0, . . . ; x1, x2) (!1 + !2 = !), (94)
where P is a monomial expression consisting of factors of the type only and
Q a monomial whose factors are of the one type μ. But the inverse of the
transformation T (cf. (10)) can be written
x01 = μ
(μ) , x02 =
(μ) ,
where 1 = −x2, 2 = x1. Then (94) becomes
X(−1)!2PQ!1
μ !2
 = (μ)k+! (95)
We now operate on both sides of (95) by
k+!, where
 = @2
@1@μ2 −
@2
@2@μ1
.
We apply (90) to the left-hand side of the result and (91) to the right-hand
side. The left-hand side accordingly becomes a sum of terms each term of which
involves necessarily ! + k determinants (), (). In fact, since the result
is evidently still of order ! in x1, x2 there will be in each term precisely !
determinant factors of type () and k of type (). There will be no factors of
type or remaining on the left since by (91) the right-hand side becomes
a constant times , and does not involve , μ. We now replace, on the left,
() by its equivalent x, () by x, etc. Then (95) gives, after division by the
constant on the right,
 = a()p()q . . . px

x . . . , (96)
where a is a constant; which was to be proved.
This theorem is sometimes called the fundamental theorem of the symbolical
theory since by it any binary invariant problem may be studied under the
Aronhold symbolical representation.

3.3 Reducibility. Elementary Complete Irreducible Systems
Illustrations of the fundamental theorem proved at the end of Section 2 will now
be given.
3.3.1 Illustrations.
It will be recalled that in (96) each symbolical letter occurs to the precise degree
equal to the order of the form to which it appertains. Note also that k + !,
the index plus the order of the concomitant, used in the proof of the theorem,

equals the weight of the concomitant. This equals the number of symbolical determinant factors of the type () plus the number of linear factors of the type x in any term of . The order ! of the concomitant equals the number of symbolical factors of the type x in any term of . The degree of the concomitant equals the number of distinct symbols , , . . . occurring in its symbolical representation.

Let

= ()p()q()r . . .
x
x . . .

be any concomitant formula for a set of forms f = m
x , g = n
x , ... No generality will be lost in the present discussion by assuming to be monomial, since each separate term of a sum of such monomials is a concomitant. In order to write down all monomial concomitants of the set of a given degree i we have only to construct all symbolical products involving precisely i symbols which fulfill the laws

p + q+. . . + = m,
p + r+. . . + = n, (97)
. . . .

where, as stated above, m is the order of f and equal therefore to the degree to which occurs in , n, the order of g, and so on.

In particular let the set consist of f = 2x
= 2x
merely. For the concomitant of degree 1 only one symbol may be used. Hence f = 2x
itself is the only concomitant of degree 1. If i = 2, we have for ,

= ()p
x
x ,

and from (97)

p + = p + = 2.

Or

p		
0	2	2
1	1	1
2	0	0

Thus the only monomial concomitants of degree 2 are

2x
2x
= f2, ()xx −()xx 0, ()2 =
1
2D.

For the degree 3

= ()p()q()r
x
x
x ,

p + q + = 2, p + r + = 2, q + r + = 2.

It is found that all solutions of these three linear Diophantine equations lead to concomitants expressible in the form fsDt, or to identically vanishing concomitants.

DEFINITION.

Any concomitant of a set of forms which is expressible as a rational integral function of other concomitants of equal or of lower degree of the set is said to be reducible in terms of the other concomitants.

It will be seen from the above that the only irreducible concomitants of a

binary quadratic f of the first three degrees are f itself and D, its discriminant.
It will be proved later that f, D form a complete irreducible system of f. By this we mean a system of concomitants such that every other concomitant of f is reducible in terms of the members of this system. Note that this system for the quadratic is finite. In another chapter we shall prove the celebrated Gordan's theorem that a complete irreducible system of concomitants exists for every binary form or set of forms and the system consists always of a finite number of concomitants. All of the concomitants of the quadratic f above which are not monomial are reducible, but this is not always the case as it will be sometimes preferable to select as a member of a complete irreducible system a concomitant which is not monomial (cf. (87)). As a further illustration let the set of forms
be f = 2x
= 2x
= . . . , g = a2
x = b2
x = . . .; let i = 2. Then employing only two
symbols and avoiding ()2 = 1
2D, etc.
= (a)p
x a
x ,
p + = p + = 2.
The concomitants from this formula are,
2x
a2
x = f Â· g, (a)xax = J, (a)2 = h,
J being the Jacobian, and h the bilinear invariant of f and .
3.3.2 Reduction by identities.
As will appear subsequently the standard method of obtaining complete irreducible systems is by transvection. There are many methods of proving concomitants reducible more powerful than the one briefly considered above, and

the interchange of equivalent symbols. One method is reduction by symbolical identities.
Fundamental identity. One of the identities frequently employed in reduction is one already frequently used in various connections, viz. formula (92). We write this
axby − aybx (ab)(xy). (98)
Every reduction formula to be introduced in this book, including Gordan's series and Stroh's series, may be derived directly from (98). For this reason this formula is called the fundamental reduction formula of binary invariant theory (cf. Chap. IV).
If we change y1 to c2, y2 to −c1, (98) becomes
(bc)ax + (ca)bx + (ab)cx 0. (99)
Replacing x by d in (99),
(ad)(bc) + (ca)(bd) + (ab)(cd) 0. (100)
From (99) by squaring,
2(ab)(ac)bxcx (ab)2c2
x + (ac)2b2
x − (bc)2a2
x. (101)
If ! is an imaginary cube root of unity, and
u1 = (bc)ax, u2 = (ca)bx, u3 = (ab)cx,
we have
(u1 + u2 + u3)(u1 + !u2 + !2u3)(u1 + !2u2 + !u3)
= (ab)3c3
x + (bc)3a3
x + (ca)3b3
x − 3(ab)(bc)(ca)axbxcx 0. (102)

Other identities may be similarly obtained.
In order to show how such identities may be used in performing reductions,
let $f = a^3_x = b^3_x = \cdots$ be the binary cubic form. Then

$= (f, f)_2 = (ab)^2 a_x b_x$,

$Q = (f,) = (ab)^2 (cb) a_x c^2_x$.

$$-(f,Q)_2 = \frac{1}{3}(ab)^2(bc)[a_x c^2_y + 2c_x c_y a_y]_{y=d} \text{ Ã— } d_x \quad (1021)$$

$$= \frac{1}{3}[(ab)^2(cd)^2(bc)a_x d_x + 2(ab)^2(ad)(cd)(bc)c_x d_x].$$

But by the interchanges a d, b c

$$(ab)^2(cd)^2(bc)a_x d_x = (dc)^2(ba)^2(cb)a_x d_x \ 0.$$

By the interchange c d the second term in the square bracket equals

$$(ab)^2(cd)c_x d_x[(ad)(bc) + (ca)(bd)],$$

or, by (100) this equals

$$(ab)^3(cd)^2 c_x d_x \ 0.$$

Hence $(f,Q)_2$ vanishes.
We may note here the result of the transvection $(,)_2$;

$$R = (,)_2 = (ab)^2(cd)^2(ac)(bd).$$

3.3.3 Concomitants of binary cubic.

We give below a table of transvectants for the binary cubic form. It shows which transvectants are reducible in terms of other concomitants. It will be inferred from the table that the complete irreducible system for the binary cubic f consists of

$$f,, Q,R,$$

one invariant and three covariants, and this is the case as will be proved later. Not all of the reductions indicated in this table can be advantageously made by the methods introduced up to the present, but many of them can. All four of the irreducible concomitants have previously been derived in this book, in terms of the actual coefficients, but they are given here for convenient reference:

$$f = a_0 x^3_1 + 3a_1 x^2_1 x_2 + 3a_2 x_1 x^2_2 + a_3 x^3_2,$$

$$= 2(a_0 a_2 - a^2_1)x^2_1 + 2(a_0 a_3 - a_1 a_2)x_1 x_2 + 2(a_1 a_3 - a^2_2)x^2_2, \text{ (cf. (35))}$$

$$Q = (a^2_0 a_3 - 3a_0 a_1 a_2 + 2a^3_1)x^3_1 + 3(a_0 a_1 a_3 - 2a_0 a^2_2 + a^2_1 a_2)x^2_1 x_2 - 3(a_0 a_2 a_3 - 2a^2_1 a_3 + a_1 a^2_2)x_1 x^2_2 - (a_0 a^2_3$$

− 3a1a2a3 + 2a32
)x32
, (cf. (39))
R =8(a0a2 − a21
)(a1a3 − a22
) − 2(a0a3 − a1a2)2. (cf. (741))
The symbolical forms are all given in the preceding Paragraph.
TABLE I
FIRST TRANSV. SECOND TRANSV. THIRD TRANSV.
(f, f) = 0 (f, f)2 = (f, f)3 = 0
(f,) = Q (f,)2 = 0
(,) = 0 (,)2 = R
(f,Q) = −1
22 (f,Q)2 = 0 (f,Q)3 = R
(,Q)= 1
2Rf (,Q)2 = 0
(Q,Q) = 0 (Q,Q)2 = 1
2R (Q,Q)3 = 0

3.4 Concomitants in Terms of the Roots
Every binary form f = amx
= bmx
= . . . is linearly factorable in some field of
rationality. Suppose
f = (r(1)
2 x1 − r(1)
1 x2)(r(2)
2 x1 − r(2)
1 x2) . . . (r(m)
2 x1 − r(m)
1 x2).
Then the coefficients of the form are the elementary symmetric functions of the
m groups of variables (homogeneous)
Pj(r(j)
1 , r(j)
2)(j = 1, 2, . . . ,m).
These functions are given by
aj = (01)jXr(1)
1 (2)
1 . . . (j)
1 (j+1)
2 . . . (m)
2 (j = 0, . . . ,m). (103)
The number of terms inPevidently equals the number of distinct terms obtainable
from its leading term by permuting all of its letters after the superscripts
are removed. This number is, then,
N = (m/)!j!(m − j)! =m Ci.
3.4.1 Theorem on linear factors.
Theorem. Any concomitant of f is a simultaneous concomitant of the linear
factors of f, i.e. of the linear forms
((1)x), ((2)x), . . . , ((m)x).
For,
f0 = (−1)m(0(1)x0)(0(2)x0) . . . (0(m)x0), (104)
and
a0j = (−1)iX 0(1)
1 0(2)
1 . . . 0(j)
1 0(j+1)
2 . . . 0(m)

2 . (1031)
Let be a concomitant of f, and let the corresponding invariant relation be
0 = (a00, . . . , a0m)i(x01, x02)! = (µ)k(a0, . . . , am)i(x1, x2)! = (µ)k.
When the primed coefficients in 0 are expressed in terms of the roots from
(1031) and the unprimed coefficients in in this invariant relation are expressed
in terms of the roots from (103), it is evident that 0 is the same function of the
primed 's that is of the unprimed 's. This proves the theorem.

3.4.2 Conversion operators.
In this Paragraph much advantage results in connection with formal manipulations
by introducing the following notation for the factored form of f:
f = (1)
x (2)
x Â· Â· Â· (m)
x . (105)
Here (j)
x = 1 + (j)
2 x2(j = 1, Â· Â· Â· ,m). The 's are related to the roots
(r(j)
1 , r(j)
2) of the previous Paragraph by the equations
(j)
1 = r(j)
2 , (j)
2 = −r(j)
1 ;
that is, the roots are (+(j)
2 ,−(j)
1)(j = 1, Â· Â· Â· ,m). The umbral expressions
a1, a2 are now cogredient to (j)
1 , (j)
2 (Chap. I, Â§2, VII, and Chap. III, (76)).
Hence,
(j) @
@a = (j)
1
@
@a1
+ (j)
2
@
@a2
is an invariantive operator by the fundamental postulate. In the same way
[Da] = (1) @
@a (2) @
@a Â· Â· Â·(m) @
@a
and
[Dabc] = [Da][Db][Dc] Â· Â· Â·
If the transformation T is looked upon as a change of reference points, the
multiplier undergoes a homographic transformation under T. are invariantive
operators. If we recall that the only degree to which any umbral pair a1, a2 can
occur in a symbolical concomitant,
 = (ab)(ac) Â· Â· Â· ,
of f is the precise degree m, it is evident that [DabcÂ·Â·Â·] operated upon gives a
concomitant which is expressed entirely in terms of the roots ((j)
2 ,−(j)
1) of f.
Illustrations follow. Let 2 be the discriminant of the quadratic

f = a2
x = b2
x = Â· Â· Â· , = (ab)2.
Then (1) @
@a = 2((1)b)(ab); [Da] = 2((1)b)((2)b).
Hence
[Dab] = −2((1)(2))2. (106)
This result is therefore some concomitant of f expressed entirely in terms of the
roots of f. It will presently appear that it is, except for a numerical factor, the

invariant itself expressed in terms of the roots. Next let be the covariant Q
of the cubic f = a3
x = ... Then
Q = (ab)2(ac)bxc2
x,
1
2
[Da]Q = ((1)b)((2)b)(c(3))bxc2
x + ((01)b)((3)b)(c(2))bxc2
x
+ ((3)b)((2)b)(x(1)bxc2
x,
1
4
[Dab]Q = ((1)b)((2)b)(c(3))(3)
x c2
x
+((1)(3))((2)(1))(c(3))(2)
x c2
x + ((1)(2))((2)(3))(c(3))(1)
x c2
x
+((1)(2))((3)(1))(c(3))(3)
x c2
x + ((1)(3))((3)(1))(c(2))(2)
x c2
x
+((1)(3))((3)(2))(c(2))(1)
x c2
x + ((3)(1))((2)(3))(c(1))(2)
x c2
x
+((3)(2))((2)(1))(c(2))(3)
x c2
x + ((3)(2))((2)(3))(c(1))(1)
x c2
x
[Dabc]Q = −25X((1)(2))2((1)(3))(3)2
x (2)
x , (107)
wherein the summation covers the permutations of the superscripts. This is
accordingly a covariant of the cubic expressed in terms of the roots.
Now it appears from (104) that each coefficient of f = amx
= . . . is of degree
m in the ’s of the roots ((j)
2 − (j)
1). Hence any concomitant of degree i
will be of degree im in these roots. Conversely, any invariant or covariant
which is of degree im in the root letters will, when expressed in terms of the

coefficients of the form, be of degree i in these coefficients. This is a property which invariants enjoy in common with all symmetric functions. Thus [Dab] above is an invariant of the quadratic of degree 2 and hence it must be the discriminant itself, since the latter is the only invariant of f of that degree (cf. Â§3). Likewise it appears from Table I that Q is the only covariant of the cubic of degree-order (3, 3), and since by the present rule [Dabc]Q is of degree-order (3, 3), (107) is, aside from a numerical multiplier, the expression for Q itself in terms of the roots.

It will be observed generally that [DabÂ·Â·Â·] preserves not only the degree-order (i, !) of , but also the weight since w =12
(im + !). If then in any case
happens to be the only concomitant of f of that given degree-order (i, w), the expression [DabÂ·Â·Â·] is precisely the concomitant expressed in terms of the roots. This rule enables us to derive easily by the method above the expressions for the irreducible system of the cubic f in terms of the roots. These are

f = a(1)
x a(2)
x a(3)
x ; a3
x.
=X(a(1)a(2))2a(3)2
x ; (ab)2axbx.
Q =X(a(1)a(2))2(a(1)a(3))a(3)2
x a(2)
x ; (ab)2(ac)bxc2
x.
R = (a(1)a(2))2(a(2)a(3))2(a(3)a(1))2; (ab)2(cd)2(ac)(bd).

Consider now the explicit form of Q:

Q = a(1)a(2)2 a(1)a(3)2
a(3)2
x a(2)
x + a(2)a(3)2 a(2)a(1)2
a(1)2
x a(3)
x
+ a(3)a(1)2 a(3)a(2)2
a(2)2
x a(1)
x + a(3)a(2)2 a(3)a(1)2
a(1)2
x a(2)
x
+ a(2)a(1)2 a(2)a(3)2
a(3)2
x a(1)
x + a(1)a(3)2 a(1)a(2)2
a(2)2
x a(3)
x

It is to be noted that this is symmetric in the two groups of letters (a(j)
1 , a(j)
2).

Also each root (value off) occurs in the same number of factors as any other root in a term of Q. Thus in the first term the superscript (1) occurs in three factors. So also does (2).

3.4.3 Principal theorem.

We now proceed to prove the principal theorem of this subject (Cayley).

Definition.

In Chapter I, Section 1, II, the length of the segment joining C(x1x2), and
D(y1, y2); real points, was shown to be
CD = μ(yx)
(y1 + y2)(x1 + x2) ,
where is the multiplier appertaining to the points of reference P, Q, and μ is
the length of the segment PQ. If the ratios x1 : x2, y1 : y2 are not real, this
formula will not represent a real segment CD. But in any case if (r(j)
a , r(j)
2),
(r(k)
a , r(k)
2), are any two roots of a binary form f = am2
, real or imaginary, we
define the difference of these two roots to be the number
[(j) (k)] = μ((j) (k))
((j)
1 + (j)
2)((k)
1 + (k)
2)
.
We note for immediate use that the expression
( ) = ((1)
1 + (1)
2)((2)
1) . . . ((m)
1 + (m)
2)
is symmetric in the roots. That is, it is a symmetric function of the two groups
of variables ((j)
1 , (j)
2)(j = 1, . . . ,m). In fact it is the result of substituting
(1,−) for (x1, x2) in
f = (−1)m((1)x)((2)x) . . . ((m)x),
and equals
( ) = (a0 − ma1 + . . . + (−1)mamm).

Obviously the reference points P, Q can be selected1 so that (1,−) is not a
root, i.e. so that ( ) 6= 0.
Theorem. Let f be any binary form, then any function of the two types of
differences
[(j) (k)], [(j)x] = μ((f)x)/((f)
1 + (j)
2)(x1 + x2),
which is homogenous in both types of differences and symmetric in the roots
((j)
1 , (j)
2)(j = 1, . . . ,m) will, when expressed in terms of x1, x2 and the coefficients
of f, and made integral by multiplying by a power of ( ) times a
power of (x1 + x2), be a concomitant if and only if every one of the products
of differences of which it consists involves all roots ((j)
1 , (j)
2) (values of j) in
equal numbers of its factors. Moreover all concomitants of f are functions
defined in this way. If only the one type of difference [(j) (k)] occurs in , it is
an invariant, if only the type [(j)x], it is an identical covariant,–a power of f
itself, and if both types occur, is a covariant. [Cf. theorem in Chap. III, Â§2,
VII.]
In proof of this let the explicit form of the function described in the theorem

be
 =Xk
[r(1)r(2)]k [r(1)r(3)]k . . . [r(1)x]k [r(2)x]k . . . ,
where
1 + 1 + . . . = 2 + 2 + . . . = . . . ,
1 + 1 + . . . = 2 + 2 + . . . = . . . ,
and is symmetric in the roots. We are to prove that is invariantive when and only when each superscript occurs in the same number of factors as every other superscript in a term of . We note first that if this property holds and we express the differences in in explicit form as defined above, the terms of will, without further algebraical manipulation, have a common denominator, and this will be of the form
Y(r)u(x1 + x2)v.
Hence Q(r)u(x1+x2)v is a sum of monomials each one of which is a product of determinants of the two types (r(j)r(k)), (r(j)x). But owing to the cogrediency of the roots and variables these determinants are separately invariant under T, hence Q(r)u(x1+x2)v is a concomitant. Next assume that in it is not true that each superscript occurs the same number of times in a term as every other superscript. Then although when the above explicit formulas for differences are introduced (x1 + x2) occurs to the same power v in every denominator in ,

[1]If the transformation T is looked upon as a change of reference points, the multiplier undergoes a homographic transformation under T.

this is not true of a factor of the type (r(j)
1 + r(j)
2). Hence the terms of
must be reduced to a common denominator. Let this common denominator be Q(r)u(x1 + x2)v. Then Q(r)u(x1 + x2)v is of the form
1 =Xk Yj
(r(j)
1 + r(j)
2)ujk (r(1)r(2))k (r(1)r(3))k . . . Ã— (r(1)x)k (r(2)x)k . . . ,
where not all of the positive integers ujk are zero. *Now is invariantive under T. Hence it must be unaltered under the special case of T; x1 = −x02, x02 = x0.
From this (j)
1 = − 0(j)
2 , (j)
2 = 0(j)
1 . Hence
01 =X
j((j)
2 − (j)
1)uj((1) (2))((1) (3)) . . . ((1)x) . . . ,
and this is obviously not identical with 1 on account of the presence of the factor . Hence 1 is not a concomitant.

All parts of the theorem have now been proved or are self-evident except that all concomitants of a form are expressible in the manner stated in the theorem. To prove this, note that any concomitant of f, being rational in the coefficients of f, is symmetric in the roots. To prove that need involve the roots in the form of differences only, before it is made integral by multiplication by ( )u(x1 +x2)v, it is only necessary to observe that it must remain unaltered when f is transformed by the following transformation of determinant unity:
x1 = x01 + cx02, x2 = x02,
and functions of determinants ((j) (k)), ((j)x) are the only symmetric functions which have this property.

As an illustration of the theorem consider concomitants of the quadratic f = ((1)x(2))((2)x). These are of the form
 =X
[(1) (2)][(1)x][(2)x].

Here owing to homogeneity in the two types of differences,
1 = 2 = . . . ; 1 + 1 = 2 + 2 = ...
Also due to the fact that each superscript must occur as many times in a term as every other superscript,
1 + 1 = 1 + 1, 2 + 2 = 2 + 2, ...
Also owing to symmetry k must be even. Hence k = 2, = = , and
( )2+(x1 + x2)2 = c{((1) (2))2}{((1)x)((2)x)} = CDf,
where C is a numerical multiplier. Now and may have any positive integral values including zero. Hence the concomitants of f consist of the discriminant

D = −(r(1)r(2))2, the form f = (r(1)x)(r(2)x) itself, and products of powers of these two concomitants. In other words we obtain here a proof that f and D form a complete irreducible system for the quadratic. We may easily derive the irreducible system of the cubic by the same method, and it may also be applied with success to the quartic although the work is there quite complicated. We shall close this discussion by determining by this method all of the invariants of a binary cubic f = −(r(1)x)(r(2)x)(r(3)x). Here
 =Xk
[r(1)r(2)]ak [r(2)r(3)]k [r(3)r(1)]k
and
ak + k = k + k = k + k.
That is,
ak = k = k = 2.
Hence
(r)4a = c{(r(1)r(2))2(r(2)r(3))2(r(3)r(1))2}a = CR.
Thus the discriminant R and its powers are the only invariants.
3.4.4 Hermite's Reciprocity Theorem.
Theorem. If a form f = amx
= bmx
= Â· Â· Â· of order m has a concomitant of
degree n and order !, then a form g = n
x = . . . of order n has a concomitant
of degree m and order !.
To prove this theorem let the concomitant of f be
I =Xk(ab)p(ac)q Â· Â· Â· ar
xbs
x Â· Â· Â· (r + s + Â· Â· Â· = !),
where the summation extends over all terms of I and k is numerical. In this the number of distinct symbols a, b, . . . is n. This expression I if not symmetric in the n letters a, b, c, . . . can be changed into an equivalent expression in the sense that it represents the same concomitant as I, and which is symmetric. To do this, take a term of I, as
k(ab)p(ac)q Â· Â· Â· ar
xbs
x Â· Â· Â· ,
and in it permute the equivalent symbols a, b, . . . in all n! possible ways, add the n! resulting monomial expressions and divide the sum by n!. Do this for all terms of I and add the results for all terms. This latter sum is an expression J equivalent to I and symmetric in the n symbols. Moreover each symbol occurs to the same degree in every term of J as does every other symbol, and this degree is precisely m. Now let
g = (1)
x (2)
x Â· Â· Â· (n)
x ,

and in a perfectly arbitrary manner make the following replacements in J:
 a, b, c, . . . , l
(1), (2), (3), . . . , (n).

The result is an expression in the roots ((j)
2 ,−(j)
1) of g,
J =XC((1)(2))p((1)(3))q Â· Â· Â· (1)r
x (2)s
x Â· Â· Â· ,
which possesses the following properties: It is symmetric in the roots, and of order !. In every term each root (value of (j)) occurs in the same number of factors as every other root. Hence by the principal theorem of this section J is a concomitant of g expressed in terms of the roots. It is of degree m in the coefficients of g since it is of degree m in each root. This proves the theorem.
As an illustration of this theorem we may note that a quartic form f has an invariant J of degree 3 (cf. (701)); and, reciprocally, a cubic form g has an invariant R of degree 4, viz. the discriminant of g (cf. (39)).

3.5 Geometrical Interpretations. Involution

In Chapter I, Section 1, II, III, it has been shown how the roots (r(i)
1 , r(i)
2)(i =
1, . . . ,m) of a binary form
f = (a0, a1, . . . , am)(x1, x2)m
may be represented by a range of m points referred to two fixed points of reference, on a straight line EF. Now the evanescence of any invariant of f implies, in view of the theory of invariants in terms of the roots, a definite relation between the points of this range, and this relation is such that it holds true likewise for the range obtained from f = 0 by transforming f by T. A property of a range f = 0 which persists for f0 = 0 is called a projective property.
Every property represented by the vanishing of an invariant I of f is therefore projective in view of the invariant equation
I(a00, . . .) = (μ)kI(a0, . . .).
Any covariant of f equated to zero gives rise to a ”derived” point range connected in a definite manner with the range f = 0, and this connecting relation is projective. The identical evanescence of any covariant implies projective relations between the points of the original range f = 0 such that the derived point range obtained by equating the covariant to zero is absolutely indeterminate.
The like remarks apply to covariants or invariants of two or more forms, and the point systems represented thereby.

3.5.1 Involution.

If
f = (a0, a1, Â· Â· Â· G x1, x2)m, g = (b0, b1, Â· Â· Â· G x1, x2)m
are two binary forms of the same order, then
f + kg = (a0 + kb0, a1 + kb1, Â· Â· Â· G x1, x2)m,
where k is a variable parameter, denotes a system of qualities which are said to form, with f and g, an involution. The * single infinity of point ranges given by k, taken with the ranges f = 0, y = 0 are said to form an involution of point ranges.
In Chapter I, Section 1, V, we proved that a point pair ((u), (v)) can be found harmonically related to any two given point pairs ((p), (r)), ((q), (s)).
If the latter two pairs are given by the respective quadratic forms f, g, the pair ((u), (v)) is furnished by the Jacobian C of f, g. If the eliminant of three quadratics f, g, h vanishes identically, then there exists a linear relation
f + kg + lh = 0,
and the pair h = 0 belongs to the involution defined by the two given pairs.
Theorem. There are, in general, 2(m−1) quantics belonging to the involution f + kg which contain a squared linear factor, and the set comprising all double roots of these quantics is the set of roots of the Jacobian of f and g.
In proof of this, we have shown in Chapter I that the discriminant of a form of order m is of degree 2(m−1). Hence the discriminant of f+kg is a polynomial in k of order 2(m − 1). Equated to zero it determines 2(m − 1) values of k for

which $f + kg$ has a double root.

We have thus proved that an involution of point ranges contains $2(m-1)$ ranges each of which has a double point. We can now show that the $2(m-1)$ roots of the Jacobian of f and g are the double points of the involution. For if $x_1u_2 - x_2u_1$ is a double factor of $f + kg$, it is a simple factor of the two forms

$$\frac{\partial f}{\partial x_1} + k\frac{\partial g}{\partial x_1}, \quad \frac{\partial f}{\partial x_2} + k\frac{\partial g}{\partial x_2},$$

and hence is a simple factor of the k eliminant of these forms, which is the Jacobian of f, g. By this, for instance, the points of the common harmonic pair of two quadratics are the double points of the involution defined by those quadratics. The square of each linear factor of C belongs to the involution $f + kg$.

In case the Jacobian vanishes identically the range of double points of the involution becomes indeterminate. This is to be expected since f is then a multiple of g and the two fundamental ranges $f = 0$, $g = 0$ coincide.

3.5.2 Projective properties represented by vanishing covariants.

The most elementary irreducible covariants of a single binary form $f = (a_0, a_1, \ldots \between x_1, x_2)^m$ are the Hessian H, and the third-degree covariant T, viz.

$$H = (f, f)^2, \quad T = (f, H).$$

We now give a geometrical interpretation of each of these.

Theorem. A necessary and sufficient condition in order that the binary form f may be the mth power of a linear form is that its Hessian H should vanish identically.

If we construct the Hessian determinant of $(r_2x_1 - r_1x_2)^m$, it is found to vanish. Conversely, assume that $H = 0$. Since H is the Jacobian of the two first partial derivatives $\frac{\partial f}{\partial x_1}$, $\frac{\partial f}{\partial x_2}$, the equation $H = 0$ implies a linear relation

$$2\frac{\partial f}{\partial x_1} - 1\frac{\partial f}{\partial x_2} = 0.$$

Also by Euler's theorem

$$x_1\frac{\partial f}{\partial x_1} + x_2\frac{\partial f}{\partial x_2} = mf,$$

and

$$\frac{\partial f}{\partial x_1}dx_1 + \partial f$$

@x2
dx2 = df.
Expansion of the eliminant of these three equations gives
df
f
= m
d(1x1 + 2x2)
1x1 + 2x2
,
and by integration
f = (1x1 + 2x2)m,
and this proves the theorem.
Theorem. A necessary and sufficient condition in order that a binary quartic
form f = a0x41
+ Â· Â· Â· should be the product of two squared linear factors is that
its sextic covariant T should vanish identically.
To give a proof of this we need a result which can be most easily proved by
the methods of the next chapter (cf. Appendix (29)) e.g. if i and J are the
respective invariants of f,
i = 2(a0a4 − 4a1a3 + 3a22
),
J = 6
a0 a1 a2
a1 a2 a3
a2 a3 a4

,

then
(T, T)6 =
1
24
(i3 − 6J2).
We also observe that the discriminant of f is 1
27 (i3 −6J2). Now write 2x
as the
square of a linear form, and
f = 2x
q2x
= a4
x = b4
x = Â· Â· Â· .
Then
H = (2x
q2x
, a4
x)2
=
1
6
[(a)2q2x
+ (qa)22x
+ 4(a)(qa)xqx]a2
x
=
1
6
[3(a)2q2x
+ 3(qa)22x

$- 2(q)^2 a^2_x] a^2_x.$

But

$$(a)^2 a^2_x = (f, 2_x)^2 = \frac{1}{6}(q)^2 2_x,$$

$$(qa)^2 a^2_x = (f, q^2_x)^2 = \frac{1}{6}[(q)^2 q^2_x + 3(qq)^2 2_x].$$

Hence

$$H = -\frac{1}{6}(q)^2 f + \frac{1}{4}(qq)^2 4_x. \quad (108)$$

This shows that when $H = 0$, f is a fourth power since $(q)^2$, $(qq)^2$ are constants. It now follows immediately that

$$T = (f, H) = \frac{1}{4}(qq)^2 (f, x)^3_x.$$

Next if f contains two pairs of repeated factors, q^2_x is a perfect square, $(qq)^2 = 0$, and $T = 0$. Conversely, without assumption that 2_x is the square of a linear form, if $T = 0$, then

$$(T, T)^6 = \frac{1}{24}(i^3 - 6J^2) = 0,$$

and f has at least one repeated factor. Let this be a_x. Then from

$$T = \frac{1}{4}(qq)^2 (f, x)^3_x = 0,$$

we have either $(qq)^2 = 0$, when q^2_x is also a perfect square, or $(f, x) = 0$, whenf $= 4_x$. Hence the condition $T = 0$ is both necessary and sufficient.

Chapter 4

REDUCTION

4.1 Gordan's Series. The Quartic

The process of making reductions by means of identities, illustrated in Chapter III, Section 3, is tedious. But we may by a generalizing process, prepare such identities in such a way that the prepared identity will reduce certain extensive types of concomitants with great facility. Such a prepared identity is the series of Gordan.

4.1.1 Gordan's series.

This is derived by rational operations upon the fundamental identity

$$a_x b_y = a_y b_x + (ab)(xy).$$

From the latter we have

$$a_x^m b_y^n = [a_y b_x + (ab)(xy)]^m b_y^{n-m} \quad (n = m)$$
$$= \sum_{k=0}^{m} \binom{m}{k} a_y^{m-k} b_x^{m-k} b_y^{n-m} (ab)^k (xy)^k. \quad (109)$$

Since the left-hand member can represent any doubly binary form of degreeorder (m, n), we have here an expansion of such a function as a power series in (xy). We proceed to reduce this series to a more advantageous form. We construct the $(n-k)$th y-polar of

$$(a_x^m, b_x^n)^k = (ab)^k a_x^{m-k} b_x^{n-k},$$

by the formula for the polar of a product (66). This gives

$$\frac{(a_x^m, b_x^n)^k}{y^{n-k}} = \frac{(ab)^k}{\binom{m+n-2k}{n-k}} \sum_{h=0}^{m-k} \binom{m-k}{m-k-h} \binom{n-k}{n-m+h} a_y^{m-k-h} a_x^h b_x^{m-k-h} b_y^{n-m+h}. \quad (110)$$

Subtracting $(ab)^k a_y^{m-k} b_x^{m-k} b_y^{n-m}$ from each term under the summation and remembering that the sum of the numerical coefficients in the polar of a product is unity we immediately obtain

$$(ab)^k a_y^{m-k} b_x^{m-k} b_y^{n-m} = (a_x^m, b_x^n)^k$$

$$\ldots y^{n-k} - \frac{(ab)^k}{\binom{m+n-2k}{n-k}} \sum_{h=1}^{m-k} \binom{m-k}{m-k-h}\binom{n-k}{n-m+h} \times a_y^{m-k-h} b_x^{m-k-h} b_y^{n-m} (a_x^h b_y^h - a_y^h b_x^h). \quad (111)$$

Aside from the factor $\binom{m}{k}$ the left-hand member of (111) is the coefficient of $(xy)^k$ in (109). Thus this coefficient is the $(n-k)$th polar of the kth transvectant of a_x^m, b_x^n minus terms which contain the factor $(ab)^{k+1}(xy)$. We now use (111) as a recursion formula, taking $k = m, m-1, \ldots$. This gives

$$(ab)^m b_y^{n-m} = (a_x^m, b_x^n)^m_{y^{n-m}},$$

$$(ab)^{m-1} a_y b_x b_y^{n-m} = (a_x^m, b_x^n)^{m-1}_{y^{n-m+1}} - \frac{1}{n-m+2}(a_x^m, b_x^n)^m_{y^{n-m}}(xy). \quad (112)$$

We now proceed to prove by induction that

$$\begin{aligned}(ab)^{k+1} a_y^{m-k-1} b_x^{m-k-1} b_y^{n-m} = {} & \gamma_0 (a_x^m, b_x^n)^{k+1}_{y^{n-k-1}} + \gamma_1 (a_x^m, b_x^n)^{k+2}_{y^{n-k-2}} (xy) + \ldots \\ & + \gamma_j (a_x^m, b_x^n)^{k+j+1}_{y^{n-k-j-1}} (xy)^j + \ldots \quad (113) \\ & + \gamma_{m-k-1} (a_x^m, b_x^n)^m_{y^{n-m}} (xy)^{m-k-1},\end{aligned}$$

where the γ's are constants. The first steps of the induction are given by (112).

Assuming (113) we prove that the relation is true when k is replaced by $k-1$.

By Taylor's theorem

$$\lambda^{h-1}+\lambda^{h-2}+\dots+\lambda+1 = t_{h-1}(\lambda-1)^{h-1}+t_{h-2}(\lambda-1)^{h-2}+\dots+t_1(\lambda-1)+t_0.$$

Hence

$$\left(a_x^h b_y^h - a_y^h b_x^h\right) = t_{h-1}(ab)^h(xy)^h + t_{h-2}(ab)^{h-1}(xy)^{h-1}a_y b_x + \dots$$
$$+\, t_{h-i}(ab)^{h-i+1}(xy)^{h-i+1}a_y^{i-1}b_x^{i-1} + \dots + t_0(ab)(xy)a_y^{h-1}b_x^{h-1}. \quad (114)$$

Hence (111) may be written

$$(ab)^k a_y^{m-k} b_x^{m-k} b_y^{n-m} = (a_x^m, b_x^n)^k_{y^{n-k}} + \sum_{h=1}^{m-k}\sum_{i=1}^{h} A_{hi}(ab)^{h-i+k+1}a_y^{m-k-h+i-1}b_x^{m-k-h+i-1}b_y^{n-m}(xy)^{h-i+1}, \quad (115)$$

in which the coefficients A_{hi} are numerical. But the terms

$$T_{hi} = (ab)^{h-i+k+1}a_y^{m-k-h+i-1}b_x^{m-k-h+i-1}b_y^{n-m} \quad (m-k \geqq h \geqq 1,\ i \leqq h)$$

for all values of h, i are already known by (112), (113) as linear combinations of polars of transvectants; the type of expression whose proof we seek. Hence since (115) is linear in the T_{hi} its terms can immediately be arranged in a form which is precisely (113) with k replaced by $k-1$. This proves the statement.

We now substitute from (113) in (109) for all values of k. The result can obviously be arranged in the form

$$a_x^m b_y^n = C_0(a_x^m, b_x^n)^0_{y^n} + C_1(a_x^m, b_x^n)^1_{y^{n-1}}(xy) + \cdots \quad (116)$$
$$+\, C_j(a_x^m, b_x^n)^j_{y^{n-j}}(xy)^j + \cdots + C_m(a_x^m, b_x^n)^m_{y^{n-m}}(xy)^m.$$

It remains to determine the coefficients C_j. By (911) of Chapter III we have, after operating upon both sides of (116) by

j and then placing $y = x$,

$$\frac{m!n!}{(m-j)!(n-j)!}(ab)^j a_x^{m-j} b_x^{n-j} = C_j \frac{j!(m+n-j+1)!}{(m+n-2j+1)!}(ab)^j a_x^{m-j} b_x^{n-j}.$$

Solving for C_j, placing the result in (116) $(j = 0, 1, \ldots, m)$, and writing the result as a summation,

$$a_x^m b_y^n = \sum_{j=0}^{m} \frac{\binom{m}{j}\binom{n}{j}}{\binom{m+n-j+1}{j}} (xy)^j (a_x^m, b_x^n)^j_{y^{n-j}}. \quad (117)$$

This is Gordan's series.

To put this result in a more useful, and at the same time a more general form let us multiply (117) by $(ab)^r$ and change m, n into $m - r$, $n - r$ respectively. Thus

$$(ab)^r a_x^{m-r} b_y^{n-r} = \sum_{j=0}^{m-r} \frac{\binom{m-r}{j}\binom{n-r}{j}}{\binom{m+n-2r-j+1}{j}} (xy)^j (a_x^m, b_x^n)^{j+r}_{y^{n-j-r}}. \quad (118)$$

If we operate on this equation by $\left(x\frac{\partial}{\partial y}\right)^k$, $\left(y\frac{\partial}{\partial x}\right)^k$, we obtain the respective formulas

$$(ab)^r a_x^{m-r} b_x^k b_y^{n-r-k} = \sum_j \frac{\binom{m-r}{j}\binom{n-r-k}{j}}{\binom{m+n-2r-j+1}{j}} (xy)^j (a_x^m, b_x^n)^{j+r}_{y^{n-j-r-k}}, \quad (119)$$

$$(ab)^r a_x^{m-r-k} a_y^k b_y^{n-r} = \sum_j \frac{\binom{m-r-k}{j}\binom{n-r}{j}}{\binom{m+n-2r-j+1}{j}} (xy)^j (a_x^m$$

, bnx
)j+r
yn−j−r+k . (120)
It is now desirable to recall the standard method of transvection; replace y1 by

c2, y2 by −c1 in (119) and multiply by cp−n+r+k
x , with the result
(ab)r(bc)n−r−kam−r
x bk
xcp−n+r+k
x
=Xj
(−1)j m−r
j n−r−k
j m+n−2r−j+1
j ((amx
, bnx
)j+r, cp
x)n−j−r−k. (121)
Likewise from (120)
(ab)r(bc)n−r(ac)kam−r−k
x cp−n+r−k
x
=Xj
(−1)j m−r−k
j m−r
j m+n−2r−j+1
j ((amx
, bnx
)j+r, cp
x)n−j−r+k. (122)
The left-hand member of equation (121) is unaltered in value except for the
factor (−1)n−k by the replacements a c,m p, r n − r − k; and likewise
(122) is unaltered except for the factor (−1)n+k by the replacements a c,m p, r n − r. The right-hand members are however altered in form by these
changes. If the changes are made in (121) and if we write f = bnx
, g = amx
, h =
cp
x, 1 = 0, 2 = n − r − k, 3 = r, we obtain
Xj m−1−3
j 2
j m+n−23−j+1
j ((f, g)3+j , h)1+2−j
= (−1)1Xj p−1−2
j 3
j n+p−22−j+1
j ((f, h)2+j , g)1+3−j , (123)
where we have
2 + 3 n, 3 + 1 m, 1 + 2 p, (1241)
together with 1 = 0.
If the corresponding changes, given above, are made in (122) and if we write
1 = k, 2 = n − r, 3 = r, we obtain the equation (123) again, precisely. Also
relations (1241) reproduce, but there is the additional restriction 2 + 3 = n.
Thus (123) holds true in two categories of cases, viz. (1) 1 = 0 with (1241),
and (2) 2 + 3 = n with (1241). We write series (123) under the abbreviation
24
f g h
n m p

1 2 3
35;

2
+

3

n,
3
+

1

m,
1
+

2

p,
(i) 1 = 0,
(ii) 1 + 2 = n.
It is of very great value as an instrument in performing reductions. We proceed to illustrate this fact by proving certain transvectants to be reducible.
Consider (,Q) of Table I.
(,Q) = ((, f),).

Here n = p = 2, m = 3, and we may take 1 = 0, 2 = 3 = 1, giving the series

f
2 3 2
0 1 1
,
that is,
((, f),) +
2
3
((, f)2,)0 = ((,), f) +
1
2
((,)2, f)0.
But
(,) = 0, (, f)2 = 0, (,)2 = R.
Hence
(,Q) = ((, f),) =
1
2Rf,
which was to be proved.
Next let f = amx
be any binary form and H = (f, f)2 its Hessian. We wish
to show that ((f, f)2, f)2 is always reducible and to perform the reduction by Gordan's series. Here we may employ
0@
f f f
m m m
0 3 11A ,
and since (f, f)2+1 = 0, this gives at once
m−1

1 3
1 2m−2
1 ((f, f)2, f)2 + m−1
3 3
3 2m−4
3 ((f, f)4, f)0 = m−3
1 1
1 2m−6
1 ((f, f)4, f)0.
Solving we obtain
((f, f)2, f)2 = m − 3
2(2m − 5)
((f, f)4, f)0 = m − 3
2(2m − 5) if, (124)
where i = (f, f)4.
Hence when m = 4 this transvectant is always reducible.

4.1.2 The quartic.

By means of Gordan's series all of the reductions indicated in Table I and the corresponding ones for the analogous table for the quartic, Table II below, can be very readily made. Many reductions for forms of higher order and indeed for a general order can likewise be made (Cf. (124)). It has been shown by Stroh[1] that certain classes of transvectants cannot be reduced by this series but the simplest members of such a class occur for forms of higher order than the

[1]Stroh; Mathematische Annalen, vol. 31.

fourth. An example where the series will fail, due to Stroh, is in connection with the decimic f = a10
x . The transvectant
((f, f)6, f)4
is not reducible by the series in its original form although it is a reducible covariant. A series discovered by Stroh will, theoretically, make all reductions, but it is rather difficult to apply, and moreover we shall presently develop powerful methods of reduction which largely obviate the necessity of its use. Stroh's series is derived by operations upon the identity (ab)cx + (bc)ax + (ca)bx = 0.

TABLE II

r = 1 r = 2 r = 3 r = 4
(f, f) 0 H 0 i
(f,H) T 1
6 if 0 J
(f, T) 1
12 (if 2 − 6H2) 0 1
4 (Jf − iH) 0
(H,H) 0 1
6 (2Jf − iH) 0 1
6 i2
(H, T) 1
6 (Jf2 − ifH) 0 1
24 (i2f − 6JH) 0
(T, T) 0 1
72 (i2f2 + 6iH2 − 12JfH) 0 0

We infer from Table II that the complete irreducible system of the quartic consists of
f,H, T, i, J.
This will be proved later in this chapter. Some of this set have already been derived in terms of the actual coefficients (cf. (701)). They are given below. These are readily derived by non-symbolical transvection (Chap. III) or by the method of expanding their symbolical expressions and then expressing the symbols in terms of the actual coefficients (Chap. III, Section 2).
f = a0x41

$$+ 4a_1x_1^3x_2 + 6a_2x_1^2x_2^2 + 4a_3x_1x_2^3 + a_4x_2^4,$$

$$H = 2[(a_0a_2 - a_1^2)x_1^4 + 2(a_0a_3 - a_1a_2)x_1^3x_2 + (a_0a_4 + 2a_1a_3 - 3a_2^2)x_1^2x_2^2 + 2(a_1a_4 - a_2a_3)x_1x_2^3 + (a_2a_4 - a_3^2)x_2^4],$$

$$T = (a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3)x_1^6 + (a_0^2a_4 + 2a_0a_1a_3 - 9a_0a_2^2 + 6a_1^2a_2)x_1^5x_2 + 5(a_0a_1a_4 - 3a_0a_2a_3 + 2a_1^2a_3)x_1^4x_2^2 + 10(a_1^2a_4 - a_0a_3^2)x_1^3x_2^3 + 5(-a_0a_3a_4 + 3a_1a_2a_4 - 2a_1a_3^2)x_1^2x_2^4 \quad (125)$$

$$+ (9a_4a_2^2 - a_4^2a_0 - 2a_1a_3a_4 - 6a_3^2a_2)x_1x_2^5 + (3a_2a_3a_4 - a_1a_4^2 - 2a_3^3)x_2^6,$$

$$i = 2(a_0a_4 - 4a_1a_3 + 3a_2^2),$$

$$J = 6\begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix} = 6(a_0a_2a_4 + 2a_1a_2a_3 - a_2^3 - a_0a_3^2 - a_1^2a_4).$$

These concomitants may be expressed in terms of the roots by the methods of Chapter III, Section 4, and in terms of the Aronhold symbols by the standard

method of transvection. To give a fresh illustration of the latter method we select $T = (f,H) = -(H, f)$. Then

(H, f) = ((ab)2a2
xb2
x, c4
x)
=
(ab)2
4 2
02
1a2
xbxby + 2
12
0axayb2
xyc
c3
x
=
1
2
(ab)2(bc)a2
xbxc3
x +
1
2
(ab)2(ac)axb2
xx3
x
= (ab)2(ac)axb2
xc3
x.

Similar processes give the others. We tabulate the complete totality of such results below. The reader will find it very instructive to develop these results in detail.

f = a4
x = b4
x = . . . ,
H = (ab)2a2
xb2
x,
T = (ab)2(ca)axb2
xc3
x,
i = (ab)4,
J = (ab)2(bc)2(ca)2.

Except for numerical factors these may also be written

f = (1)
x (2)
x (3)
x (4)
x ,
H =X((1)
x (2)
x)2(3)2
x (4)2
x ,
T =X((1)
x (2)
x)2((1)
x (3)
x)(2)

x (3)2
x (4)3
x , (126)
i =X((1)
x (2)
x)2((3)
x (4)
x)2,
J =X((1)
x (2)
x)2((3)
x (4)
x)2((3)
x (1)
x)((2)
x (4)
x).

It should be remarked that the formula (90) for the general rth transvectant of Chapter III, Section 2 may be employed to great advantage in representing concomitants in terms of the roots.

With reference to the reductions given in Table II we shall again derive in detail only such as are typical, to show the power of Gordan's series in performing reductions. The reduction of (f,H)2 has been given above (cf. (124)).

We have
(−T,H)3 = ((H, f),H)3 = (H, T)3.
Here we employ the series
0@
H f H
4 4 4
0 3 11A .

This gives
3 Xj=0 3
j 3
j 7−j
j ((H, f)1+j ,H)3−j =
1 Xj=0 1
j 1
j 3−j
j ((H,H)3+j , f)1−j .
Substitution of the values of the transvectants (H, f)r, H,H)4 gives
(H, T)3 =
1
24
(−6JH + i2f).
The series for (T, T)2 = ((f,H), T)2 is

f H T
4 4 6
0 2 1
,
or
((f,H), T)2 + ((f,H)2, T) = ((f, T)2,H) +
2
3
((f, T)8, h)0.
But
(T, T)2 = −
1

72
$(i^2f^2 + 6iH^2 - 12JHf)$,
Hence, making use of the third line in Table II,
$(T, T)^2 = -$
1
72
$(i^2f^2 + 6iH^2 - 12JHf)$,
which we wished to prove. The reader will find it profitable to perform all of the reductions indicated in Table II by these methods, beginning with the simple cases and proceeding to the more complicated.

4.2 Theorems on Transvectants

We shall now prove a series of very far-reaching theorems on transvectants.

4.2.1 Monomial concomitant a term of a transvectant

Every monomial expression, , in Aronhold symbolical letters of the type peculiar to the invariant theory, i.e. involving the two types of factors (ab), ax;
 =Y(ab)pacq . . . a
xb
xc
x . . . ,
is a term of a determinate transvectant.

In proof let us select some definite symbolical letter as a and in all determinant factors of which involve a set $a_1 = -y_2$, $a_2 = y_1$. Then may be separated into three factors, i.e.
0 = PQa
x,

where Q is an aggregate of factors of the one type by, Q = b3
yct
y..., and P is a
symbolical expression of the same general type as the original but involving one less symbolical letter,
p = (bc)u(bd)v...b
x c
x...

Now PQ does not involve a. It is, moreover, a term of some polar whose index r is equal to the order of Q in y. To obtain the form whose rth polar contains the term PQ it is only necessary to let y = x in PQ since the latter will then go back into the original polarized form (Chap. III, Section 1, I). Hence is a term of the result of polarizing (PQ)y=xr times, changing y into a and multiplying this result by ap
x. Hence by the standard method of transvection is a term of the transvectant
((PQ)y=x, ar+p
x)r(r + p = m). (127)

For illustration consider
 = (ab)2(ac)(bc)axbxc2
x
Placing a y in (ab)2(ac) we have
0 = −b2
yc2
y(bc)bxc2
x Â· ax
Placing y x in 0 we obtain
00 = −(bc)b3
xc3
xax.
Thus is a term of
A = (−(bc)b3
xc3

x, $a^4_x)^3$.

In fact the complete transvectant A is

$$+A = -\frac{1}{20}(bc)(ca)^3 a_x b^3_x - \frac{9}{20}(bc)(ca)^2(ba) a_x b^2_x c_x - \frac{9}{20}(bc)(ca)(ba)^2 a_x b_x c^2_x - \frac{1}{20}(bc)(ba)^3 a_x c^3_x$$

and is its third term.

DEFINITION.

The mechanical rule by which one obtains the transvectant $(ab)a^{m-1}_x b^{m-1}_x$ from the product $a^m_x b^m_x$, consisting of folding one letter from each symbolical form a^m_x, b^m_x into a determinant (ab) and diminishing exponents by unity, is called convolution. Thus one may obtain $(ab)^2(ac)a_x b^2_x c^3_x$ from $(ab)a^3_x b^3_x c^4_x$ by convolution.

4.2.2 Theorem on the difference between two terms of a transvectant.

Theorem. (1) The difference between any two terms of a transvectant is equal to a sum of terms each of which is a term of a transvectant of lower index of forms obtained from the forms in the original transvectant by convolution.

(2) The difference between the whole transvectant and one of its terms is equal to a sum of terms each of which is a term of a transvectant of lower index of forms obtained from the original forms by convolution (Gordan).

In proof of this theorem we consider the process of constructing the formula for the general rth transvectant in Chapter III, Section 5. In particular we examine the structure of a transvectant-like formula (89). Two terms of this or of any transvectant are said to be adjacent when they differ only in the arrangement of the letters in a pair of symbolical factors. An examination of a formula such as (89) shows that two terms can be adjacent in any one of three ways, viz.:

(1) P((i)(j))((h)(k)) and P((t)(k))((h)(j)),

(2) $P((i)(j))(h)_x$ and $P((h)(j))(i)_x$,

(3) $P((i)(j))(k)_x$ and $P((i)(k))(j)$

x ,
where P involves symbols from both forms f, g as a rule, and both types of symbolical factors.
The differences between the adjacent terms are in these cases respectively
(1) P((i)(h))((j)(k)),
(2) P((i)(h))(j)
x ,
(3) P((k)(j))(i)
x .
These follow directly from the reduction identities, i.e. from formulas (99), (100).
Now, taking f, g to be slightly more comprehensive than in (89), let
f = A(1)
x (2)
x . . . (m)
x ,
g = B(1)
x (2)
x . . . (n)
x ,
where A and B involve only factors of the first type (). Then formula (90) holds true;
(f, g) =
1
! m
n
X"((1)(1))((2)(2) . . . (()())
(1)
x (2)
x . . . ()
x (1)
x (2)
x . . . ()
x
fg#.
and the difference between any two adjacent terms of (f, g) is a term in which at least one factor of type () is replaced by one of type (0) or of type (0). There then remain in the term only − 1 factors of type (). Hence this difference is a term of a transvectant of lower index of forms obtained from the original forms f, g by convolution.
For illustration, two adjacent terms of ((ab)2a2
xb2
x, c4
x)2 are
(ab)2(ac)2b2
xc2
x, (ab)2(ac)(bc)axbxc2
x.
The difference between these terms, viz. (ab)3(ac)bxc3
x, is a term of
((ab)3axbx, c4
x),

and the first form of this latter transvectant may be obtained from (ab)2a2
xb2
x
by convolution.
Now let t1, t2 be any two terms of (f, g) . Then we may place between t1, t2 a series of terms of (f, g) such that any term of the series,

t1, t11, t12, . . . , t1i, t2
is adjacent to those on either side of it. For it is always possible to obtain t2 from t1 by a finite number of interchanges of pairs of letters,–a pair being composed either of two 's or else of two 's. But
t1 − t2 = (t1 − t11) + (t11 − t12) + . . . + (t1 − t2),
and all differences on the right are differences between adjacent terms, for which the theorem was proved above. Thus the part (1) of the theorem is proved for all types of terms.
Next if t is any term of (f, g) , we have, since the number of terms of this transvectant is
r!m
r n
r,
(f, g)r − t =
1
r! m
r n
rXt0 − t (128)
=
1
r! m
r n
rX(t0 − t),
and by the first part of the theorem and on account of the form of the righthand member of the last formula this is equal to a linear expression of terms of transvectants of lower index of forms obtained from f, g by convolution.

4.2.3 Difference between a transvectant and one of its terms.

The difference between any transvectant and one of its terms is a linear combination of transvectants of lower index of forms obtained from the original forms by convolution.
Formula (128) shows that any term equals the transvectant of which it is a term plus terms of transvectants of lower index. Take one of the latter terms and apply the same result (128) to it. It equals the transvectant of index s < of which it is a term plus terms of transvectant of index < s of forms obtained from the original forms by convolution. Repeating these steps we arrive at transvectants of index 0 between forms derived from the original forms by convolution, and so after not more than applications of this process the right-hand side of (128) is reduced to the type of expression described in the theorem.
Now on account of the Theorem I of this section we may go one step farther. As proved there every monomial symbolical expression is a term of a determinate

transvectant one of whose forms is the simple f = amx
of degree-order (1,m).
Since the only convolution applicable to the form amx
is the vacuous convolution
producing amx
itself, Theorem III gives the following result:
Let be any monomial expression in the symbols of a single form f, and let some symbol a occur in precisely r determinant factors. Then equals a linear combination of transvectants of index 5 r of amx
and forms obtained from
(PQ)x=y (cf. (127)) by convolution.
For illustration
= (ab)2(bc)2a2
xc2
x = ((ab)2a2
xb2

xc4
x)2 − ((ab)3axbx, c4
x) +
1
3
((ab)4, c4
x)0.
It may also be noted that (PQ)y=x and all forms obtained from it by convolution are of degree one less than the degree of in the coefficients of f. Hence by reasoning inductively from the degrees 1, 2 to the degree i we have the result:
Theorem. Every concomitant of degree i of a form f is given by transvectants of the type
(Ci−1, f)i,
where the forms Ci−1 are all concomitants of f of degree i − 1. (See Chap. III, Â§2, VII.)

4.3 Reduction of Transvectant Systems

We proceed to apply some of these theorems.

4.3.1 Reducible transvectants of a special type. (Ci−1, f)i.

The theorem given in the last paragraph of Section 2 will now be amplified by another proof. Suppose that the complete set of irreducible concomitants of degrees < i of a single form is known. Let these be
f, 1, 2, Â· Â· Â· , k,
and let it be required to find all irreducible concomitants of degree i. The only concomitant of degree unity is f = amx
. All of degree 2 are given by
(f, f) = (ab)am−
x bm−
x ,
where, of course, is even. A covariant of degree i is an aggregate of symbolical products each containing i symbols. Let Ci be one of these products, and a one of the symbols. Then by Section 2 Ci is a term of a transvectant
(Ci−1, xmx
)i,
where Ci−1 is a symbolical monomial containing i − 1 symbols, i.e. of degree i − 1. Hence by Theorem II of Section 2,
Ci = (Ci−1, f)i +X(Â¯ Ci−1f)j0 (j0 < j), (129)

where Â¯ Ci−1 is a monomial derived from Ci−1 by convolution. Now Ci−1, Â¯ Ci−1 being of degree i − 1 are rational integral expressions in the irreducible forms f, 1, . . . , k. That is they are polynomials in f, 1, . . . , k, the terms of which are of the type
i−1 = faa1
1 . . . ak
k .
Hence Ci is a sum of transvectants of the type
(i−1, f)i(j 5 m),
and since any covariant of f, of degree i is a linear combination of terms of the type of Ci, all concomitants of degree i are expressible in terms of transvectants of the type
(i−1, f)i, (130)
where i−1 is a monomial expression in f, 1, . . . , k, of degree i − 1, as just explained.
In order to find all irreducible concomitants of a stated degree i we need now to develop a method of finding what transvectants of (130) are reducible in terms of f, 1, . . . , k. With this end in view let i−1 = , where , are also monomials in f, 1, . . . , k, of degrees < i−1. Let be a*form of order n1;
p = pn1
x , and = n2
x . Then assume that j 5 n2, the order of . Hence we have

(i−1, f)j = (pn1
x n2
x , amx
)j .
Then in the ordinary way by the standard method of transvection we have the following:
(i−1, f)j = K{pn1
x n2−j
x jy
}y=aam−j
x + Â· Â· Â·
= Kp(, f)j + Â· Â· Â· . (131)
Hence if p2 now represents (, f)j , then pp2 is a term of (i−1, f), so that
(i−1, f)j = pp2 +X(i−1, f)j02 (j0 < j). (132)
Evidently p, p2 are both covariants of degree /lti and hence are reducible in terms of f, 1, Â· Â· Â· , k. Now we have the right to assume that we are constructing the irreducible concomitants of degree i by proceeding from transvectants of a stated index to those of the next higher index, i.e. we assume these transvectants to be ordered according to increasing indices. This being true, all of the transvectants (i−1, f)j0 at the stage of the investigation indicated by (132) will be known in terms of f, 1, Â· Â· Â· , k or known to be irreducible, those that are so, since j0/ltj. Hence (132) shows (i−1, f) to be reducible since it is a polynomial in f, 1, Â· Â· Â· , k and such concomitants of degree i as are already known.
The principal conclusion from this discussion therefore is that irreducible concomitants of degree i are obtained only from transvectants (i−1, f)j for which no factor of order = j occurs in i−1. Thus for instance if m = 4, (f2, f)j is reducible for all values of j since f2 contains the factor f of order 4 and j cannot exceed 4.
We note that if a form is an invariant it may be omitted when we form i−1, for if it is present (i−1, f)j will be reducible by (80).

4.3.2 Fundamental systems of cubic and quartic.
Let m = 3 (cf. Table I). Then f = a3
x is the only concomitant of degree 1.
There is one of degree 2, the Hessian (f, f)2 = . Now all forms 2 of (2, f)j are included in
2 = f,
and either = 2, = 0, or = 0, = 1. But (f2, f)j is reducible for all values of j since f2 contains the factor f of order 3 and j 3. Hence the only transvectants which could give irreducible concomitants of degree 3 are
(, f)j (j = 1, 2).
But (, f)2 = 0 (cf. Table I). In fact the series
0@
f f f
3 3 3
1 2 11A
gives 1
2 ((f, f)2, f)2 = −((f, f)2, f)2 −(, f)2 = 0. Hence there is one irreducible covariant of degree 3, e.g.
(, f) = −Q.
Proceeding to the degree 4, there are three possibilities for 3 in (3, f)j .
These are 3 = f3, f,Q. Since j 3 (f3, f)j , (f, f)j(j = 1, 2, 3) are all reducible by Section 3, I. Of (Q, f)j(j = 1, 2, 3), (Q, f)2 = 0, as has been proved before (cf. (102)), and (Q, f) = 1
22 by the Gordan series (cf. Table I)
0@
f f
3 2 3
0 1 11A .

Hence $(Q, f)^3 = -R$ is the only irreducible case. Next the degree 5 must be treated. We may have

4 = f4, f2, fQ,R,2.

But R is an invariant, is of order 2, and Q of order 3. Hence since j 3 in (4, f)j the only possibility for an irreducible form is (2, f)j , and this is reducible by the principle of I if j < 3. But

(2, f)3 = (2x
02
x , a3
x)3 = (a)2(0a)0x = (02
x , (a)2ax) = 0.

For (a)2ax = (, f)2 = 0, as shown above. Hence there are no irreducible concomitants of degree 5. It immediately follows that there are none of degree > 5, either, since 5 in (5, f)j is a more complicated monomial than 4 in the same forms f,,Q and all the resulting concomitants have been proved reducible.

Consequently the complete irreducible system of concomitants of f, which may be called the fundamental system (Salmon) of f is

f,, Q,R.

Next let us derive the system for the quartic f; m = 4. The concomitants of degree 2 are $(f, f)^2 = H$, $(f, f)^4 = i$. Those of degree 3 are to be found from $(H, f)^j (j = 1, 2, 3, 4)$.

Of these (f,H) = T and is irreducible; $(f,H)^4 = J$ is irreducible, and, as has been proved, $(H, f)^2 = \frac{1}{6} if$ (cf. (124)). Also from the series

$$\begin{pmatrix} f & f & f \\ 4 & 4 & 4 \\ 1 & 3 & 11 \end{pmatrix},$$

$(H, f)^3 = 0$. For the degree four we have in (3, f)j

3 = f3, fH, T,

all of which contain factors of order = j 4 except T. From Table II all of the transvectants $(T, f)^j (j = 1, 2, 3, 4)$ are reducible or vanish, as has been, or may be proved by Gordan's series. Consider one case; $(T, f)^4$. Applying the series

$$\begin{pmatrix} f & H & f \\ 4 & 4 & 4 \\ 1 & 3 & 11 \end{pmatrix},$$

we obtain

$$((f,H), f)^4 = -((f,H)^2, f)^3 - \frac{3}{10}((f,H)^3, f)^2.$$

But $((f,H)^2, f)^3 = \frac{1}{6} i(f, f)^3 = 0$; and $(f,H)^3 = 0$ from the proof above. Hence

$$((f,H), f)^4 = (T, f)^4 = 0.$$

There are no other irreducible forms since 4 in (4, f)i will be a monomial in f, H, T more complicated than 3. Hence the fundamental system of f consists of

f,H, T, i, J;

It is worthy of note that this has been completely derived by the principles of this section together with Gordan's series.

4.3.3 Reducible transvectants in general.

In the transvectants studied in (I) of this section, e.g. (i−1, f)i, the second form is simple, $f = a^m_x$, of the first degree. It is possible and now desirable to extend those methods of proving certain transvectants to be reducible to the more general case where both forms in the transvectants are monomials in other concomitants of lesser degree.

Consider two systems of binary forms, an (A) system and a (B) system. Let the forms of these systems be

(A): $A_1, A_2, \ldots, A_k$, of orders $a_1, a_2, \ldots, a_k$ respectively;

and

(B): $B_1, B_2, \ldots, B_l$, of orders $b_1, b_2, \ldots, b_k$ respectively.

Suppose these forms expressed in the Aronhold symbolism and let

= A1
1 A2
2 . . .Ak
k , = B1
1 B2
2 , . . .Bl
l .

Then a system (C) is said to be the system derived by transvection from (A) and (B) when it includes all terms in all transvectants of the type

(,)i. (133)

Evidently the problem of reducibility presents itself for analysis immediately. For let

= , = μ,

and suppose that j can be separated into two integers,

$j = j_1 + j_2$,

such that the transvectants

(, μ)i1 , (,)i2

both exist and are different from zero. Then the process employed in proving formula (132) shows directly that (,)i contains terms which are products of terms of (, μ)i1 and terms of (,)i2 , that is, contains reducible terms.

In order to discover what transvectants of the (C) system contain reducible terms we employ an extension of the method of Paragraph (I) of this section. This may be adequately explained in connection with two special systems

(A) = f, (B) = i,

where f is a cubic and i is a quadratic. Here

(C) = (,)i = (f, i)i.

Since f must not contain a factor of order j, we have

3 − 3 < j 3; j = 3, 3 − 1, 3 − 2.

Also

2 − 2 < j 2; j = 2, 2 − 1.

Consistent with these conditions we have

(f, i), (f, i)2, (f, i2)3, (f2, i2)4, (f2, i3)5, (f2, i3)6, (f3, i4)7, (f3, i4)8, (f3, i5)9, ...

Of these, (f2, i2)4 contains terms of the product (f, i)2(f, i)2, that is, reducible terms. Also (f2, i3)5 is reducible by (f, i)2 (f, i2)3. In the same way (f3, i4)7, . . . all contain reducible terms. Hence the transvectants of (C) which do not contain reducible terms are six in number, viz.

f, i, (f, i), (f, i)2, (f, i2)3, (f2, i3)6.

The reader will find it very instructive to find for other and more complicated (A) and (B) systems the transvectants of (C) which do not contain reducible terms. It will be found that the irreducible transvectants are in all cases finite in number. This will be proved as a theorem in the next chapter.

4.4 Syzygies

We can prove that m is a superior limit to the number of functionally independent invariants and covariants of a single binary form f = amx of order m. The totality of independent relations which can and do subsist among the quantities

x1, x2, x01, x02, 0i , i(i = 0, . . . ,m), 1, 2, μ1, μ2,M = (μ)

are m + 4 in number. These are

0i = m−i
iμ

(i = 0, . . . ,m); x1 = 1x01 + μ1x02, x2 = 2x01 + μ2x02;
M = 1μ2 − 2μ1.

When one eliminates from these relations the four variables 1, 2, μ1, μ2 there result at most m relations. This is the maximum number of equations which can exist between 0i , i(i = 0, . . . ,m), x1, x2, x01, x02, and M. That is, if a greater number of relations between the latter quantities are assumed, extraneous conditions, not implied in the invariant problem, are imposed upon the coefficients and variables. But a concomitant relation

(00, . . . , 0m, x01, x02) = Mk(0, . . . , m, x1, x2)

is an equation in the transformed coefficients and variables, the untransformed coefficients and variables and M. Hence there cannot be more than m algebraically independent concomitants as stated.

Now the fundamental system of a cubic contains four concomitants which are such that no one of them is a rational integral function of the remaining three. The present theory shows, however, that there must be a relation between the four which will give one as a function of the other three although this function is not a rational integral function. Such a relation is called a syzygy (Cayley). Since the fundamental system of a quartic contains five members these must also be connected by one syzygy. We shall discover that the fundamental system of a quintic contains twenty-three members. The number of syzygies for a form of high order is accordingly very large. In fact it is possible to deduce a complete set of syzygies for such a form in several ways. There is, for instance, a class of theorems on Jacobians which furnishes an advantageous method of constructing syzygies. We proceed to prove these theorems.

4.4.1 Reducibility of ((f, g), h).

Theorem. If f, g, h are three binary forms, of respective orders n, m, p all greater than unity, the iterated Jacobian ((f, g), h) is reducible.

The three series 0@
f g h
n m p
0 1 11A ,
0@
h f g
p n m
0 1 11A ,
0@
g h f
m p n
0 1 11A give the respective results
((f, g), h) = ((f, h), g) + p − 1
n + p − 2
(f, h)2g −
m − 1
m + n − 2
(f, g)2h,
−((h, f), g) = −((h, g), f) −
m − 1
m + p − 2
(h, g)2f + n − 1
n + p − 2
(h, f)2g,
((g, h), f) = ((g, f), h) + n − 1
m + n − 2
(g, f)2h −
p − 1
m + p − 2
(g, h)2f.

We add these equations and divide through by 2, noting that (f, g) = −(g, f), and obtain

$$((f, g), h) = \frac{n - m}{2(m + n - 2)}(f, g)^2 h + \frac{1}{2}(f, h)^2 g - \frac{1}{2}(g, h)^2 f. \quad (134)$$

This formula constitutes the proof of the theorem. It may also be proved readily by transvection and the use of reduction identity (101).

4.4.2 Product of two Jacobians.

Theorem. If $e = a_x^m$, $f = b_x^n$, $g = c_x^p$, $h = d_x^q$ are four binary forms of orders greater than unity, then

$$(e, f)(g, h) = -\frac{1}{2}(e, g)^2 fh + \frac{1}{2})e, h)^2 fg + \frac{1}{2}(f, g)^2 eh - \frac{1}{2}(f, g)^2 eg. \quad (135)$$

We first prove two new symbolical identities. By an elementary rule for expanding determinants

$$\begin{vmatrix} a_1^2 & a_1a_2 & a_2^2 \\ b_1^2 & b_1b_2 & b_2^2 \\ c_1^2 & c_1c_2 & c_2^2 \end{vmatrix} = -(ab)(bc)(ca).$$

Hence

$$\begin{vmatrix} a_1^2 & a_1a_2 & a_2^2 \\ b_1^2 & b_1b_2 & b_2^2 \\ c_1^2 & c_1c_2 & c_2^2 \end{vmatrix} \begin{vmatrix} d_2^2 & -2d_2d_2 & d_1^2 \\ e_2^2 & -2e_2e_1 & e_1^2 \\ f_2^2 \end{vmatrix}$$

$2 \; -2f_2 f_1 \; f_2$
1

$$= 2(ab)(bc)(ca)(de)(ef)(fd) = \begin{vmatrix} (ad)^2 & (ae)^2 & (af)^2 \\ (bd)^2 & (be)^2 & (bf)^2 \\ (cd)^2 & (ce)^2 & (cf)^2 \end{vmatrix}. \quad (136)$$

In this identity set $c_1 = -x_2$, $c_2 = x_1$, $f_1 = -x_2$, $f_2 = x_1$.
Then (136) gives the identity.

$$2(ab)(de)a_x b_x d_x e_x = \begin{vmatrix} (ad)^2 & (ae)^2 & a_x^2 \\ (bd)^2 & (be)^2 & b_x^2 \\ d_x^2 & e_x^2 & 0 \end{vmatrix}. \quad (137)$$

We now have

$$(e, f)(g, h) = (a, b)(c, d)a_x^{m-1} b_x^{n-1} c_x^{p-1} d_x^{q-1} = \frac{1}{2} a_x^{m-2} b_x^{n-2} c_x^{p-2} d_x^{q-2} \begin{vmatrix} (ac)^2 & (ad)^2 & a_x^2 \\ (bc)^2 & (bd)^2 & b_x^2 \\ c_x^2 & d_x^2 & 0 \end{vmatrix}$$

by (137). Expanding the right-hand side we have formula (135) immediately.

4.5 The square of a Jacobian.

The square of a Jacobian $J = (f, g)$ is given by the formula

$$-2J^2 = (f, f)_2 g^2 + (g, g)_2 f^2 - 2(f, g)_2 fg. \quad (138)$$

This follows directly from (135) by the replacements

$e = f, f = g, g = f, h = g.$

4.5.1 Syzygies for the cubic and quartic forms.

In formula (138) let us make the replacements $J = Q$, $f = f$, $g =$, where f is a cubic, is its Hessian, and Q is the Jacobian $(f,)$. Then by Table I

$$S = 2Q^2 + 3 + RF^2 = 0. \quad (139)$$

This is the required syzygy connecting the members of the fundamental system of the cubic.

Next let f, H, T, i, J be the fundamental system of a quartic f. Then, since T is a Jacobian, let $J = T$, $f = f$, $g = H$ in (138), and we have

$$-2T^2 = H^3 - 2(f,H)_2 fH + (H,H)_2 f^2.$$

But by Table II

$$(f,H)_2 = \frac{1}{6} if, \quad (H,H)_2 =$$

$\frac{1}{6}(2Jf - iH)$.

Hence we obtain

$$S = 2T^2 + H^3 - \frac{1}{2}if^2H + \frac{1}{3}Jf^3 = 0. \quad (140)$$

This is the syzygy connecting the members of the fundamental system of the quartic.

Of the twenty-three members of a system of the quintic nine are expressible as Jacobians (cf. Table IV, Chap. VI). If these are combined in pairs and substituted in (135), and substituted singly in (138), there result 45 syzygies of the type just derived. For references on this subject the reader may consult Meyer's "Bericht ueber den gegenwÂ¨artigen Stand der Invariantentheorie" in the Jahresbericht der Deutschen Mathematiker-Vereinigung for 1890-91.

4.5.2 Syzygies derived from canonical forms.

We shall prove that the binary cubic form,

$$f = a_0x_1^3 + 3a_1x_1^2x_2 + 3a_2x_1x_2^2 + a_3x_2^3,$$

may be reduced to the form,

$$f = X^3 + Y^3,$$

by a linear transformation with non-vanishing modulus. In general a binary quantic f of order m has m + 1 coefficients. If it is transformed by

$$T : x_1 = 1x'_1 + \mu_1x'_2,\ x_2 = 2x'_1 + \mu_2x'_2,$$

four new quantities 1, μ_1, 2, μ_2 are involved in the coefficients of f'. Hence no binary form of order m with less than m − 3 arbitrary coefficients can be the transformed of a general quantic of order TO by a linear transformation. Any quantic of order m having just m − 3 arbitrary quantities involved in its coefficients and which can be proved to be the transformed of the general form f by a linear transformation of non-vanishing modulus is called a canonical form of f. We proceed to reduce the cubic form f to the canonical form $X^3 + Y^3$.

Assume

$$f = a_0x_1^3 + \ldots = p_1(x_1 + 1x_2)^3 + p_2(x_1 + 2x_2)^3 = X^3 + Y^3. \quad (1401)$$

This requires that f be transformable into its canonical form by the inverse of the transformations

$$S : X = p_1^{\frac{1}{3}}\,1 + p_1^{\frac{1}{3}}\,1\,1_1,\quad Y = p_2^{\frac{1}{3}}\,1 + p_2^{\frac{1}{3}}\,2\,2_2.$$

We must now show that p_1, p_2, 1, 2 may actually be determined, and that the determination is unique. Equating coefficients in (1401) we have

$$p_1 + p_2 = 0,$$

$$1p_1 + 2p_2 = 1, \quad (1402)$$

2

1p1 + 2
2p2 = 2,
3
1p1 + 3
2p2 = 3.
Hence the following matrix, M, must be of rank 2:
M =
1 1 2
1 3
1
1 2 2
2 3
2
0 1 2 3

From M = 0 result

1 1 2
1
1 2 2
2
0 1 2

= 0,
1 1 2
1
1 2 2
2
1 2 3

= 0.
Expanding the determinants we have
P0 + Q1 + R2 = 0,
P1 + Q2 + R3 = 0,
Also, evidently
P + Qi + R2
i = 0(i = 1, 2)
Therefore our conditions will all be consistent if 1, 2 are determined as the
roots, 1 Ã· 2, of
1
2

0 1 2
1 2 3
2
2 −12 2
1

= 0.
This latter determinant is evidently the Hessian of f, divided by 2. Thus the
complete reduction of f to its canonical form is accomplished by solving its
Hessian covariant for the roots 1, 2, and then solving the first two equations
of (1402) for p1, p2. The inverse of S will then transform f into X3 + Y 3. The
determinant of S is
D = (p1 Â· p2) 1
3 (2 − 1),
and D 6= 0 unless the Hessian has equal roots. Thus the necessary and sufficient
condition in order that the canonical reduction be possible is that the discriminant
of the Hessian (which is also the discriminant, R, of the cubic f) should

not vanish. If R = 0, a canonical form of f is evidently X^2Y .
Among the problems that can be solved by means of the canonical form are,
(a) the determination of the roots of the cubic f = 0 from

$$X^3 + Y^3 = (X + Y)(X + !Y)(X + !^2Y).$$

! being an imaginary cube root of unity, and (b) the determination of the syzygy among the concomitants of f. We now solve problem (b). From Table I, by substituting $_0 = _3 = 1$, $_1 = _2 = 0$, we have the fundamental system of the canonical form:

$$X^3 + Y^3,\ 2XY, X^3 - Y^3, -2.$$

Now we may regard the original form f to be the transformed form of $X^3 + Y^3$ under S. Hence, since the modulus of S is D, we have the four invariant relations

$$f = X^3 + Y^3,$$
$$ = 2D^2XY,$$
$$Q = D^3(X^3 - Y^3),$$
$$R = -D^6 Â· 2.$$

It is an easy process to eliminate D, X, Y from these four equations. The result is the required syzygy:

$$f^2R + 2Q^2 + ^3 = 0$$

A general binary quartic can be reduced to the canonical form (Cayley)

$$X^4 + Y^4 + 6mX^2Y^2;$$

a ternary cubic to the form (Hesse)

$$X^3 + Y^3 + Z^3 + 6mXYZ.$$

An elegant reduction of the binary quartic to its canonical form may be obtained by means of the provectant operators of Chapter III, Section 1, V. We observe that we are to have identically

$$f = (a_0, a_1, \ldots, a_4)(x_1, x_2)^4 = X_1^4 + X_2^4 + 6mX_1^2X_2^2,$$

where X_1, X_2 are linear in x_1, x_2;

$$X_1 = _1x_1 + _2x_2, X_2 = _1x_1 + _2x_2.$$

Let the quadratic X_1X_2 be $q = (A_0, A_1, A_2)(x_1, x_2)^2$. Then

$$@q Â· X_j^4 = (A_0, A_1, A_2)\left(@\over @x_2, -{@ \over @x_1}\right)^2 X_j^4 = 0\ (j = 1, 2).$$

$$6m@q Â· X_1^2X_2^2 = 12 Â· 2(4A_0A_2 - A_1^2)mX_1X_2 = 12X_1X_2.$$

Equating the coefficients of x_1^2, x_1x_2, x_2^2 in the first equation above, after operating on both sides by @q, we now have

$$A_0a_2 - A_1a_1 + A_2a_0 = A_0,$$
$$A_0a_3 - A_1a_2 + A_2a_1 = {1 \over 2}A_0,$$
$$A_0a_4 - A_1a_3 + A_2a_2 = A_2.$$

Forming the eliminant of these we have an equation which determines , and therefore m, in terms of the coefficients of the original quartic f. This eliminant is

a_0 a_1 a_2 −
a_1 a_2 + 1
2 a_3
a_2 − a_3 a_4

= 0,
or, after expanding it,
3 −
1
2i −
1
3J = 0,
where i, J are the invariants of the quartic f determined in Chapter III, Â§1, V. It follows that the proposed reduction of f to its canonical form can be made in three ways.

A problem which was studied by Sylvester,[2] the reduction of the binary sextic to the form
$X_1^6 + x_2^6 + X_3^6 + 30mX_1^2X_2^2X_3^2$,
has been completely solved very recently by E. K. Wakeford.[3]

4.6 Hilbert's Theorem

We shall now prove a very extraordinary theorem due to Hilbert on the reduction of systems of quantics, which is in many ways closely connected with the theory of syzygies. The proof here given is by Gordan. The original proof of Hilbert may be consulted in his memoir in the Mathematische Annalen, volume 36.

4.6.1 Theorem

Theorem. If a homogeneous algebraical function of any number of variables be formed according to any definite laws, then, although there may be an infinite number of functions F satisfying the conditions laid down, nevertheless a finite number $F_1, F_2, \ldots, F_r$ can always be found so that any other F can be written in the form
$$F = A_1F_1 + A_2F_2 + Â· Â· Â· + A_rF_r,$$
where the A' s are homogeneous integral functions of the variables but do not necessarily satisfy the conditions for the F's.

An illustration of the theorem is the particular theorem that the equation of any curve which passes through the intersections of two curves $F_1 = 0$, $F_2 = 0$ is of the form
$$F = A_1F_1 + A_2F_2 = 0.$$
Here the law according to which the F's are constructed is that the corresponding curve shall pass through the stated intersections. There are an infinite number of functions satisfying this law, all expressible as above, where A_1, A_2

[2] Cambridge and Dublin Mathematical Journal, vol. 6 (1851), p. 293.
[3] Messenger of Mathematics, vol. 43 (1913-14), p. 25.

are homogeneous in x_1, x_2, x_3 but do not, as a rule, represent curves passing through the intersections.

We first prove a lemma on monomials in n variables.

Lemma 1. If a monomial $x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n}$, where the k's are positive integers, be formed so that the exponents $k_1 \ldots, k_n$ satisfy prescribed conditions, then, although the number of products satisfying the given conditions may be infinite, nevertheless a finite number of them can be chosen so that every other is divisible by one at least of this finite number.

To first illustrate this lemma suppose that the prescribed conditions are

$$2k_1 + 3k_2 - k_3 - k_4 = 0, \quad k_1 + k_4 = k_2 + k_3. \tag{141}$$

Then monomials satisfying these conditions are

$$x_1^2 x_2^2 x_3^5 x_4^5,\ x_1^2 x_3^3 x_4,\ x_2 x_3 x_4^2,\ x_1^2 x_2 x_4^3 x_3^4,\ \ldots$$

and all are divisible by at least one of the set $x_1^2 x_3^3 x_4$, $x_2 x_3 x_4^2$.

Now if $n = 1$, the truth of the lemma is self-evident. For all of any set of positive powers of one variable are divisible by that power which has the least exponent. Proving by induction, assume that the lemma is true for monomials of $n - 1$ letters and prove it true for n letters.

Let $K = x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n}$ be a representative monomial of the set given by the prescribed conditions and let $P = x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}$ be a specific product of the set. If K is not divisible by P, one of the numbers k must be less than the corresponding number a. Let $k_r < a_r$. Then k_r has one of the series of values

$$0, 1, 2, \ldots, a_r - 1,$$

that is, the number of ways that this can occur for a single exponent is finite and equal to

$$N = a_1 + a_2 + \ldots + a_n.$$

The cases are

$$\begin{aligned} &k_1 \text{ equals one of the series } 0, 1, \cdots, a_1 - 1; \ (a_1 \text{ cases}),\\ &k_2 \text{ equals one of the series } 0, 1, \cdots, a_2 - 1; \ (a_2 \text{ cases}),\\ &\text{etc.}\end{aligned} \tag{142}$$

Now let $k_r = m$ and suppose this to be case number p of (142). Then the $n - 1$ remaining exponents $k_1, k_2, \cdots, k_{-1}, k_{+1}, \cdots, k_n$ satisfy definite conditions which could be obtained by making $k_r = m$ in the original conditions. Let

$$K_p = x_1^{k_1} x_2^{k_2} \cdots x^m \cdots x_n^{k_n} = x^m K'_p$$

be a monomial of the system for which $k_r = m$. Then K'_p contains only $m - 1$ letters and its exponents satisfy definite conditions which are such that $x^m K'_p$ satisfies the original conditions. Hence by hypothesis a finite number of monomials of the type K'_p, say,

$$L_1, L_2, \cdots, L_{a_p},$$

exist such that all monomials K'_p are divisible by at least one L. Hence $K_p = x^m K'_p$ is divisible by at least one L, and so by at least one of the monomials

$$M_p^{(1)} = x^m L_1, M_p^{(2)} = x^m L_2, \cdots, M_p^{(a_p)}$$

p = xm Lap .
Also all of the latter set of monomials belong to the original system. Thus in the case number p in (142) K is divisible by one of the monomials
M(1)
p ,M(2)
p , Â· Â· Â· ,M(ap)
p ,
Now suppose that K is not divisible by P. Then one of the cases (142) certainly arises and so K is always divisible by one of the products
M(1)
1 ,M(2)
1 , Â· Â· Â· ,M(1)
1 ,M(1)
2 ,M(2)
2 , Â· Â· Â· ,M(2)
2 , Â· Â· Â· ,M(N)
N ,
or else by P. Hence if the lemma holds true for monomials in n − 1 letters, it holds true for n letters, and is true universally.
We now proceed to the proof of the main theorem. Let the variables be
1, . . . , n and let F be a typical function of the system described in the theorem.
Construct an auxiliary system of functions of the same variables under the law that a function is an function when it can be written in the form
 = AF (143)
where the A's are integral functions rendering homogeneous, but not otherwise restricted except in that the number of terms in must be finite.
Evidently the class of functions is closed with respect to linear operations.
That is,
B = B11 + B22 + . . . = BAF = A0F
is also an function. Consider now a typical function. Let its terms be ordered in a normal order. The terms will be defined to be in normal order if the terms of any pair,
S = a1
1 a2
2 . . . an
n , T = b1
1 b2
2 . . . bn
n .
are ordered so that if the exponents a, b of S and T are read simultaneously from left to right the term first to show an exponent less than the exponent in the corresponding position in the other term occurs farthest to the right. If the normal order of S, T is (S, T), then T is said to be of lower rank than S. That is, the terms of are assumed to be arranged according to descending rank and there is a term of highest and one of lowest rank. By hypothesis the functions are formed according to definite laws, and hence their first terms satisfy definite laws relating to their exponents. By the lemma just proved we can choose a finite number of functions, 1, 2, . . . p such that the first term of any other
 is divisible by the first term of at least one of this number. Let the first term of a definite be divisible by the first term of m1 and let the quotient be P1.

Then − P1m1 is an function, and its first term is of lower rank than the first term of . Let this be denoted by
 = P1m1 + (1).
Suppose next that the first term of (1) is divisible by m2 ; thus,
(1) = P2m2 + (2),
and the first term of (2) is of lower rank than that of (1). Continuing, we obtain
(r−1) = Prmr + (r).

Then the first terms of the ordered set
, (1), (2), . . . , (r), . . .
are in normal order, and since there is a term of lowest rank in we must have for some value of r
$(r) = P_{r+1}m_{r+1}$.
That is, we must eventually reach a point where there is no function (r+1) of the same order as and whose first term is of lower rank than the first term of (r). Hence

$$= P_1m_1 + P_2m_2 + . . . + P_{r+1}m_{r+1} \quad (144)$$

and all 's on the right-hand side are members of a definite finite set
1, 2, . . . , p.
But by the original theorem and (143), every F is itself an function. Hence by (144)

$$F = A_1F_1 + A_2F_2 + . . . + A_rF_r \quad (145)$$

where $F_i(i = 1, . . . , r)$ are the F functions involved linearly in 1, 2, . . . , p.
This proves the theorem.

4.6.2 Linear Diophantine equations.

If the conditions imposed upon the exponents k consist of a set of linear Diophantine equations like (141), the lemma proved above shows that there exists a set of solutions finite in number by means of which any other solution can be reduced. That is, this fact follows as an evident corollary.

Let us treat this question in somewhat fuller detail by a direct analysis of the solutions of equations (141). The second member of this pair has the solutions

	k_1,	k_2,	k_3,	k_4,
(1)	0	0	1	1
(2)	0	1	0	1
(3)	1	0	1	0
(4)	1	1	0	0
(5)	2	1	1	0
(6)	0	0	1	1

. . . .

Of these the fifth is obtained by adding the first and the fourth; the sixth is reducible as the sum of the third and the fourth, and so on. The sum or difference of any two solutions of any such linear Diophantine equation is evidently again a solution. Thus solutions (1), (2), (3), (4) of $k_1+k_4 = k_2+k_3$ form the complete set of irreducible solutions. Moreover, combining these, we see at once that the general solution is

(I) $k_1 = x + y, k_2 = x + z, k_3 = y + w, k_4 = z + w.$

Now substitute these values in the first equation of (141)

$$2k_1 + 3k_2 - 3k_3 - k_4 = 0.$$

There results

$$5x + y + 2z = 2w$$

By the trial method illustrated above we find that the irreducible solutions of the latter are

$x = 2, w = 5,\ y = 2, w = 1;\ z = 1, w = 1;\ x = 1, y = 1, w = 3,$

where the letters not occurring are understood to be zero. The general solution is here

(II) $x = 2a + d, y = 2b + d, z = c, w = 5a + b + c + 3d,$

and if these be substituted in (I) we have

$$k_1 = 2a + 2b + 2d\ k_2 = 2a + c + d\ k_3 = 5a + 3b + c + 4d\ k_4 = 5a + b + 2c + 3d$$

Therefore the only possible irreducible simultaneous solutions of (141) are

	k_1,	k_2,	k_3,	k_4
(1)	2	2	5	5
(2)	2	0	3	1
(3)	0	1	1	2
(4)	2	1	4	3

But the first is the sum of solutions (3) and (4); and (4) is the sum of (2) and

(3). Hence (2) and (3) form the complete set of irreducible solutions referred to in the corollary. The general solution of the pair is

$k_1 = 2, k_2 = , k_3 = 3, k_4 = + 2.$

The corollary may now be stated thus:

Corollary 1. Every simultaneous set of linear homogeneous Diophantine equations possesses a set of irreducible solutions, finite in number.

A direct proof without reference to the present lemma is not difficult.[4] Applied to the given illustration of the above lemma on monomials the above analysis shows that if the prescribed conditions on the exponents are given by (141) then the complete system of monomials is given by

x2
1 x62
x3+
3 x+2
4 ,

where and range through all positive integral values independently. Every monomial of the system is divisible by at least one of the set

x21
x33
x4, x2x3x24
,

which corresponds to the irreducible solutions of the pair (141).

4.6.3 Finiteness of a system of syzygies.

A syzygy S among the members of a fundamental system of concomitants of a form (cf. (140)) f,

$I_1, I_2, \ldots, I_\mu$

is a polynomial in the I's formed according to the law that it will vanish identically when the I's are expressed explicitly in terms of the coefficients and variables of f. The totality of syzygies, therefore, is a system of polynomials (in the invariants I) to which Hilbert's theorem applies. It therefore follows at once that there exists a finite number of syzygies,

$S_1, S_2, \ldots, S$

such that any other syzygy S is expressible in the form

$S = C_1S_1 + C_2S_2 + \cdots + CS$ (146)

Moreover the C's, being also polynomials in the I's are themselves invariants of f. Hence

Theorem. The number of irreducible syzygies among the concomitants of a form f is finite, in the sense indicated by equation (146).

[4] Elliott, Algebra of Qualities, Chapter IX.

4.7 Jordan's Lemma

Many reduction problems in the theory of forms depend for their solution upon a lemma due to Jordan which may be stated as follows:

Lemma 2. If $u_1+u_2+u_3 = 0$, then any product of powers of u_1, u_2, u_3 of order n can be expressed linearly in terms of such products as contain one exponent equal to or greater than $\frac{2}{3}n$.

We shall obtain this result as a special case of a considerably more general result embodied in a theorem on the representation of a binary form in terms of other binary forms.

Theorem. If $a_x, b_x, c_x, \ldots$ are r distinct linear forms, and $A, B, C, \ldots$ are binary forms of the respective orders , , , . . . where

$+ + + \ldots = n - r + 1$

then any binary form f of order n can be expressed in the form

$f = a_x^{n-} A + b_x^{n-} B + C_x^{n-} C + \ldots,$

and the expression is unique.

As an explicit illustration of this theorem we cite the case $n = 3$, $r = 2$.
Then + = 2, = = 1.

$f = a2$
$x(p_{00}x_1 + p_{01}x_2) + b2$
$x(p_{10}x_1 + p_{11}x_2)$ (147)

Since f, a binary cubic, contains four coefficients it is evident that this relation (147) gives four linear nonhomogeneous equations for the determination of the four unknowns p_{00}, p_{01}, p_{10}, p_{11}. Thus the theorem is true for this case provided the determinant representing the consistency of these linear equations does not vanish. Let $a_x = a_1x_1 + a_2x_2$, $b_x = b_1x_1 + b_2x_2$, and $D = a_1b_2 - a_2b_1$. Then the aforesaid determinant is

a21
0 b21
0
2a1a2 a21
2b1b2 b21
a22
2a1a2 b22
2b1b2
0 a22
0 b22 This equals D4, and D 6= O on account of the hypothesis that a_x and b_x are distinct. Hence the theorem is here true. In addition to this we can solve for the py and thus determine A,B explicitly. In the general case the number of unknown coefficients on the right is

 + + + . . . + r = n + 1

Hence the theorem itself may be proved by constructing the corresponding consistency determinant in the general case;[5] but it is perhaps more instructive to proceed as follows:

[5] Cf. Transactions Amer. Math. Society, Vol. 15 (1914), p. 80.

It is impossible to find r binary forms A,B,C, . . . of orders , , , , . . . where

 + + + . . . = n − r + 1,

such that, identically,

E = an−
x A + bn−
x B + cn−
x C + . . . 0.

In fact suppose that such an identity exists. Then operate upon both sides of this relation + 1 times with

 = a2
@
@x1 − a1
@
@x2
($a_x = a_1x_1 + a_2x_2$).

Let gn
x be any form of order n and take $a_2 = 0$. Then
+1gn
x = k(a1 Â· g2)+1gn−−a
x
= k1a+1
1 gn−−1
1 g+1
2 xn−−1 − 2 + k2a+1
1 gn−−2
1 g+2
2 xn−−2
2

+ . . . + kn−a+1
1 gn
2 xn−−1
2 ,
where the k's are numerical. Hence +1gn
x cannot vanish identically in case
a2 = 0, and therefore not in the general case a2 6= 0, except when the last n−
coefficients of gn
x vanish: that is, unless gn
x contains an−
x as a factor. Hence
+1E = bn−−−1
x B0 + cn−−−1
x C0 + . . . ,
where B0,C0 are of orders , , . . . respectively. Now +1E is an expression of
the same type as E, with r changed into r−1 and n into n−−1, as is verified
by the equation
+ + . . . = (n − − 1) − (r − 1) + 1 = n − r + 1 −
Thus if there is no such relation as E 0 for r − 1 linear forms ax, bx, . . . ,
there certainly are none for r linear forms. But there is no relation for one
form (r = 1) save in the vacuous case (naturally excluded) where A vanishes
identically. Hence by induction the theorem is true for all values of r.
Now a count of coefficients shows at once that any binary form f of order
n can be expressed linearly in terms of n + 1 binary forms of the same order.
Hence f is expressible in the form
f = an−
x A + bn−
x B + cn−
x D + Â· Â· Â· .
That the expression is unique is evident. For if two such were possible, their
difference would be an identically vanishing expression of the type E 0, and,
as just proved, none such exist. This proves the theorem.

4.7.1 Jordan's lemma.
Proceeding to the proof of the lemma, let u3 = −(u1 + u2), supposing that
u1, u2 replace the variables in the Theorem I just proved. Then u3, u1, u2 are
three linear forms and the Theorem I applies with r = 3, + + = n − 2.
Hence any homogeneous expression f in u1, u2, u3 can be expressed in the form
un−
1 A + un−
2 B + un−
3 C,
or, if we make the interchanges
n − n − n −
µ
in the form
u
1 A + uµ
2B + u
3 C, (148)
where
+ µ + = 2n + 2. (149)
Again integers , µ, may always be chosen such that (149) is satisfied and
=
2
3n, µ =
2
3n, =
2

$\frac{2}{3}n$.

Hence Jordan's lemma is proved.

A case of three linear forms u_i for which $u_1 + u_2 + u_3 = 0$ is furnished by the identity

$$(ab)c_x + (bc)a_x + (ca)b_x = 0.$$

If we express A in (148) in terms of u_1, u_2 by means of $u_1 + u_2 + u_3 = 0$, B in terms of u_2, u_3, and C in terms of u_3, u_1, we have the conclusion that any product of order n of $(ab)c_x$, $(bc)a_x$, $(ca)b_x$ can be expressed linearly in terms of

$(ab)^n c^n_x$, $(ab)^{n-1}(bc)c^{n-1}_x a_x$, $(ab)^{n-2}(bc)^2 c^{n-2}_x a^2_x$, . . . , $(ab)(bc)^{n-} c_x a^{n-}_x$,

$(bc)^n a^n_x$, $(bc)^{n-1}(ca)a^{n-1}_x b_x$, $(bc)^{n-2}(ca)^2 a^{n-2}_x b^2_x$, . . . , $(bc)^\mu(ca)^{n-\mu} a^\mu_x b^{n-\mu}_x$ (150)

$(ca)^n b^n_x$, $(ca)^{n-1}(ab)b^{n-1}_x c_x$, $(ca)^{n-2}(ab)^2 b^{n-2}_x c^2_x$, . . . , $(ca)(ab)^{n-} b_x c^{n-}_x$,

where

$= \frac{2}{3}n$, $\mu = \frac{2}{3}n$, $= \frac{2}{3}n$.

It should be carefully noted for future reference that this monomial of order n in the three expressions $(ab)c_x$, $(bc)a_x$, $(ca)b_x$ is thus expressed linearly in terms of symbolical products in which there is always present a power of a determinant of type (ab) equal to or greater than $\frac{2}{3}n$. The weight of the coefficient of the

leading term of a covariant is equal to the number of determinant factors of the type (ab) in its symbolical expression. Therefore (150) shows that if this weight w of a covariant of f does not exceed the order of the form f all covariants having leading coefficients of weight w and degree 3 can be expressed linearly in terms of those of grade not less than $\frac{2}{3}w$. The same conclusion is easily shown to hold for covariants of arbitrary weight.

4.8 Grade

The process of finding fundamental systems by passing step by step from those members of one degree to those of the next higher degree, illustrated in Section 3 of this chapter, although capable of being applied successfully to the forms of the first four orders fails for the higher orders on account of its complexity. In fact the fundamental system of the quintic contains an invariant of degree 18 and consequently there would be at least eighteen successive steps in the process. As a proof of the finiteness of the fundamental system of a form of order n the process fails for the same reason. That is, it is impossible to tell whether the system will be found after a finite number of steps or not.

In the next chapter we shall develop an analogous process in which it is proved that the fundamental system will result after a finite number of steps. This is a process of passing from the members of a given grade to those of the next higher grade.

4.8.1 Definition.

The highest index of any determinant factor of the type (ab) in a monomial symbolical concomitant is called the grade of that concomitant. Thus (ab)4(ac)2b2
xc4
x
is of grade 4. The terms of covariants (84), (87) are each of grade 2.

Whereas there is no upper limit to the degree of a concomitant of a form f order n, it is evident that the maximum grade is n by the theory of the Aronhold symbolism. Hence if we can find a method of passing from all members of the fundamental system of f of one grade to all those of the next higher grade, this will prove the finiteness of the system, since there would only be a finite number of steps in this process. This is the plan of the proof of Gordan's theorem in the next chapter.

4.8.2 Grade of a covariant.

Theorem. Every covariant of a single form f of odd grade 2 − 1 can be transformed into an equivalent covariant of the next higher even grade 2.

We prove, more explicitly, that if a symbolical product contains a factor (ab)2−1 it can be transformed so as to be expressed in terms of products each

containing the factor (ab)2. Let A be the product. Then by the principles of Section 2A is a term of
((ab)2−1an+1−2
x bn+1−2
x ,).

Hence by Theorem III of Section 2.

A = ((ab)2−1an+1−2
x bn+1−2
x ,) +XK(((ab)2−1a Â¯ n+1−2
x bn+1−2
x , Â¯)0 ,
(151)

where 0 < and Â¯ is a concomitant derived from by convolution, K being numerical. Now the symbols are equivalent. Hence
= (ab)2−1an+1−2
x bn+1−2
x = −(ab)2−1an+1−2
x bn+1−2
x = 0.

Hence all transvectants on the right-hand side of (151), in which no convolution in occurs, vanish. All remaining terms contain the symbolical factor (ab)2, which was to be proved.

DEFINITION. A terminology borrowed from the theory of numbers will now be introduced. A symbolical product, A, which contains the factor (ab)r is said to be congruent to zero modulo (ab)r;
A 0(mod (ab)r).

Thus the covariant (84)
C =
1
3
(ab)2(ba)2a2
xx +
2

3
$(ab)^2(a\eta)(b\eta)a_x b_x x$

gives

C
2
3
$(ab)^2(a\eta)(b\eta)a_x b_x x \pmod{(b\eta)^2}$.

4.8.3 Covariant congruent to one of its terms

Theorem. Every covariant of $f = a^n_x = b^n_x = \ldots$ which is obtainable as a covariant of $(f, f)^{2k} = g^{2n-4k}_{1x} = (ab)^{2k}a^{n-2k}_x b^{n-2k}_x$ (Chap. II, Â§4) is congruent to any definite one of its own terms modulo $(ab)^{2k+1}$.

The form of such a concomitant monomial in the g symbols is

$A = (g_1g_2)^p(g_1g_3)^q \ldots g_{1x} g_{2x} \ldots$

Proceeding by the method of Section 2 of this chapter change g_1 into y; i.e. $g_{11} = y_2$, $g_{12} = -y_1$. Then A becomes a form of order $2n-4k$ in y, viz. $\alpha^{2n-4k}_y = \beta^{2n-4k}_y = \ldots$. Moreover

$A = (\alpha^{2n-4k}_y, g^{2n-4k}_{1y})^{2n-4k} = (\alpha^{2n-4k}_y, (ab)^{2k}a^{n-2k}_y b^{n-2k}_y)^{2n-4k}$,

by the standard method of transvection. Now this transvectant A is free from y. Hence there are among its terms expressed in the symbols of f only two types of adjacent terms, viz. (cf. Â§2, II)

$(da)(eb)P, (db)(ea)P$.

The difference between A and one of its terms can therefore be arranged as a succession of differences of adjacent terms of these two types and since P involves $(ab)^{2k}$ any such difference is congruent to zero modulo $(ab)^{2k+1}$, which proves the theorem.

4.8.4 Representation of a covariant of a covariant.

Theorem. If $n = 4k$, any covariant of the covariant

$g^{2n-4k}_x = (ab)^{2k}a^{n-2k}_x b^{n-2k}_x$

is expressible in the form

$XC_{2k+1} + (ab)^{\frac{n}{2}}(bc)^{\frac{n}{2}}(ca)^{\frac{n}{2}}$, (152)

where C_{2k+1} represents a covariant of grade $2k + 1$ at least, the second term being absent (= 0) if n is odd.

Every covariant of g^{2n-4k}_x of a stated degree is expressible as a linear combination of transvectants of g^{2n-4k}_x with covariants of the next lower degree (cf. Â§2, III). Hence the theorem will be true if proved for $T = (g^{2n-4k}_x, g^{2n-4k}$

x)
the covariants of second degree of this form. By the foregoing theorem T is congruent to any one of its own terms mod (ab)2k+1. Hence if we prove the present theorem for a stated term of T, the conclusion will follow. In order to select a term from T we first find T by the standard transvection process (cf. Chap. III, Â§2). We have after writing s = n − 2k for brevity, and as
xbs
x = 2s
x
T = (ab)2k(cd)2k
 Xt=0   s
t   s
      −t 2s
  cs−t
x ds−+t
x Â· (c)t(d)−t2t−
x . (153)
Now the terms of this expression involving may be obtained by polarizing
2s
x t times with respect to y, −t times with respect to z, and changing y into c and z into d. Performing these operations upon as
xbs
x we obtain for T,
T =
 Xt=0
t Xu=0
−t Xv=0
Ktuv(ab)2k(cd)2k(ac)u(ad)v(bc)t−u(bd)−t−v
Ã—as−u−v
x bs−+u+v
x cs−t
x ds−+t
x , (154)
where Ktuv is numerical. Evidently is even.
We select as a representative term the one for which t = , u = v = 0.

This is
 = (ab)2k(bc)(cd)2kan−2k
x bn−2k−
x cn−2k−
x dn−2k
x .
Assume n = 4k. Then by Section 6,
  = (ab)2k(bc)(ca)2kan−4k
x bn−2k−
x cn−2k−
x
can be expressed in terms of covariants whose grade is greater than 2k unless
 = 2k = n
2 . Also in the latter case   is the invariant
  = (ab)n2
(bc)n2
(ca)n2
.
It will be seen at once that n must then be divisible by 4. Next we transform by (cd)ax = (ad)cx − (ac)dx. The result is
0 =
2k Xi=0 2
i (ab)2k(bc)(ca)i(ad)2k−ian−4k

x bn−2k−
x cn−−i
x dn−2k+i
x .
(I) Now if > k, we have from Section 6 that is of grade > 2
3 Â· 3k, i.e.
> 2k, or else contains (ab)n2
(bc)n2
(ca)n2
i.e.
=XC2k+1 + (ab)n2
(bc)n2
(ca)n2
T. (155)
(II) Suppose then leqqk. Then in 0, since i = 2k has been treated under
above, we have either
(a)i = k,
or
(b)2k − i > k.
In case (a) (155) follows directly from Section 6. In case (b) the same conclusion follows from the argument in (I). Hence the theorem is proved.

Chapter 5

GORDAN'S THEOREM

We are now in position to prove the celebrated theorem that every concomitant of a binary form f is expressible as a rational and integral algebraical function of a definite finite set of the concomitants of f. Gordan was the first to accomplish the proof of this theorem (1868), and for this reason it has been called Gordan's theorem. Unsuccessful attempts to prove the theorem had been made before Gordan's proof was announced.

The sequence of introductory lemmas, which are proved below, is that which was first given by Gordan in his third proof (cf. Vorlesungen uber Invariantentheorie, Vol. 2, part 3). 1 The proof of the theorem itself is somewhat simpler than the original proof. This simplification has been accomplished by the theorems resulting from Jordan's lemma, given in the preceding chapter.

5.1 Proof of the Theorem

We proceed to the proof of a series of introductory lemmas followed by the finiteness proof.

5.1.1 Lemma

Lemma 3. If (A) : A1,A2, . . . ,Ak is a system of binary forms of respective orders a1, a2, . . . , ak, and (B) : B1,B2, . . . ,Bl, a system of respective orders by b1, b2, . . . , bl, and if
= A1
1 A2
2 . . .Ak
k , = B1
1 B2
2 . . .Bl
l
denote any two products for which the a's and the 's are all positive integers (or zero), then the number of transvectants of the type of
= (,)j
which do not contain reducible terms is finite.

1Cf. Grace and Young; Algebra of Invariants (1903).

To prove this, assume that any term of contains symbols of the forms A not in second order determinant combinations with a symbol of the B forms, and symbols of the B's not in combination with a symbol of the A's. Then evidently we have for the total number of symbols in this term, from (A) and (B) respectively,

$a_{11} + a_{22} + . . . + a_{kk} = + j,$

$b_{11} + b_{22} + . . . + b_{ll} = + j.$ (156)

To each positive integral solution of the equations (156), considered as equations in the quantities , , , , j, will correspond definite products, and a definite index j, and hence a definite transvectant . But as was proved (Chap. IV, Section 3, III), if the solution corresponding to (,)j is the sum of those corresponding to (1, 1)j1 and (2, 2)j2 , then certainly contains reducible terms. In other words transvectants corresponding to reducible solutions contain reducible terms. But the number of irreducible solutions of (156) is finite (Chap. IV, Section 5, II). Hence the number of transvectants of the type which do not contain reducible terms is finite. A method of finding the irreducible transvectants was given in Section 3, III of the preceding chapter.

Definitions.

A system of forms (A) is said to be complete when any expression derived by convolution from a product of powers of the forms (A) is itself a rational integral function of the forms (A).

A system (A) will be called relatively complete for the modulus G consisting of the product of a number of symbolical determinants when any expression derived by convolution from a product is a rational integral function of the forms (A) together with terms containing G as a factor.

As an illustration of these definitions we may observe that

$f = a^3$

x = Â· Â· Â· , = $(ab)^2 x_x b_x$,Q = $(ab)^2(ca)b_x c^2$

x,

$R = (ab)^2(cd)^2(ac)(bd)$

is a complete system. For it is the fundamental system of a cubic f, and hence any expression derived by convolution from a product of powers of these four concomitants is a rational integral function of f,, Q,R.

Again f itself forms a system relatively complete modulo $(ab)^2$.

Definition.

A system (A) is said to be relatively complete for the set of moduli G_1,G_2, Â· Â· Â· when any expression derived from a product of powers of A forms by convolution is a rational integral function of A forms together with terms containing at least one of the moduli G_1,G_2, Â· Â· Â· as a factor.

5.1. PROOF OF THE THEOREM 117

In illustration it can be proved (cf. Chap. IV, Â§7, IV) that in the complete system derived for the quartic

$H = (ab)^2a^2$

xb^2

x,

any expression derived by convolution from a power of H is rational and integral in H and

$G_1 = (ab)^4,G_2 = (bc)^2(ca)^2(ab)^2.$

Thus H is a system which is relatively complete with regard to the two moduli

$G_1 = (ab)^4,G_2 = (bc)^2(ca)^2(ab)^2.$

Evidently a complete system is also relatively complete for any set of moduli. We call such a system absolutely complete.

Definitions.

The system (C) derived by transvection from the systems (A), (B) contains an infinite number of forms. Nevertheless (C) is called a finite system when all its members are expressible as rational integral algebraic functions of a finite number of them.

The system (C) is called relatively finite with respect to a set of moduli G_1,G_2, . . . when every form of (C) is expressible as a rational integral algebraic function of a finite number of the forms (C) together with terms containing at least one of the moduli G_1,G_2, . . . as a factor.

The system of all concomitants of a cubic f is absolutely finite, since every

concomitant is expressible rationally and integrally in terms of f,, Q,R.
Lemma 4. If the systems (A), (B) are both finite and complete, then the system (C) derived from them by transvection is finite and complete.
We first prove that the system (C) is finite. Let us first arrange the transvectants
= (,)j
in an ordered array
1, 2, Â· Â· Â· , r, Â· Â· Â· , (157)
the process of ordering being defined as follows:
(a) Transvectants are arranged in order of ascending total degree of the product in the coefficients of the forms in the two systems (A), (B).
(b) Transvectants for which the total degree is the same are arranged in order of ascending indices j; and further than this the order is immaterial.
Now let t, t0 be any two terms of . Then
(t − t0) = (Â¯ , Â¯)j0 (j0 < j),

where Â¯ is a form derived by convolution from . But by hypothesis (A), (B) are complete systems. Hence Â¯ , Â¯ are rational and integral in the forms A, B respectively,
Â¯ = F(A), Â¯ = G(B).
Therefore (Â¯ , Â¯)j can be expressed in terms of transvectants of the type (i.e. belonging to (C)) of index less than j and hence coming before in the ordered array (157). But if we assume that the forms of (C) derived from all transvectants before can be expressed rationally and integrally in terms of a finite number of the forms of (C),
C1,C2, . . . ,Cr,
then all C's up to and including those derived from
= (,)j
can be expressed in terms of
C1,C2, . . . ,Cr, t.
But if contains a reducible term t = t1t2 then since t1, t2 must both arise from transvectants before in the ordered array no term t need be added and all C's up to and including those derived from are expressible in terms of
C1,C2, . . . ,Cr.
Thus in building by this procedure a system of C's in terms of which all forms of (C) can be expressed we need to add a new member only when we come to a transvectant in (157) which contains no reducible term. But the number of such transvectants in (C) is finite. Hence, a finite number of C's can be chosen such that every other is a rational function of these.
The proof that (C) is finite is now finished, but we may note that a set of C's in terms of which all others are expressible may be chosen in various ways, since t in the above is any term of . Moreover since the difference between any two terms of is expressible in terms of transvectants before in the ordered array we may choose instead of a single term t of an irreducible = (,)j , an aggregate of any number of terms or even the whole transvectant and it will remain true that every form of (C) can be expressed as a rational integral algebraic function of the members of the finite system so chosen.
We next prove that the finite system constructed as above is complete.
Let
C1,C2, . . . ,Cr
be the finite system. Then we are to prove that any expression Â¯X derived by convolution from
X = C1
1 C2
2 . . .Cr
r

is a rational integral algebraic function of C1, . . . ,Cr. Assume that Â¯X contains second-order determinant factors in which a symbol from an (A) form is in combination with a symbol belonging to a (B) form.

Then Â¯X is a term of a transvectant (Â¯ , Â¯), where Â¯ contains symbols from system (A) only, and Â¯ contains symbols from (B) only. Then Â¯ must be derivable by convolution from a product of the A's and Â¯ from a product of B forms. Moreover

˙Â¯X = (Â¯ , Â¯) +X(Â¯ , Â¯)0 (0 <),

and Â¯ , Â¯ having been derived by convolution from Â¯ , Â¯ , respectively, are ultimately so derivable from , . But

Â¯ = F(A), Â¯ = G(B)

and so Â¯X is expressed as an aggregate of transvectants of the type of

= (,)i.

But it was proved above that every term of is a rational integral function of C1, . . . ,Cr.

Hence Â¯X is such a function; which was to be proved.

5.1.2 Lemma

Lemma 5. If a finite system of forms (A), all the members of which are covariants of a binary form f, includes f and is relatively complete for the modulus G0; and if, in addition, a finite system (B) is relatively complete for the modulus G and includes one form B1 whose only determinantal factors are those constituting G0, then the system (C) derived by transvection from (A) and (B) is relatively finite and complete for the modulus G.

In order to illustrate this lemma before proving it let (A) consist of one form

f = a3

x = Â· Â· Â· , and (B) of two forms

= (ab)2axbx,R = (ab)2(ac)(bd)(cd)2.

Then (A) is relatively complete for the modulus G0 = (ab)2. Also B is absolutely complete, for it is the fundamental system of the Hessian of f. Hence the lemma states that (C) should be absolutely complete. This is obvious. For (C) consists of the fundamental system of the cubic,

f,, Q,R,

and other covariants of f.

We divide the proof of the lemma into two parts.

Part 1. First, we prove the fact that if P be an expression derived by convolution from a power of f, then any term, t, of = (P,)i can be expressed as an aggregate of transvectants of the type

= (,)i,

in which the degree of is at most equal to the degree of P. Here and are products of powers of forms (A), (B) respectively, and by the statement of the lemma (A) contains only covariants of f and includes f itself.

This fact is evident when the degree of P is zero. To establish an inductive proof we assume it true when the degree of P is < r and note that

t = (P,)i + (Â¯ P, Â¯)i0 (i0 < i),

and, inasmuch as P and Â¯ P are derived by convolution from a power of f,

P = F(A) + G0Y F(A) (modG0),

Â¯ P = F0(A) + G0Y 0 F0(A) (modG0).

Also

Â¯ = (B) + GZ (B)(mod G).

Hence t contains terms of three types (a), (b), (c).

(a) Transvectants of the type (F(A), (B))i, the degree of F(A) being r, the degree of P.

(b) Transvectants of type (G0Y,)k, G0Y being of the same degree as P.

(c) Terms congruent to zero modulo G.

Now for (a) the fact to be proved is obvious. For (b), we note that G0Y can be derived by convolution from B1fs, where s < r. Hence any term of (G0Y,)k can be derived by convolution from B1fs and is expressible in the form

X(P0,B1),

where P0 is derived by convolution from fs and is of degree < r. But by hypothesis every term in these latter transvectants is expressible as an aggregate

X(,)i(modulo G),

inasmuch as
B1 (B)(modulo G).
But in this (,)i is of degree 5 s < r. Hence
t X(,)i(mod G),
and the desired inductive proof is established.
As a corollary to the fact just proved we note that if P contain the factor
G0, then any term in
(P,)i

can be expressed in the form
X(,)i (158)
where the degree of is less than that of P.
Part 2. We now present the second part of the proof of the original lemma,
and first to prove that (C) is relatively finite modulo G.
We postulate that the transvectants of the system (C) are arranged in an
ordered array defined as follows by (a), (b), (c).
(a) The transvectants of (C) shall be arranged in order of ascending degree
of , assuming the transvectants to be of the type = (,)j .
(b) Those for which the degree of is the same shall be arranged in order
of ascending degree of .
(c) Transvectants for which both degrees are the same shall be arranged in
order of ascending index j; and further than this the ordering is immaterial.
Let t, t0 be any two terms of . Then
t0 − t = (,)j(j0 < j).
Also by the hypotheses of the lemma
= F(A) + G0Y, = (B) + GZ.
Hence
t0 − t (F(A), (B))f + (G0Y, (B))f (modG).
Now transvectants of the type (F(A), (B))f belong before in the ordered
array since j0 < j and the degree of F(A) is the same as that of . Again
(G0Y, (B))f can by the above corollary (158) be expressed in the form
X(0, 0)j ,
where the degree of 0 is less than that of G0Y and hence less than that of .
Consequently, t0 − t can be written
t0 − t =X(00, 00)j +X(0, 0)j(mod G),
where the degree of 00 is the same as that of and where j0/ltj, and where
the degree of 0 is less than that of . Therefore if all terms of transvectants
coming before
= (,)i
in the ordered array are expressible rationally and integrally in terms of
C1,C2, . . . ,Cq,
except for terms congruent to zero modulo G, then all terms of transvectants
up to and including can be so expressed in terms of
C1,C2, . . . ,Cq, t,

where t is any term of . As in the proof of lemma 2, if contains a reducible
term t = t1t2, t does not need to be added to
C1,C2, . . . ,Cq,
since then t1, t2 are terms of transvectants coming before /tau in the ordered
array. Hence, in building up the system of C's in terms of which all forms of
(C) are rationally expressible modulo G, by proceeding from one transvectant
to the next in the array, we add a new member to the system only when we
come to a transvectant containing no reducible term. But the number of such
irreducible transvectants in (C) is finite. Hence (C) is relatively finite modulo G.
Note that C1, . . . ,Cq may be chosen by selecting one term from each irreducible
transvectant in (C).
Finally we prove that (C) is relatively complete modulo G. Any term X
derived by convolution from
X = C1

1 C

2 . . .Cq
q ,
is a term of a transvectant (,), where, as previously, Â¯ is derived by convolution from a product of A forms and from a product of B forms. Then
X = (,) + (,)00 < .
That is, X is an aggregate of transvectants (,), = P can be derived by convolution from a power of f, and
(B) (mod G).
Thus,
Â¨X
X(P, (B)) (mod G)
X(P,) (mod G)
X(,)i (mod G)
where is of degree not greater than the degree of P, by the first part of the proof. But all transvectants of the last type are expressible as rational integral functions of a finite number of C's modulo G. Hence the system (C) is relatively complete, as well as finite, modulo G.

5.1.3 Corollary

Corollary 2. If the system (B) is absolutely complete then (C) is absolutely complete.

Corollary 3. If (B) is relatively complete for two moduli G1, G2 and contains a form whose only determinantal factors are those constituting G0, then the system (C) is relatively complete for the two moduli G1, G2.

5.1.4 Theorem

Theorem. The system of all concomitants of a binary form f = anx
= . . . of
order n is finite.

The proof of this theorem can now be readily accomplished in view of the theorems in Paragraphs III, IV of Chapter IV, Section 7, and lemma 3 just proved.

The system consisting of f itself is relatively complete modulo (ab)2. It is a finite system also, and hence it satisfies the hypotheses regarding (A) in lemma 3. This system (A) = f may then be used to start an inductive proof concerning systems satisfying lemma 3. That is we assume that we know a finite system Ak−1 which consists entirely of covariants of f, which includes f, and which is relatively complete modulo (ab)2k. Since every covariant of f can be derived from f by convolution it is a rational integral function of the forms in Ak−1 except for terms involving the factor (ab)2k. We then seek to construct a subsidiary finite system Bk−1 which includes one form B1 whose only determinant factors are (ab)2k = G0, and which is relatively complete modulo (ab)2k+2 = G. Then the system derived by transvection from Ak−1 and Bk−1 will be relatively finite and complete modulo (ab)2k+2. That is, it will be the system Ak. This procedure, then, will establish completely an inductive process by which we construct the system concomitants of f relatively finite and complete modulo (ab)2k+2 from the set finite and complete modulo (ab)2k, and since the maximum grade is n we obtain by a finite number of steps an absolutely finite and complete system of concomitants of f. Thus the finiteness of the system of all concomitants of f will be proved.

Now in view of the theorems quoted above the subsidiary system Bk−1 is easily constructed, and is comparatively simple. We select for the form B1 of the lemma
B1 = (ab)2kan−2k
x bn−2k
x = hx.

Next we set apart for separate consideration the case (c) n = 4k. The remaining cases are (a) n > 4k, and (b) n < 4k.

(a) By Theorem IV of Section 7 in the preceding chapter if $n > 4k$ any form derived by convolution from a power of h_k is of grade $2k+1$ at least and hence can be transformed so as to be of grade $2k + 2$ (Chap. IV, §7, II). Hence h_k itself forms a system which is relatively finite and complete modulo $(ab)^{2k+2}$ and is the system B_{k-1} required.

(b) If $n < 4k$ then h_k is of order less than n. But in the problem of constructing fundamental systems we may proceed from the forms of lower degree to those of the higher. Hence we may assume that the fundamental system of any form of order $< n$ is known. Hence in this case (b) we know the fundamental system of h_k. But by III of Chapter IV, Section 7 any concomitant of h_k is congruent to any one of this concomitant's own terms modulo $(ab)^{2k+1}$. Hence if we select one term from each member of the known fundamental system of h_k we have a system which is relatively finite and complete modulo $(ab)^{2k+2}$; that is, the required system B_{k-1}.

(c) Next consider the case $n = 4k$. Here by Section 7, IV of the preceding chapter the system $B_{k-1} = h_k$ is relatively finite and complete with respect to two moduli

$$G_1 = (ab)^{2k+2}, G_2 = (ab)^{2k}(bc)^{2k}(cd)^{2k},$$

and G_2 is an invariant of f. Thus by corollary 2 of lemma 3 the system, as C_k derived by transvection from A_{k-1} and B_{k-1} is relatively finite and complete with respect to the two moduli G_1, G_2. Hence, if $\bar{C}_k$ represents any form of the system C_k obtained from a form of C_k by convolution,

$$\bar{C}_k = F_1(C_k) + G_2P_1 \pmod{(ab)^{2k+2}}.$$

Here P_1 is a covariant of degree less than the degree of $\bar{C}_k$. Hence P_1 may be derived by convolution from f, and so

$$P_1 = F_2(C_k) + G_2P_2 \pmod{(ab)^{2k+2}},$$

and then P_2 is a covariant of degree less than the degree of P_1. By repetitions of this process we finally express $\bar{C}_k$ as a polynomial in

$$G_2 = (ab)^{2k}(bc)^{2k}(ca)^{2k},$$

whose coefficients are all covariants of f belonging to C_k, together with terms containing $G_1 = (ab)^{2k+2}$ as a factor, i.e.

$$\bar{C}_k \equiv F_1(C_k) + G_2F_2(C_k) + G_2^2F_3(C_k) + \cdots + G_2 F_r(C_k) \pmod{G_1}.$$

Hence if we adjoin G_2 to the system C_k we have a system A_k which is relatively finite and complete modulo $(ab)^{2k+2}$.

Therefore in all cases (a), (b), (c) we have been able to construct a system A_k relatively finite and complete modulo $(ab)^{2k+2}$ from the system A_{k-1} relatively finite and complete modulo $(ab)^{2k}$. Since A_0 evidently consists of f itself the required induction is complete.

Finally, consider what the circumstances will be when we come to the end of the sequence of moduli

$$(ab)^2, (ab)^4, (ab)^6, \cdots.$$

If n is even, $n = 2g$, the system A_{g-1} is relatively finite and complete modulo $(ab)^{2a} = (ab)^n$. The system B_{g-1} consists of the invariant $(ab)^n$ and hence is absolutely finite and complete. Hence, since A_g is absolutely finite and complete, the irreducible transvectants of A_g constitute the fundamental system of f. Moreover A_g consists of A_{g-1} and the invariant $(ab)^n$.

If n is odd, $n = 2g + 1$, then A_{g-1} contains f and is relatively finite and complete modulo $(ab)^{2g}$. The system B_{g-1} is here the fundamental system of the quadratic $(ab)^{2g}a_xb_x$ e.g.

$$B_{g-1} = \{(ab)^{2g}a_xb_x, (ab)^{2g}(ac)(bd)(cd)^{2g}\}.$$

This system is relatively finite and complete modulo $(ab)^{2g+1}$. But this modulus is zero since the symbols are equivalent. Hence B_{g-1} is absolutely finite and complete and by lemma 3 A_g will be absolutely finite and complete. Then the set of irreducible transvectants in A_g is the fundamental system of f.

Gordan's theorem has now been proved.

5.2 Fundamental Systems of the Cubic and Quartic by the Gordan Process

It will now be clear that the proof in the preceding section not only establishes the existence of a finite fundamental system of concomitants of a binary form f of order n, but it also provides an inductive procedure by which this system may be constructed.

5.2.1 System of the cubic.

For illustration let n = 3,

f = a3

x = b3

x = Â· Â· Â· .

The system A0 is f itself. The system B0 is the fundamental system of the single form

h1 = (ab)2axbx,

since h1 is of order less than 3. That is,

B0 = {(ab)2axbx,D}

where D is the discriminant of h1. Then A1 is the system of transvectants of the type of

 = (f, h

1 ,D)j .

But B0 is absolutely finite and complete. Hence AI is also.

Now D belongs to this system, being given by = = j = 0, = 1. If j > 0 then is reducible unless = 0, since D is an invariant. Hence, we have to consider which transvectants

 = (fa, h

1)j

are irreducible. But in Chapter IV, Section 3 II, we have proved that the only one of these transvectants which is irreducible is Q = (f, h1). Hence, the irreducible members of A1 consist of

A1 = {f, h1, Q,D},

or in the notation previously introduced,

A1 = {f,, Q,R}.

But B0 is absolutely complete and finite. Hence these irreducible forms of A1 constitute the fundamental system of f.

5.2.2 System of the quartic.

Let f = a4

x = b4

x = Â· Â· Â· . Then A0 = {f}. Here B0 is the single form

h1 = (ab)2a2

xb2

x,

and B0 is relatively finite and complete (modd(ab)4, (ab)2(bc)2(ca)2). The system C1 of transvectants

 = (f, h

1)j

is relatively finite and complete (modd(ab)4, (ab)2(bc)2(ca)2). In if j > 1, contains a term with the factor (ab)2(ac)2 which is congruent to zero with respect to the two moduli. Hence j = 1, and by the theory of reducible transvectants (Chap. IV, Section 3, III)

4 − 4 < j 4,

or = 1, = 1. The members of C1 which are irreducible with respect to the two moduli are therefore

f, h1, (f, h1).

Then

A1 = {f, h1, (f, h1), J = (ab)2(bc)2(ca)2}.

Next B1 consists of i = (ab)4 and is absolutely complete. Hence, writing h1 = H, (f, h1) = T, the fundamental system of f is f,H, T, i, J.

Chapter 6

FUNDAMENTAL SYSTEMS

In this chapter we shall develop, by the methods and processes of preceding chapters, typical fundamental systems of concomitants of single forms and of sets of forms.

6.1 Simultaneous Systems

In Chapter V, Section 1, II, it has been proved that if a system of forms (A) is both finite and complete, and a second system (B) is also both finite and complete, then the system (S) derived from (A) and (A) by transvection is finite and complete. In view of Gordan's theorem this proves that the simultaneous system of any two binary quantics f, g is finite, and that this simultaneous system may be found from the respective systems of f and g by transvection. Similarly for a set of n quantics.

6.1.1 Linear form and quadratic.

The complete system of two linear forms consists of the two forms themselves and their eliminant. For a linear form l = lx, and a quadratic f, we have

(A) = l, (B) = {f,D}.

Then S consists of the transvectants

S = {(fD, l)}.

Since D is an invariant S is reducible unless = 0. Also 5 , and unless
 = , (f, l) is reducible by means of the product

(f, l)(1, f−)0.

Hence = . Again, by

(f−1, l−2)−2(f, l2)2,

S is reducible if > 2. Hence the fundamental system of f and l is

S = {f,D, l, (f, l), (f, l2)2}.

When expressed in terms of the actual coefficients these forms are

l = a0x1 + a1x2 = lx = l0x = . . . ,
f = b0x21
+ 2b1x1x2 + b2x22
= a2
x = b2
x = . . .
D = 2(b0b2 = b21
= (ab)2,
(f, l) = (b0a1 − b1a0)x1 + (b1a1 − b2a0)x2 = (al)ax,
(f, l2)2 = b0a21
− 2b1a0a1 + b2a20
= (al)(al0).

6.1.2 Linear form and cubic.

If l = lx and f = a3
x = b3
x = . . . , then (cf. Table I),

(A) = {l}; (B) = {f,, Q,R},

and

S = (fQR, l).

Since R is an invariant = 0 for an irreducible transvectant. Also = as in
(I). If 6= 0 then, by the product

(f, l3)3(f−1Q, l−3)−3

S is reducible unless   3, and if   3 S is reducible by

(f, l)(f−1Q, 1)0

unless = = 0, = 1. Thus the fundamental system of f and l is

S = {f,, Q,R, l, (f, l), (f, l2)2, (f, l3)3,
(, l), (, l2)2, (Q, l), (Q, l2)2, (Q, l3)3}.

6.1.3 Two quadratics.

Let f = a2
x = a02
x ; g = b2
x = b02
x = ... Then
(A) = {f,D1}, (B) = {g,D2}, S = (fD
1 , gD
2).
Here = = 0. Also
2   2 − 1,
2   2 − 1,

and consistent with these we have the fundamental system
S = {f, g,D1,D2, (f, g), (f, g)2}. (6.1)
Written explicitly, these quantities are
f = a0x21
+ 2a1x1x2 + a22
x22
= a2
x = a02
x = Â· Â· Â· ,
g = b0x21
+ 2b1x1x2 + b2x22
= b2
x = b02
x = Â· Â· Â· ,
D1 = 2(a0a2 − a21
) = (aa0)2,
D2 = 2(b0b2 − b21
) = (bb0)2,
J = (f, g)
= (a0b1 − a1b0)x21
+ (a0b2 − a2b0)x1x2 + (a1b2 − a2b1)x22
= (ab)axbx,
h = (f, g)2 = a0b2 − 2a1b1 + a2b0 = (ab)2.
6.1.4 Quadratic and cubic.
Consider next the simultaneous system of f = a2
x = a02
x = Â· Â· Â· , g = b3
x = b03
x =
Â· Â· Â· . In this case
(A) = {f,D}, (B) = {g,, Q,R}, S = (fD, gabQcRd).
In order that S may be irreducible, = d = 0. Then in case > 2 and b 6= 0,
S = (f, gabQc) is reducible by means of the product
(f,)2(f−1, gab−1Qc)−2.
Hence only three types of transvectants can be irreducible;
(f,), (f,)2, (f, gaQc).
The first two are, in fact irreducible. Also in the third type if we take c = 0, the
irreducible transvectants given by (f, ga) will be those determined in Chapter
IV, Section 3, III, and are
f, g, (f, g), (f, g)2, (f2, g)3, (f3, g2)6.
If c > 1, we may substitute in our transvectant (f, gaQc) the syzygy
Q2 = −
1
2
(3 + Rg2);
and hence all transvectants with c > 1 are reducible. Taking a = 0, c = 1 we
note that (f,Q) is reducible because it is the Jacobian of a Jacobian. Then the

only irreducible cases are
(f,Q)2, (f2,Q)3.
Finally if c = 1, a 6= 0, the only irreducible transvectant is
(f3, gQ)6.
Therefore the fundamental system of a binary cubic and a binary quadratic
consists of the fifteen concomitants given in Table III below.

TABLE III
ORDER
DEGREE 0 1 2 3
1 f g
2 D (f, g)2 (f, g)
3 (f,)2 (f2, g)3 (f,) Q
4 R (f,Q)2
5 (f3, g2)6 (f2,Q)3
7 (f3, gQ)6

6.2 System of the Quintic

The most powerful process known for the discovery of a fundamental system
of a single binary form is the process of Gordan developed in the preceding
chapter. In order to summarize briefly the essential steps in this process let
the form be f. Construct, then, the system A0 which is finite and complete
modulo (ab)2, i.e. a system of forms which are not expressible in terms of forms
congruent to zero modulo (ab)2. Next construct A1, the corresponding system
modulo (ab)4, and continue this step by step process until the system which is
finite and complete modulo (ab)n is reached. In order to construct the system
Ak which is complete modulo (ab)2k+2 from Ak−1, complete modulo (ab)2k, a
subsidiary *system Bk−1 is introduced. The system Bk−1 consists of covariants
of = (ab)2kan−2k
x bn−2k
x . If 2n−4k < n then Bk−1 consists of the fundamental
system of . If 2n−4k > n, Bk−1 consists of itself, and if 2n−4k = n, Bk−1
consists of and the invariant (ab)n2
(bc)n2
(ca)n2
. The system derived from Ak−1,
Bk−1 by transvection is the system Ak.

6.2.1 The quintic.

Suppose that n = 5; f = a5
x = b5
x = Â· Â· Â· . Here, the system A0 is f itself. The
system B0 consists of the one form H = (ab)2a3
xb3
x. Hence the system A1 is the
transvectant system given by
(f,H).
By the standard method of transvection, if > 2 this transvectant always
contains a term of grade 3 and hence, by the theorem in Chapter IV, it may
be transformed so that it contains a series of terms congruent to zero modulo
(ab)4, and so it contains reducible terms with respect to this modulus. Moreover
(f,H)2 is reducible for forms of all orders as was proved by Gordan's series in
Section 1 of Chapter IV. Thus A1 consists of f,H, (f,H) = T.
Proceeding to construct B1 we note that i = (ab)4axbx is of order < 5.
Hence B1 consists of its fundamental system:
B1 = {i,D},

where D is the discriminant of i. Hence A2 which is here the fundamental
system of f is the transvectant system given by
 = (fHT, iD").
The values = = = = = 0, " = I give D. Since D is an invariant is

reducible if 6= 0 and " 6= 0. Hence " = 0.
If > 1, is reducible by means of such products as
(fHT, i)(H−1, i−1)−1.
Hence
(i) = 0
(ii) = 0, = 0, = 1.
By Chapter IV, Section 4, IV,
T2 = −
1
2{(f, f)2h2 − 2(f,H)2fH + (H,H)2f2}.
Hence
T2 −
1
2H3(mod(ab)4).
But if > 1, the substitution of this in raises above 1 and hence gives a reducible transvectant. Thus = 0 or 1 (of. Chap. V (158)).
Thus we need to consider in detail the following sets only:
(i) = 1 or 2, = 0, = 0,
(ii) = 0, = 0, = 1,
(iii) = 1, = 0, = 1,
(iv) = 0, = 1, = 0.
In (i) we are concerned with (f, i). By the method of Section 3, Chapter IV,
2 − 1 5 5 2,
5 − 4 5 5 5,
and consistent with this pair of relations we have
i, f, (f, i), (f, i)2, (f, i2)3, (f, i2)4, (f, i3)5,
(f2, i3)6, (f2, i4)7, (f2, i4)8, (f2, i6)9, (f2, i5)10.
Of these, (f2, i3)6 contains reducible terms from the product
(f, i2)4(f, i)2,

and in similar fashion all these transvectants are reducible except the following eight:
f, i, (f, i), (f, i)2, (f, i2)3, (f, i2)4, (f, i8)5, (f2i5)10.
In (ii) we have (T, i). But T = −(ab)2(bc)a3
xb2
xc4
x, and (T, i) contains the
term t = −(ab)2(bc)(bi)a3
xbxc4
x. Again
(bc)(bi)cxix =
1
2
[(bc)2i2
x + (bi)2c2
x − (ci)2b2
x].
Hence t involves a term having the factor f. The analysis of the remaining cases proceeds in precisely the same way as in Cases (i), (ii). In Case (ii) the irreducible transvectants prove to be
(T, i)2, (T, i2)4, (T, i3)6, (T, i4)8, (T, i5)9.
Case (iii) gives but one irreducible case, viz. (fT, i7)14.
In Case (iv) we have
(H, i), (H, i)2, (H, i2)3, (H, i2)4, (H, i3)5, (H, i3)6.
Table IV contains the complete summary. The fundamental system of f consists of the 23 forms given in this table.
TABLE IV
ORDER

DEGREE 0 1 2 3 4 5 6 7 9
1 f
2 i H
3 (i, f)2 (i, f) T
4 D (i,H)2 (i,H)
5 (i2, f)4 (i2, f)3 (i, T)2
6 (i2,H)4 (i2,H)3
7 (i3, f)5 (i2, T)4
8 (i3,H)6 (i3,H)5
9 (i3, T)6
11 (i4, T)8
12 (i5, f2)10
13 (i5, T)9
18 (i7, fT)14

6.3 Resultants in Aronhold's Symbols

In order to express the concomitants derived in the preceding section in symbolical form the standard method of transvection may be employed and gives

readily any concomitant of that section in explicit symbolical form. We leave details of this kind to be carried out by the reader. However, in this section we give a derivation, due to Clebsch, which gives the symbolical representation of the resultant of two given forms. In view of the importance of resultants in invariant theories, this derivation is of fundamental consequence.

6.3.1 Resultant of a linear form and an n-ic.

The resultant of two binary forms equated to zero is a necessary and sufficient condition for a common factor.

Let

f = anx

; = x = 1x1 = 2x2 = 0.

Then x1 : x2 = −2 : 1. Substitution in f evidently gives the resultant, and in the form

R = (a)n

6.3.2 Resultant of a quadratic and an n-ic.

Let

 = 2x

= pxqx.

The resultant R = 0 is evidently the condition that f have either px or qx as a factor. Hence, by I,

R = (ap)n(bq)n.

Let us express R entirely in terms of a, b, Â· Â· Â· , and , , Â· Â· Â· symbols.

We have, since a, b are equivalent symbols,

R =
1
2{(ap)n(bq)n + (aq)n(bp)n}.

Let (ap)(bq) = μ, (aq)(bp) = , so that

R = μn + n
2

Theorem. If n is even, R = μn+n
2 is rationally and integrally expressible in terms of 2 = (μ −)2 and = μ. If n is odd, (μ +)−1R is so expressible.

In proof write

Sk = μk + (−1)n−kk.

Then

R =
1
2Sn.

Moreover it follows directly that

Sn = (μ −)Sn+1 + μSn−2,

Sn−1 = (μ −)Sn−2 + μSn−3,
. . . .
S2 = (μ −)S1 + μS0.
Also for n even
S1 = μ − , S0 = 2,
and for n odd
S1 = μ + , S0 = 0.
Now let
 = S2 + zS3 + z2S4 + Â· Â· Â·
= S1 + S0 + zS2 + zS1 + z2S3 + z2S2 + . . .
Then we have
 = (S1 + z
) + (S0 + zS1 + z2
),
and
 =
(+ z)S1 + S0
1 − z + z2 .
Then Sn is the coefficient of xn−2 in the expansion of
. Now
1
1 − z − z2 =
1
1 − z
+ z2
(1 − z)2 + 2z4
(1 − z)3 + Â· Â· Â·
= 1 + z + 2z2 + 3z3 + Â· Â· Â·
+(1+2z + 32z2 + 43z3 + Â· Â· Â·)z2
+(1+
2 Â· 3
1 Â· 2z +
3 Â· 4
1 Â· 22z2 +
4 Â· 5
1 Â· 23z3)2z4
+. . . . .
= K0 + K1z + K2z2 + K3z3 + Â· Â· Â· ,
where
K0 = 1,K2 = 2 + ,K4 = 4 + 32 + 2,
K1 = ,K3 = 3 + 2,K5 = 5 + 43 + 32,
...
Kh = h + (h − 1)h−2 +
(h − 2)(h − 3)
1 Â· 2 2h−4
+
(h − 3)(h − 4)(h − 5)
1 Â· 2 Â· 3 3h−6 + Â· Â· Â·
But
 = {(S1 + S0) + zS1}{K0 + K1z + K2z2 + Â· Â· Â· }.

In this, taking the coefficient of zn−2,
2R = Sn = (S1 + S0)Kn−2 + S1Kn−3.
But,
Kn−2 + Kn−3 = Kn−1.
Hence,
R +
1
2{S1Kn−1 + S0Kn−2}.

Hence according as n is even or odd we have
2R = n + nn−2 + n(n − 3)
1 Â· 2 2n−4 + n(n − 4)(n − 5)
1 Â· 2 Â· 3 3n−6 + Â· Â· Â· ,
2R = (µ +){n−1 + (n − 2)n−3 +
(n − 3)(n − 4)
1 Â· 2 2n−5 +
(n − 4)(n − 5)(n − 6)
1 Â· 2 Â· 3 3n−7 + Â· Â· Â· },
which was to be proved.
Now if we write
 = pxqx = 2x
= 2x
= Â· Â· Â· ,
we have
p1q1 = 2
1, p1q2 + p2q1 = 212, p2q2 = 2
2.
Then
µ + = (ap)(bq) + (aq)(bp)
= (a1p2 − a2p1)(b1q2 − b2q1) + (a1q2 − a2q1)(b1p2 − b2p1)
= 2[a1b1a22
− a1b212 − a2b112 + a2b22
1]
= 2(a)(b),
µ = = (ap)(aq)(bp)(bq)
= paqa Â· pbqb = (a)2(ab)2,
(µ −)2 = 2 = {(ap)(bq) − (aq)(bp)}2 = (ab)2(pq)2
= −2(ab)2()2 = −2(ab)2D.
Let the symbols of be 0, 00, Â· Â· Â· ; 0, 00, Â· Â· Â· , , Â· Â· Â· . Then we can write for the general term of R,
n−2kk = (µ −)n−2k(µ)k = (−2)n2
−kD
n2
−k(ab)n−2k
Ã— (a0)2(b0)2(a00)2(b00)2 Â· Â· Â· (a(k))2(b(k))2
= (−2)n2
−kD
n2
−kAk.

Evidently Ak is itself an invariant. When we substitute this in 2R above we
write the term for which k = 1
2n last. This term factors. For if
B = (a0)2(a00)2 Â· Â· Â· (a(n2
))2
= (b0)2(b00)2 Â· Â· Â· (b(n2
))2,
then

n2
= B2.
Thus when n is even,
R = (−D)n2
Â· 2n−2
2 A0 + n(−D) n−2
2 2n−4
2 A1
+ n(n − 3)

1 Â· 2
(−D) n−4
2 2n−6
2 A2
+ n(n − 4)(n − 5)
1 Â· 2 Â· 3
(−D) n−6
2 2n−8
2 A3 (159)
+ Â· Â· Â· −
n2
4 DAn2
−1 + B2.
We have also,
n−2k−1k(µ +) = 2(−2) n−1
2 D
n−1
2 kAk,
where Ak is the invariant,
Ak = (ab)n−1−2k(a)(b) Â· (a0)(a00)2 Â· Â· Â· (a(k))2 Â· (b0)2(b00)2 Â· Â· Â· (b(k))2.
In this case,
R = (−2D) n−1
2 A0 + (n − 2)(−2D) n−3
2 A1
+
(n − 3)(n − 4)
1 Â· 2
(−2d) n−5
2 A2 (1591)
+ Â· Â· Â· −
n2 − 1
4 DAn−3
2
+ An−1
2
.
Thus we have the following:
Theorem. The resultant of a form of second order with another form of even order is always reducible in terms of invariants of lower degree, but in the case of a form of odd order this is not proved owing to the presence of the term An−1
2
.

A few special cases of such resultants will now be given; (a), (b), (c), (d).
(a)n = 1 :R = A0,A0 = (a)2
(b)n = 2 :R = −DA0 + B2,A0 = (ab)2,B = (a)2.
R = −()2(ab)2 + (a)2(b)2.
(c)n = 3 :R = −2DA0 + A1,A0 = (ab)2(a)(b).
A1 = (a)(b)(a)2(b)2.
R = −2()2(ab)2(a)(b) + (a)(b)(a)2(b)2.
(d)n = 4 :R = 2D2A0 − 4DA1 + B2,A0 = (ab)4.
A1 = (ab)2(a)2(b)2.
B = (a)2(a0)2.
R = 2()2(00)2(ab)4 − 4()2(ab)2(a0)2(b0)2
+ (a)2(a0)2(b)2(b0)2.
6.4 Fundamental Systems for Special Groups of Transformations
In the last section of Chapter I we have called attention to the fact that if the group of transformations to which a form f is subjected is the special group

given by the transformations

$$x_1 = \frac{\sin(\omega -)}{\sin \omega} x'_1 + \frac{\sin(\omega -)}{\sin \omega} x'_2; \quad x_2 = \frac{\sin}{\sin \omega} x'_1 + \frac{\sin}{\sin \omega} x'_2,$$

then

$$q = x_1^2 + 2x_1x_2 \cos \omega + x_2^2$$

is a universal covariant. Boole was the first to discover that a simultaneous concomitant of q and any second binary quantic f is, when regarded as a function of the coefficients and variables of f, a concomitant of the latter form alone under the special group. Indeed the fundamental simultaneous system of q and f taken in the ordinary way is, from the other point of view, evidently a fundamental system of f under the special group. Such a system is called a Boolean system of f. We proceed to give illustrations of this type of fundamental system.

6.4.1 Boolean system of a linear form.

The Boolean system for a linear form,

$$l = a_o x_1 + a_1 x_2,$$

is obtained by particularizing the coefficients of f in Paragraph I, Section 1 above by the substitution b_0, b_1, b_2 1, $\cos \omega$ 1 .

Thus this fundamental system is

$$l = a_0x_1 + a_1x - 2,$$
$$q = x_1^2 + 2x_1x_2 \cos \omega + x_2^2,$$
$$a = \sin^2 \omega,$$
$$b = (a_0 \cos \omega - a_1)x_1 + (a_0 - a_1 \cos \omega)x_2,$$
$$c = a_0^2 - 2a_0a - 1 \cos \omega + a_1^2.$$

6.4.2 Boolean system of a quadratic.

In order to obtain the corresponding system for a quadratic form we make the above particularization of the b coefficients in the simultaneous system of two quadratics (cf. Section 1, III above).

Thus we find that the Boolean system of f is

$$f = a_0x_1^2 + 2a_1x_1x_2 + a_2x_2^2,$$
$$q = x_1^2 + 2x_1x_2 \cos \omega + x_2^2,$$
$$D = 2(a_0a_2 - a_1^2),$$
$$d = \sin^2 \omega,$$
$$e = a_0 + a_2 - 2a_1 \cos \omega,$$
$$F = (a_0 \cos \omega - a_1)x_1^2 + (a_0 - a_1)x_1x_2 + (a_1 - a_2 \cos \omega)x_2^2.$$

6.4.3 Formal modular system of a linear form.

If the group of transformations is the finite group formed by all transformations T_p whose coefficients are the positive residues of a prime number p then, as was mentioned in Chapter I,

$$L = x_1^p x_2 - x_1 x_2^p$$

is a universal covariant. Also one can prove that all other universal covariants of the group are covariants of L. Hence the simultaneous system of a linear form l and L, taken in the algebraic sense as the simultaneous system of a linear form and a form of order $p + 1$ will give formal modular invariant formations of l. We derive below a fundamental system of such concomitants for the case $p = 3$. Note that some forms of the system are obtained by polarization. Let $f = a_0x_1 + a_1x_2$; $p = 3$. The algebraical system of f is f itself. Polarizing this,

$$C = x^3 \frac{\partial}{\partial x} f = a_0x_1^3 + a_1x_2^3, \quad x^9 \frac{\partial}{\partial x} f = a_0x_1^9 + a_1x_2^9 = C',$$

$$D = a^3 \frac{\partial}{\partial a} f = a_0^3 x_1 + a_1^3 x_2.$$

The fundamental system of universal covariants of the group T_3 is

$$L = x_1^3 x_2 - x_1 x_2^3, \quad Q = x_1^6 + x_1^4 x_2^2 + x_1^2 x_2^4 + x_2^6 = ((L,L)_2, L).$$

The simultaneous system of f and L is (cf. §1, II)

$$(L, f^r)_r \ (r = 1, \ldots, 4); \quad (Q, f^s)_s \ (s = 1, \ldots, 6).$$

Of these some belong to the above polar system and some are reducible; as $(Q, f^2)_2 \equiv fC \pmod 3$. But

$$A = (L, f^4)_4 \equiv a_0^3 a_1 - a_0 a_1^3,$$

$$B = (Q, f^6)_6 \equiv a_0^6 + a_0^4 a_1^2 + a_0^2 a_1^4 + a_1^6,$$

$$E = (Q, f^3)_3 \equiv a_1(a_0^2 - a_1^2)x_1^3 - a_0^3 x_1^2 x_2 + a_1^3 x_1 x_2^2 + a_0(a_0^2 - a_1^2$$

)x32
(mod 3).
The polars x3 @
@x D f3, x3 @
@x E DL a3 @
@a A 0, a3 @
@a B A2 (mod 3),
are reducible. The polar C0 is also reducible. In fact,
C0 CQ − fL2 (mod 3).
The formal fundamental system of f modulo 3 is
A,B,C,D,E, f,L,Q.

6.5 Associated Forms

Consider any two covariants 1, 2 of a binary form f(x1, x2) of order m. Let the first polars of these be
= n−1
1x 1y, µ = p−1
2 x2y
or
= 1y1 + 2y2, µ = µ1y1 + µ2y2 (1591)
where
i =
1
n
@1
@xi
, µi =
1
p
@2
@xi
(i = 1, 2).
Let the equations (1591) be solved for y1, y2. Then if J is the Jacobian of the two covariants 1, 2, the result of substituting y1, y2 for x1, x2 in f(x1, x2) is
f(y1, y2) =
1
Jm(A0m + A1m−1µ + . . . + Amµm)

and the forms A0,A1, Â· Â· Â· ,An are covariants of f, as will be proved below. But the inverse of (1591) constitutes a linear transformation on the variables y1, y2 in which the new variables are , µ. Hence if
(a0, a1, Â· Â· Â· , am; y1, y2)
is any covariant of f with x1, x2 replaced by the cogredient set y1, y2, and if f(y1, y2) above is taken as the transformed form, the corresponding invariant relation is
C0(a0, a1, Â· Â· Â· , am; y1, y2) = JkA0
Jm,
A1
Jm, Â· Â· Â· ,
Am
Jm
; , µ,
where C0 is a constant. Now let (y) = (x), and this relation becomes, on account of the homogeneity in the coefficients,
(a0, a1, Â· Â· Â· , am; a1, x2) = C
J (A0,A1, Â· Â· Â· ,Am; 1, 2).
Thus every covariant of f is expressible rationally in terms of the m + 3 covariants of the system
A0,A1,A2, Â· Â· Â· ,Am, 1, 2.
Such a system of covariants in terms of which all covariants of a form are rationally

expressible is called a system of associated forms (Hermite). The expression
for f(y1, y2) above is called a typical representation of f.
Now we may select for 2 in this theory the universal covariant
2y = x1y2 − x2y1,
and then the coefficient covariants A0,A1, Â· Â· Â· can be given in explicit symbolical
form. First, however, we obtain the typical representation of f as an expansion
based upon a formal identity. From
 = 1y1 + 2y2, µ = µ1y1 + µ2y2,
i.e. = y, µ = µy; and f = amy, we have the identity
(µ)ay = (aµ) − (a)µ.
If we raise both sides of this identity to the mth power we have at once the
symbolical representation of the typical representation of f, in the form
(µ)mf(y1, y2) = B0m − mB1m−1µ + Â· Â· Â· + (−1)mBmµm,
where
B0 = (aµ)m,B1 = (aµ)m−1(a),B2 = (aµ)m−2(a)2, Â· Â· Â· ,Bm = (a)m.

Also
(µ)m = Jm.
Now with µ = (xy) we have
J = 1x1 + 2x2 = 1,
by Euler's theorem. Moreover we now have
B0 = amx
= f,B1 = am−1
x (a),B2 = am−2
x (a)2, Â· Â· Â· ,
for the associated forms, and
(a0, a1, Â· Â· Â· ; y1, y2) =
1

1
(B0,−B1,B2, Â· Â· Â· ; , µ),
and
(a0, a1, Â· Â· Â· ; x1, x2) =
1

1
(f,−B1,B2, Â· Â· Â· ; , 0).
Again a further simplification may be had by taking for the form f itself.
Then we have
B0 = f,B1 = (ab)am−1
x bm−1
x = 0,B2 = (ab)(ac)am−2
x bm−1
x cm−1
x , Â· Â· Â·
and the following theorem:
Theorem. If in the leading coefficient of any covariant we make the replacements
 a0, a1, a2, a3, Â· Â· Â· ,
1(= f), −B1(= 0), B2, −B3, Â· Â· Â· ,
and divide by a properly chosen power of (= f) we have an expression for
as a rational function of the set of m associated forms
1(= f),B1(= 0),B2,B3, Â· Â· Â· .
For illustration let m = 3, f being a binary cubic. Let be the invariant R.
Then since
B2 = (ab)(ac)bxcx axbxcx =
1
2
(ab)2axbxc3
x =

1
2
  f,B3 = fQ,
where is the Hessian, and Q the cubic covariant of f, the typical representation
of f is
f2f(y) = 3 +
3
2
2 + Q3.
If one selects for the invariant
−
1
2R = (a0a3 − a1a2)2 − 4(a0a2 − a21
)(a1a3 − a22
),

and substitutes a0, a1, a2, a3
f−2, 0, 1
2f−2, −Qf−2,
there results
−
1
2R = (f−4Q)2 +
1
2f−33f6.
That is,
−Rf2 = 2Q2 + 3.
This is the syzygy connecting the members of the fundamental system of the
cubic f (cf. Chap. IV, Â§4). Thus the expression of R in terms of the associated
forms leads to a known syzygy.

Chapter 7
COMBINANTS AND RATIONAL CURVES

7.1 Combinants

In recent years marked advances have been made in that branch of algebraic
invariant theory known as the theory of combinants.

7.1.1 Definition.

Let f, g, h, Â· Â· Â· be a set of m binary forms of order n, and suppose that m < n;
f = a0xn1
+ Â· Â· Â· , g = b0xn1
+ Â· Â· Â· , h = c0xn1
+ Â· Â· Â· .
Let
(a1, a1, Â· Â· Â· ; b0, Â· Â· Â· ; c0, Â· Â· Â· ; x1, x2)
be a simultaneous concomitant of the set. If is such a function that when
f, g, h, Â· Â· Â· are replaced by
f0 = 1f + 1g + 1h + Â· Â· Â· , g0 = 2f + 2g + 2h + Â· Â· Â· ,
h0 = 3f + 3g + 3h + Â· Â· Â· , Â· Â· Â· (160)
the following relation holds:
(a00, a01, Â· Â· Â· ; b00, Â· Â· Â· ; c00, Â· Â· Â· ; x1, x2)
= (Â· Â· Â·)k(a0, a1, Â· Â· Â· ; b0, Â· Â· Â· ; c0, Â· Â· Â· ; x1, x2); (161)
where
D (Â· Â· Â·) =
1, 1, 1, Â· Â· Â· 2, 2, 2, Â· Â· Â· 3, 3, 3, Â· Â· Â· 4, 4, 4, Â· Â· Â·
Â· Â· Â· Â·

,

then is called a combinant of the set (Sylvester).
We have seen that a covariant of f in the ordinary sense is an invariant function under two linear groups of transformations. These are the group given by T and the induced group (231) on the coefficients. A combinant is not only invariantive under these two groups but also under a third group given by the relations
a00 = 1a0 + 1b0 + 1c0 + Â· Â· Â· ,
a01 = 1a1 + 1b1 + 1c1 + Â· Â· Â· ,
. . . (162)
b00 = 2a0 + 2b0 + 2c0 + Â· Â· Â· ,
. . .
As an illustration of a class of combinants we may note that all transvectants of odd index of f and g are combinants of these forms. Indeed
(1f + 1g, 2f + 2g)2r+1
= 12(f, f)2r+1 + ()(f, g)2r+1 + 12(g, g)2r+1 (163)
= ()(f, g)2r+1,
by (79) and (81). Hence (f, g)2r+1 is a combinant. Included in the class (163) is the Jacobian of f and g, and the bilinear invariant of two forms of odd order (Chap. III, V).
7.1.2 Theorem on Aronhold operators.
Theorem. Every concomitant, , of the set f, g, h, Â· Â· Â· which is annihilated by each one of the complete system of Aronhold's polar operators
a
@
@b, a
@
@c, b
@
@c, b
@
@a, Â· Â· Â·
is a combinant of the set.
Observe first that is homogeneous, and in consequence
a
@
@a = i1,b
@
@b= i2, Â· Â· Â· ,
where i1 is the partial degree of in the coefficients a of f, i2 the degree of in the coefficients of g, and so forth. Since (a @
@b) = 0, then (a0 @
@b0)0 = 0. Thus
+ (1a0 + 1b0 + Â· Â· Â· + 1e0) @0
@(2a0 + 2b0 + Â· Â· Â· + 2e0)
+ (1a1 + 1b1 + Â· Â· Â· + 1e1) @0
@(2a1 + 2b1 + Â· Â· Â· + 2e1)
(164)
Â· Â·
+ (1an + 1bn + Â· Â· Â· + 1en) @0
@(2an + 2bn + Â· Â· Â· + 2en)
= 0.

1
n Xi=0
ai
@0
@(2ai + Â· Â· Â· + 2ei)
+ Â· Â· Â· + 1
n Xi=0

ei
@0
@(2ai + Â· Â· Â· + 2ei)
= 0. (165)
1
n Xi=0
@0
@(2ai + Â· Â· Â· + 2ei)
@(2ai + Â· Â· Â· + 2ei)
@2
+ Â·
Â·
+ 1
n Xi=0
@0
@(2ai + Â· Â· Â· + 2ei)
@(2ai + Â· Â· Â· + 2ei)
@2
= 0. (166)
Hence
120 1
@
@2
+ 1
@
@2
+ Â· Â· Â· + 1
@
@20 = 0,
and generally,
st0 s
@
@t
+ s
@
@t
+ Â· Â· Â· + s
@
@t0 = 0(s
<>
t)
= is0(s = t),
(167)
where i is the total degree of in all of the coefficients. In (167) we have m2
equations given by (s, t = 1, Â· Â· Â· ,m). We select the following m of these and
solve them for the derivatives @0
@1
, Â· Â· Â· :
1
@0
@1
+ 1
@0
@1
+ Â· Â· Â· + 1
@0
@1
=i10,
2

@0
@1
+ 2
@0
@1
+ Â· Â· Â· + 2
@0
@1
=0, (168)
Â· Â·
m
@0
@1
+ m
@0
@1
+ Â· Â· Â· + m
@0
@1
=0.
Solution of these linear equations gives
@0
@1
=
@D
@1
(Â· Â· Â·) i10,
@0
@1
=
@D
@1
(Â· Â· Â·) i10, Â· Â· Â· ,
@0
@1
=
@D
@1
(Â· Â· Â·) i10.
But we know that
d0 = @0
@1
d1 + @0
@1
d1 + Â· Â· Â· + @0
@1
d1.
Hence
d0 = @D
@1
d1 + @D
@1
d1 + Â· Â· Â· + @D
@1
d1i10
D
= dD
D
i10.

Hence we can separate the variables and integrate:
d0
0
= i1
dD
D
,
0 = Di1F(a0, . . .), (169)
where F is the constant of integration. To determine F, particularize the relations (162) by taking all coefficients , , Â· Â· Â· zero except
1 = 2 = Â· Â· Â· = m = 1.
Then a00 = a0, a01 = a1, Â· Â· Â· , b0i = bi, etc., and (169) becomes
F.
Hence
0 = Di1,
which proves the theorem.
It is to be noted that the set (168) may be chosen so that the differentiations are all taken with respect to k, k, Â· Â· Â· in (168). Then we obtain in like manner
0 = Dik.
Thus
i1 = i2 = Â· Â· Â· = im.
That is, a combinant is such a simultaneous concomitant that its partial degrees in the coefficients of the several forms are all equal. This may be proved independently as the
7.1.3 Partial degrees.
Theorem. A combinant is of equal partial degrees in the coefficients of each form of the set.
We have
a
@
@bb
@
@a− b
@
@aa
@
@b = 0.
Hence
a
@
@a− b
@
@b (i1 − i2) = 0.
Thus i1 = i2. Similarly ij = ik(j, k = 1, 2, Â· Â· Â· ,m).

7.1.4 Resultants are combinants.
Theorem. The resultant of two binary forms of the same order is a combinant.
Let
f = f(x1, x2), g = g(x1, x2).
Suppose the roots of f are (r(i)
1 , r(i)
2)(i = 1, Â· Â· Â· , n), and of g (s(i)
1 , s(i)
2)(i =
1, Â· Â· Â· , n). Then the resultant may be indicated by
R = g(r(1)
1 , r(1)
2)g(r(2)

1 , r(2)
2) Â· Â· Â· g(r(n)
1 , r(n)
2),
and by
R = f(s(1)
1 , s(1)
2)f(s(2)
1 , s(2)
2) Â· Â· Â· f(s(n)
1 , s(n)
2).
Hence
a
@
@bR = f(r(1)
1 , r(1)
2)g(r(2)
1 , r(2)
2) Â· Â· Â· g(r(n)
1 , r(n)
2) = 0,
b
@
@aR = g(s(1)
1 , s(1)
2)f(s(2)
1 , s(2)
2) Â· Â· Â· f(s(n)
1 , s(n)
2) = 0.
Thus R is a combinant by Theorem II.

Gordan has shown1 that there exists a fundamental combinant of a set of forms. A fundamental combinant is one of a set which has the property that its fundamental system of concomitants forms a fundamental system of combinants of the set of forms. The proof of the Theorem II of this section really proves also that every combinant is a homogeneous function of the determinants of order m,

ak1 bk1 ck1 Â· Â· Â· lk1
ak2 bk2 ck2 Â· Â· Â· lk2
...
...
...
. . .
...
akm bkm ckm Â· Â· Â· lkm

,
that can be formed from the coefficients of the forms of the set. This also follows from (162). For the combinant is a simultaneous invariant of the linear forms
ak + bk + ck + Â· Â· Â· + lk(k = 0, 1, Â· Â· Â· , n), (170)
and every such invariant is a function of the determinants of sets of m such linear forms. Indeed if we make the substitutions
= 10 + 20 + Â· Â· Â· + m0,
= 10 + 20 + Â· Â· Â· + m0,
Â· Â· Â· Â· Â· Â· Â· Â· Â· Â· Â· Â· Â· Â· Â·

1Mathematische Annalen, Vol. 5.

in (170) we obtain

$a_{0k} = 1a_k + 1b_k + 1c_k + \cdots,$

$b_{0k} = 2a_k + 2b_k + 2c_k + \cdots,$

$\cdots\cdots\cdots\cdots\cdots$

and these are precisely the equations (162).

For illustration, if the set of n-ics consists of

$f = a_0x_1^2 + 2a_1x_1x_2 + a_2x_2^2,$

$g = b_0x_1^2 + 2b_1x_1x_2 + b_2x_2^2,$

any combinant of the set is a function of the three second order determinants $(a_0b_1 - a_1b_0)$, $(a_0b_2 - a_2b_0)$, $(a_1b_2 - a_2b_1)$.

Now the Jacobian of f and g is

$J = (a_0b_1 - a_1b_0)x_1^2 + (a_0b_2 - a_2b_0)x_1x_2 + (a_1b_2 - a_2b_1)x_2^2.$

Hence any combinant is a concomitant of this Jacobian. In other words J is the fundamental combinant for two quadratics. The fundamental system of combinants here consists of J and its discriminant. The latter is also the resultant of f and g.

The fundamental system of combinants of two cubics f, g, is (Gordan)

= (f, g), = (f, g)3, = (#, #)2, (#, #)4, (, #), (, #)4.

The fundamental combinants are # and , the fundamental system consisting of the invariant and the system of the quartic # (cf. Table II).

7.1.5 Bezout's form of the resultant.

Let the forms f, g be quartics,

$f = a_0x_1^4 + a_1x_1^3x_2 + \cdots,$

$g = b_0x_1^4 + b_1x_1^3x_2 + \cdots.$

From f = 0, g = 0 we obtain, by division,

$$\frac{a_0}{b_0} = \frac{a_1x_1^3 + a_2x_1^2x_2 + a_3x_1x_2^2 + a_4x_2^3}{b_1x_1^3 + b_2x_1^2x_2 + b_3x_1x_2^2 + b_4x_2^3},$$

$$\frac{a_0x_1 + a_1x_2}{b_0x_1 + b_1x_2} = \frac{a_2x_1^2 + a_3x_1x_2 + a_4x_2^2}{b_2x_1^2 + b_3x_1x_2 + b_4x_2^4},$$

$$\frac{a_0x_1^2 + a_1x_1x_2 + a_2x_2^2}{b_0x_1^2 + b_1x_1x_2 + b_2x_2^2} = \frac{a_3x_1 + a_4x_2}{b_3x_1 + b_4x_2},$$

a0x31
+ a1x21
x2 + a2x1x22
+ a3x32
b0x31
+ b1x21
x2 + b2x1x22
+ b3x32
= a4
b4
.

Now we clear of fractions in each equation and write
ai bi
ak bk= pik.
We then form the eliminant of the resulting four homogeneous cubic forms. This is the resultant, and it takes the form
R =
p01 p02 p03 p04
p02 p08 + p12 p04 + p13 p14
p03 p04 + p13 p14 + p23 p24
p04 p14 p24 p34

Thus the resultant is exhibited as a function of the determinants of the type peculiar to combinants. This result is due to Bezout, and the method to Cauchy.
7.2 Rational Curves
If the coordinates of the points of a plane curve are rational integral functions of a parameter the curve is called a rational curve. We may adopt a homogeneous parameter and write the parametric equations of a plane quartic curve in the form
x1 = a104
1 + a118
12 + Â· Â· Â· + a144
2 = f1(1, 2),
x2 = a204
1 + a213
12 + Â· Â· Â· + a244
2 = f2(1, 2), (1701)
x3 = a304
1 + a313
12 + Â· Â· Â· + a344
2 = f3(1, 2).
We refer to this curve as the R4, and to the rational plane curve of order n as the Rn.
7.2.1 Meyer's translation principle.
Let us intersect the curve R4 by two lines
ux = u1x1 + u2x2 + u3x3 = 0,
vx = v1x1 + v2x2 + v3x3 = 0.
The binary forms whose roots give the two tetrads of intersections are
uf = (a10u1 + a20u2 + a30u3)4
1 + (a11u1 + a21u2 + a31u3)3
12
+ (a12u1 + a22u2 + a32u3)2
1u3)2
12
2 + (a18u1 + a23u2 + a33u3)13
2
+ (a14u1 + a24u2 + a34u3)4
2,

and the corresponding quartic vf . A root ((i)
1 , (i)
2) of uf = 0 substituted in
(1701) gives one of the intersections (x(i)
1 , x(i)
2 , x(i)
3) of ux = 0 and the R4.

Now uf = 0, vf = 0 will have a common root if their resultant vanishes.
Consider this resultant in the Bezout form R. We then have, by taking
aiu = a1iu1 + a2iu2 + a3iu3 (i = 0, Â· Â· Â· , 4),
pik = aiuakv − aivaku.
Thus
pik = (uv)1(a2ia3k − a2ka3i) + (uv)2(a3ia1k − a1ia3k)
+ (uv)3(a1ia2k − a1ka2i),
where (uv)1 = u2v3 − u3v2, (uv)2 = u3v1 − u1v3, (uv)3 = u1v2 − u2v1. Hence
pik =
(uv)1 (uv)2 (uv)3
a1i a2i a3i
a1k a2k a3k

.
But if we solve ux = 0, vx = 0 we obtain
x1 : x2 : x3 = (uv)1 : (uv)2 : (uv)3.
Therefore
pik =
x1 x2 x3
a1i a2i a3i
a1k a2k a3k

(i, k = 0, Â· Â· Â· , 4),
where is a constant proportionality factor. We abbreviate
pik = |xaiak|.
Now substitute these forms of pik in the resultant R. The result is a ternary form in x1, x2, x3 whose coefficients are functions of the coefficients of the R4. Moreover the vanishing of the resulting ternary form is evidently the condition that ux = 0, vx = 0 intersect on the R4. That is, this ternary form is the cartesian equation of the rational curve. Similar results hold true for the Rn as an easy extension shows.
Again every combinant of two forms of the same order is a function of the determinants
pik =
ai ak
bi bk .
Hence the substitution
pik = |xaiak|,
made in any combinant gives a plane curve. This curve is covariantive under ternary collineations, and is called a covariant curve. It is the locus of the intersection of ux = 0, vx = 0 when these two lines move so as to intersect the rational curve in two point ranges having the projective property represented by the vanishing of the combinant in which the substitutions are made.

7.2.2 Covariant curves.
For example two cubics
f = a0x31
+ a1x21
x3 + Â· Â· Â· , g = b + 0x31
+ b1x21
x2 + Â· Â· Â· ,

have the combinant

$$K = (a_0b_3 - a_3b_0) - \frac{1}{3}(a_1b_2 - a_2b_1).$$

When $K = 0$ the cubics are said to be apolar. The rational curve R_3 has, then, the covariant curve

$$K(x) \equiv |xa_0a_3| - \frac{1}{3}|xa_1a_2| = 0.$$

This is a straight line. It is the locus of the point (u_x, v_x) when the lines $u_x = 0$, $v_x = 0$ move so as to cut R_3 in apolar point ranges. It is, in fact, the line which contains the three inflections of R_3, and a proof of this theorem is given below. Other theorems on covariant curves may be found in W. Fr. Meyer's Apolarität und Rationale Curven (1883). The process of passing from a binary combinant to a ternary covariant here illustrated is called a translation principle. It is easy to demonstrate directly that all curves obtained from combinants by this principle are covariant curves.

Theorem. The line $K(x) = 0$ passes through all of the inflexions of the rational cubic curve R_3.

To prove this we first show that if g is the cube of one of the linear factors of f = (1) x x1 + (1) 2 x2)3,

g = ((1) 1 x1 + (1) 2 x2)3,

then the combinant K vanishes identically. In fact we then have

b0 = (1)3 1 , b1 = 3(1)2 1 (1) 2 , · · · ,

and

a0 = (1) 1 (2) 1 (3) 1 , a1 =X(1) 1 (2) 1 (3) 2 , · · · .

When these are substituted in K it vanishes identically.

Now assume that u_x is tangent to the R_3 at an inflexion and that v_x passes through this inflexion. Then u_f is the cube of one of the linear factors of v_f, and hence $K(x)$ vanishes, as above. Hence $K(x) = 0$ passes through all inflexions.

The bilinear invariant of two binary forms f, g of odd order $2n + 1 = m$ is

$$K_m = a_0b_m - ma_1b_{m-1} + \binom{m}{2}a_2b_{m-2} + \cdots + ma_{m-1}b_1 - a_mb_0,$$

or

$$K_m = p_{0m} - mp_{1m-1} + \binom{m}{2}p_{2m-2} - \cdots + (-1)^n\binom{m}{n}p_{nn+1},$$

where $f = a_0x_1^m + ma_1x_1^{m-1}x_2 + \cdots$.

If two lines $u_x = 0$, $v_x = 0$ cut a rational curve R_m of order $m = 2n + 1$ in two ranges given by the respective binary forms

$$u_f,\ v_f$$

of order m, then in order that these ranges may have the projective property

$K_m = 0$ it is necessary and sufficient that the point (u_x, v_x) trace the line

$$K_m(x) \equiv \sum_{i=1}^{n} (-1)^i |a_i a_{m-i} x| \binom{m}{i} = 0.$$

This line contains all points on the R_m where the tangent has m points in common with the curve at the point of tangency. The proof of this theorem is a direct extension of that above for the case $m = 3$, and is evidently accomplished with the proof of the following:

Theorem. A binary form, f, of order m is apolar to each one of the m, m-th powers of its own linear factors.

Let the quantic be

$$f = a_x^m = a_0 x_1^m + \cdots = \prod_{j=1}^{m} (r_2^{(j)} x_1 - r_1^{(j)} x_2).$$

The condition for apolarity of f with any form $g = b_x^m$ is

$$(ab)^m = a_0 b_m - m a_1 b_{m-1} + \cdots + (-1)^m a_m b_0 = (f, g)^m = 0$$

But if g is the perfect m-th power,

$$g = (r_2^{(j)} x_1 - r_1^{(j)} x_2)^m = (x r^{(j)})^m,$$

we have (cf. (88))

$$(f, g)^m = (a_x^m, (x r^{(j)})^m)^m = (-1)^m a^m_{r^{(f)}},$$

which vanishes because $(r_1^{(j)}, r_2^{(j)})$ is a root of f.

To derive another type of combinant, let f, g be two binary quartics,

$$f = a_0 x_1^4 + 4a_1 x_1^3 x_2 + \cdots. \quad g = b_0 x_1^4 + 4b_1 x_1^3 x_2 + \cdots.$$

Then the quartic $F \equiv f + kg = A_0 x_1^4 + \cdots$, has the coefficient

$$A_i = a_i + k b_i \quad (i = 0, 1, \cdots, 4).$$

The second degree invariant $i_F = A_0 A_4 - 4A_1 A_3 + 3A_2^2$ of F now takes the form

$$i + \delta i \cdot k + \frac{\delta^2 i}{|2} k^2 = i_F,$$

where δ is the Aronhold operator

$$\delta = b_0 \frac{\partial}{\partial a_0} + b_1 \frac{\partial}{\partial a_1} + b_2 \frac{\partial}{\partial a_2} + b_3 \frac{\partial}{}$$

∂a_3
$+ b_4$
∂
∂a_4
,
and
$i = a_0a_4 - 4a_1a_3 + 3a_2^2$
.
The discriminant of iF, e.g.,
G (i)2 − 2i(2i),
is a combinant of the two quartics f, g. Explicitly,

$$G = p_{04}^2 + 16p_{13}^2 - 8p_{03}p_{14} - 8p_{01}p_{34} + 12p_{02}p_{24} - 48p_{12}p_{23}.$$

Applying the translation principle to $G = 0$ we have the covariant curve

$$G(x) = |a_0a_4x|^2 + \frac{1}{16}|a_1a_3x|^2 - \frac{1}{2}|a_0a_3x|\,|a_1a_4x| - \frac{1}{2}|a_0a_1x|\,|a_3a_4x| + \frac{1}{3}|a_0a_2x|\,|a_2a_4x| - \frac{1}{12}|a_1a_2x|\,|a_2a_3x| = 0.$$

If $iF = 0$ the quartic F is said to be self-apolar, and the curve $G(x) = 0$ has the property that any tangent to it cuts the R_4 in a self-apolar range of points.

Chapter 8

SEMINVARIANTS. MODULAR INVARIANTS

8.1 Binary Semivariants

We have already called attention, in Chapter I, Section 1, VIII, to the fact that a complete group of transformations may be built up by combination of several particular types of transformations.

8.1.1 Generators of the group of binary collineations.

The infinite group given by the transformations T is obtainable by combination of the following particular linear transformations:

$t : x_1 = x, x_2 = \mu y,$
$t_1 : x = x' + y', y = y',$
$t_2 : x' = x'_1, y' = x'_1 + x'_2.$

For this succession of three transformations combines into
$x_1 = (1 +) x'_1 + x'_2, x_2 = \mu x'_1 + \mu x'_2,$
and evidently the four parameters,
$\mu_2 = \mu, {}_2 = \mu, \mu_1 = , {}_1 = (1 +),$
are independent. Hence the combination of t, t_1, t_2 is
$T : x_1 = {}_1x'_1 + \mu_1x'_2, x_2 = {}_2x'_1 + \mu_2x'_2.$

In Section 4 of Chapter VI some attention was given to fundamental systems of invariants and covariants when a form is subjected to special groups of transformations T_p. These are the formal modular concomitants. Booleans are also of this character. We now develop the theory of invariants of a binary form f subject to the special transformations t_1.

8.1.2 Definition.

Any homogeneous, isobaric function of the coefficients of a binary form f whose coefficients are arbitrary variables, which is left invariant when f is subjected to the transformation t_1 is called a seminvariant. Any such function left invariant by t_2 is called an anti-seminvariant.

In Section 2 of Chapter I it was proved that a necessary and sufficient condition that a homogeneous function of the coefficients of a form f of order m be an invariant is that it be annihilated by

$$O = ma_1 \frac{\partial}{\partial a_0} + (m-1)a_2 \frac{\partial}{\partial a_1} + \cdots + a_m \frac{\partial}{\partial a_{m-1}},$$

$$= a_0 \frac{\partial}{\partial a_1} + 2a_1 \frac{\partial}{\partial a_2} + \cdots + ma_{m-1} \frac{\partial}{\partial a_m}.$$

We now prove the following:

8.1.3 Theorem on annihilator

.

Theorem. A necessary and sufficient condition in order that a function I, homogeneous and isobaric in the coefficients of $f = a_m x$, may be a seminvariant of f is that it satisfy the linear partial differential equation

$$I = 0.$$

Transformation of $f = a_0 x_1^m + m a_1 x_1^{m-1} x_2 + \cdots$ by t_1 gives $f' = a'_0 x'^m_1 + m a'_1 x'^{m-1}_1 x'_2 + \cdots$, where

$$a'_0 = a_0,$$
$$a'_1 = a_1 + a_0 v,$$
$$a'_2 = a_2 + 2a_1 v + a_0 v^2,$$
$$\cdots\cdots\cdots$$
$$a'_m = a_m + m a_{m-1} v + \binom{m}{2} a_{m-2} v^2 + \cdots + a_0 v^m.$$

Hence

$$\frac{\partial a'_0}{\partial v} = 0, \quad \frac{\partial a'_1}{\partial v} = a'_0, \quad \frac{\partial a'_2}{\partial v} = 2a'_1, \quad \frac{\partial a'_3}{\partial v} = 3a'_2, \cdots,$$

$$\frac{\partial a'_m}{}$$

@v
= ma0m−1.
Now we have
@I(a00, a01, Â· Â· Â·)
@v
= @I
@a00
@a00
@v
+ @I
@a01
@a01
@v
+ Â· Â· Â· + @I
@a0m
@a0m
@v
= (a00
@
@a01
+ 2a01
@
@a02
+ Â· Â· Â· + ma0m−1
@
@a0m
)I =
0I(a00, Â· Â· Â·). (172)
But @I(a00,Â·Â·Â·)
@v = 0 is a necessary and sufficient condition in order that I(a00, Â· Â· Â· , a0m)
may be free from v, i.e. in order that I(a00, Â· Â· Â·) may be unaffected when we
make v = 0. But when v = 0, a0j = aj and
I(a00, Â· Â· Â· , a0m) = I(a0, Â· Â· Â· , am).

Hence @I
@v =
0(a00, Â· Â· Â·) = 0 is the condition that I(a00, Â· Â· Â·) be a seminvariant.
Dropping primes,
I(a0, Â· Â· Â·) = 0 is a necessary and sufficient condition that
I(a0, Â· Â· Â·) be a seminvariant.
8.1.4 Formation of seminvariants.
We may employ the operator advantageously in order to construct the seminvariants
of given degree and weight. For illustration let the degree be 2 and
the weight w. If w is even every seminvariant must be of the form
I = a0aw + 1a1aw−1 + 2a2aw−2 + Â· Â· Â· + 12
wa2
12
w.
Then by the preceding theorem

I = (w + 1)a0aw−1 + ((w − 1)1 + 22)a1aw−2 + Â· Â· Â· 0.
Or
w+1 = 0, (w−1)1+22 = 0, (w−2)2+33 = 0, Â· Â· Â· , 1
2w + 112
w−1+w12
w = 0.
Solution of these linear equations for 1, 2, Â· Â· Â· gives
I = a0aw − w
1a1aw−1 + w

2a2aw−2 − Â· Â· Â·
+(−1) 1
2w−1 w
1
2w − 1a12
w−1a1
2w+1 +
1
2
(−1) 1
2w w
1
2wa2
12
w.
Thus there is a single seminvariant of degree 2 for every even weight not exceeding
m.
For an odd weight w we would assume
I = a0aw + 1a1aw−1 + Â· Â· Â· + 12
(w−1)a12
(w−1)a12
(w+1).
Then
I = 0 gives
w+1 = 0, (w−1)1+22 = 0, Â· Â· Â· ,
1
2
(w+3)12
(w−3)+
1
2
(w−1)1
2 (w−1) = 0, 12
(w−1) = 0.
Hence 1 = 2 = Â· Â· Â· = 12
(w−1) = 0, and no seminvariant exists.
Thus the complete set of seminvariants of the second degree is
A0 = a20
,
A2 = a0a2 − a21,
A4 = a0a4 − 4a1a3 + 3a22
,
A6 = a0a6 − 6a1a5 + 15a2a4 − 10a23
,
A8 = a0a8 − 8a1a7 + 28a2a6 − 56a3a5 + 35a24
.
. . . .

The same method may be employed for the seminvariants of any degree and
weight. If the number of linear equations obtained from
I = 0 for the determination
is just sufficient for the determination of 1, 2, 3, Â· Â· Â· and if these
equations are consistent, then there is just one seminvariant I of the given degree
and weight. If the equations are inconsistent, save for 0 = 1 = 2 = Â· Â· Â· = 0,
there is no seminvariant. If the number of linear equations is such that one
can merely express all ’s in terms of r independent ones, then the result of
eliminating all possible ’s from I is an expression
I = 1I1 + 2I2 + Â· Â· Â· + rIr.
In this case there are r linearly independent seminvariants of the given degree

and weight. These may be chosen as
I1, I2, Â· Â· Â· , Ir.
8.1.5 Roberts' Theorem.
Theorem. If C0 is the leading coefficient of a covariant of f = a0xm1
+ Â· Â· Â· of
order !, and C! is its last coefficient, then the covariant may be expressed in
the forms
C0x!
1 + OC0
1! x!−1
1 x2 + O2C0
2! x!−2
1 x22
+ Â· Â· Â· + O!C0
!! x2, (173)

!C!
!! x!
1 +

!−1C!
(! − 1)! x!−1
1 x2 + Â· Â· Â· +

C!
1! x1x!−1
2 + C!x!
2 . (174)
Moreover, C0 is a seminvariant and C! an anti-seminvariant.
Let the explicit form of the covariant be
K = C0x!
1 + !
1C1x!−1
1 x2 + Â· Â· Â· + C!x!
2 .
Then by Chapter I, Section 2, XII,
 − x2
@
@x1K 0.
Or

C0x! + !(
C1 − C0)x!−1
1 x2 + !
2(
C2 − 2C1)x!−2
1 x22
+ Â· Â· Â·
+ !(
C!−1 − ! −Â¯ 1C!−2)x1x!−1
2 + (
C! − !C!−1)x!
2 0.

Hence the separate coefficients in the latter equation must vanish, and therefore

C0 = 0,

C1 = C0,

C2 = 2C1,
....

C!−1 = (! − 1)C!−2,

C! = !C!−1.
The first of these shows that C0 is a seminvariant. Combining the remaining ones, beginning with the last, we have at once the determination of the coefficients indicated in (174).
In a similar manner O − x1
@
@x2K 0,
and this leads to
OC0 = !C1, OC1 = (! − 1)C2, Â· Â· Â· , OC!−1 = C!,OC! = 0;
Ci = 1
!(!−1)(!−2)Â·Â·Â·(!−i+1)OiC0(i = 0, Â· Â· Â· , !).
This gives (173).
It is evident from this remarkable theorem that a covariant of a form f is completely and uniquely determined by its leading coefficient. Thus in view of a converse theorem in the next paragraph the problem of determining covariants is really reduced to the one of determining its seminvariants, and from certain points of view the latter is a much simpler problem. To give an elementary illustration let f be a cubic. Then
0 = 3a1
@
@a0
+ 2a2
@
@a1
+ a3
@
@a2
,
and if C0 is the seminvariant a0a2 − a21
we have
OC0 = a0a3 − a1a2, O2C0 = 2(a1a3 − a22
), O3C0 = 0.
Then 2K is the Hessian of f, and is determined uniquely from C0.
8.1.6 Symbolical representation of seminvariants.
The symbolical representation of the seminvariant leading coefficient C0 of any covariant K of f, i.e.
K = (ab)p(ac)q Â· Â· Â· ar
xbs
xct
x Â· Â· Â· (r + s + t + Â· Â· Â· = !),
is easily found. For, this is the coefficient of x1 in K, and in the expansion of
(ab)p(ac)q Â· Â· Â· (a1x1 + a2x2)r(b1x1 + b2x2)s Â· Â· Â·

the coefficient of x!
1 is evidently the same as the whole expression K except that
a1 replaces ax, b1 replaces bx, and so forth. Hence the seminvariant leader of K is
C0 = (ab)p(ac)q Â· Â· Â· ar
1bs
1ct
1 Â· Â· Â· (r + s + t + Â· Â· Â· a positive number).
(175)
In any particular case this may be easily computed in terms of the actual coefficients

of f (cf. Chap. III, Â§2, I).
Theorem. Every rational integral seminvariant of f may be represented as a
polynomial in expressions of the type C0, with constant coefficients.
For let be the seminvariant and
(a00, Â· Â· Â·) = (a0, Â· Â· Â·)
the seminvariant relation. The transformed of
f = (1x1 + 2x2)m
by
t1 : x1 = x01 + x02, x2 = x02,
is
f0 = [1x01 + (1 + 2)x02]m.
If the a0, a1, Â· Â· Â· in (a0, Â· Â· Â·) are replaced by their symbolical equivalents it
becomes a polynomial in 1, 2, 1, 2, Â· Â· Â· say F(1, 2, 1, 2, . . .). Then
(a00, Â· Â· Â·) = F(1, 1 + 2, 1, 1 + 2, Â· Â· Â·)
= F(1, 2, 1, 2, Â· Â· Â·).
Expansion by Taylor's theorem gives
 1
@
@2
+ 1
@
@2
+ 1
@
@2
+ Â· Â· Â·F(1, 2, 1, 2, Â· Â· Â·) = 0.
Now a necessary and sufficient condition that F should satisfy the linear partial
differential relation
F = 1
@
@2
+ 1
@
@2
+ Â· Â· Â·F = 0,
is that F should involve the letters 2, 2, . . . only in the combinations
(), (), (), Â· Â· Â· .
In fact, treating F = 0 as a linear equation with constant coefficients (1, 1, Â· Â· Â· being unaltered
under t1) we have the auxiliary equations
d2
1
= d2
1
= d2
1
= Â· Â· Â· = dF
0 .

Hence F is a function of (), (), Â· Â· Â· with constant coefficients which may
involve the constants 1, 1, Â· Â· Â· . In other words, since (a0) = F(a1, Â· Â· Â·) is
rational and integral in the a's F is a polynomial in these combinations with
coefficients which are algebraical rational expressions in the 1, 1, Â· Â· Â· . Also
every term of such an expression is invariant under t1, i.e. under
01 = 1, 02 = 1 + 2, Â· Â· Â· ,
and is of the form
 0 = ()p1 ()p2 Â· Â· Â·
1
1 Â· Â· Â· ,
required by the theorem.

We may also prove as follows: Assume that F is a function of (), (),
(), Â· Â· Â· and of any other arbitrary quantity s. Then
1
@F
@2
= 1
@F
@(
@()
@2
+ 1
@F
@()
@()
@2
+ Â· Â· Â· + 1
@F
@s
@s
@2
,
1
@F
@beta2
= 1
@F
@(
@()
@2
+ 1
@F
@()
@()
@2
+ Â· Â· Â· + 1
@F
@s
@s
@2
,
etc. But
1
@F
@()
@()
@2
= −11
@F
@() ,
1
@F
@()
@()
@2
= +11
@F
@() ,
Â· Â· Â·
Hence by summing the above equations we have

F = @F
@s 1
@s
@2
+ 1
@s
@2
+ Â· Â· Â·= @F
@s
s = 0.
Since s is entirely arbitrary we can select it so that s 6= 0. Then @F
@s = 0, and
F, being free from s, is a function of the required combinations only.
Theorem. Every seminvariant of f of the rational integral type is the leading
coefficient of a covariant of f.
It is only required to prove that for the terms 0 above w = + + Â· Â· Â· is
constant, and each index
, , Â· Â· Â·
is always a positive integer or zero. For if this be true the substitution of
x, x, Â· Â· Â· for 1, 1, Â· Â· Â· respectively in the factors
1
1 Â· Â· Â· of 0 and the other
terms of F, gives a covariant of order ! whose leading coefficient is (a0, Â· Â· Â·).
We have X 0 =X()p1 ()p2 Â· Â· Â·
1
1 Â· Â· Â· = (a0, Â· Â· Â·).

If the degree of is i, the number of symbols involved in 0 is i and its degree
in these symbols im. The number of determinant factors () Â· Â· Â· is, in general,
w = p1 + p2 + Â· Â· Â· + p1
2 i(i−1),
and this is the weight of . The degree in the symbols contributed to 0 by the
factors () Â· Â· Â· is evidently 2w, and we have , , Â· Â· Â· all positive and
im = 2w,
that is,
! = im − 2w = 0.
For a more comprehensive proof let
d = 2
@
@1
+ 2
@
@1
+ Â· Â· Â· .
Then
d − d = 1
@
@1
+ 1
@
@1
+ Â· Â· Â· − 2
@
@2 − 2
@
@2
+ Â· Â· Â· .
Hence, since 0 is homogeneous in the symbols we have by Euler's theorem,
(d − d) 0 = (w + ! − w) 0 = ! 0,

(d2 − d2) 0 = (d − d)d 0 + d(d − d) 0 = 2(! − 1)d 0,
...
...
(dk − dk) 0 = k (! − k + 1)dk−1 0(k = 1, 2, . . .),
Â· Â· Â· Â· Â· Â·
But
0 = 0, hence dk 0 = k(! − k + 1)dk−1 0.
Also
dai = dam−i
1 ai
2 = (m − i)ai−1 = Oai (i = 0, 1, . . . ,m − 1),
d = @
@a0
da0 + @
@a1
da1 + Â· Â· Â· + @
@am−1
dam−1 = O.
Hence dk 0 is of weight w + k. Then
dim−w+1 0 = 0.
For this is of weight im+1 whereas the greatest possible weight of an expression of degree i is im, the weight of ai
m.
Now assume ! to be negative. Then dim−w 0 = 0 because
dim−w+1 0 = (im − w + 1)[! − (im − w + 1) + 1]dim−w 0 = 0.
Next dim−w−1 0 = 0 because
dim−w 0 = (im − w)[! − (im − w) + 1]dim−w−1 0 = 0.
Proceeding in this way we obtain 0 = 0, contrary to hypothesis. Hence the theorem is proved.

8.1.7 Finite systems of binary seminvariants.
If the binary form f = a0xm1
+ ma1xm−1
1 + Â· Â· Â· be transformed by
x1 = x01 + x02, x2 = x02,
there will result,
f0 = C0x0m
1 + mC1x0m−1
1 x02 + m
2 C2x0m−2
1 x02
2 + Â· Â· Â· + Cmx0m
2 ,
in which
Ci = a0i + ia1i−1 + i
2a2i−2 + Â· Â· Â· + iai−1 + ai. (176)
Since
C0 =
a0 = 0, C0 is a seminvariant. Under what circumstances will all
of the coefficients Ci(i = 0, Â· Â· Â· ,m) be seminvariants? If C1 is a seminvariant

C1 =
(a0 + a1) = a0
 + a0 = 0.
That is,
 = −1. We proceed to show that if this condition is satisfied
Ci = 0
for all values of i.
Assume

= −1 and operate upon Ci by
. The result is capable of simplification
by

s = ss−1
= −ss−1,
and is

Ci = −ia0i−1 − i
1(i − 1)a1i−2 − Â· Â· Â· − i
r(i − r)ari−r−1 − Â· Â· Â·
−iai−1 + i
1a0i−1 + 2i
2a1i−2 + Â· Â· Â·
+ i
r + 1(r + 1)ari−r−1 + Â· Â· Â· + iai−1.
But
i
r + 1(r + 1) = i(i − 1) Â· Â· Â· (i − r + 1)(i − r)
r!
= i
r(i − r).
Hence
Ci = 0.
Now one value of for which
= −1 is = −a1
a0
. If f be transformed by
x1 = x01 −
a1
a0
x02, x2 = x02,

then C1 0, and all of the remaining coefficients Ci are seminvariants. Moreover,
in the result of the transformation,
i ai−1
0 Ci = ai−1
0 ai − i
1ai−2
0 ai−1a1 + i
2ai−3
0 ai−2a21
− Â· Â· Â·
+ (−1)i−2i
2a0a2ai−2
1 + (−1)i−1(i − 1)ai
1
=
i−2 Xr=2
(−1)ri
rai−r−1
0 ai−rar
1 + (−1)i−1(i − 1)ai
1.
This gives the explicit form of the seminvariants. The transformed form itself
may now be written
f0 = 0x0m
1 + m
2 2

0
x_0m-2
1 $x_0 2$
2 + m
3 3
20
x_0m-3
1 $x_0 3$
2 + Â· Â· Â· +
m
m−1
0
x_0m
2 .

Theorem. Every seminvariant of f is expressible rationally in terms of 0, 2, 3, Â· Â· Â· , m. One obtains this expression by replacing a_1 by 0, a_0 by 0, and $a_i (i 6= 0, 1)$ by
i
i−1
0
in the original form of the seminvariant. Except for a power of 0 = a_0 in the denominator the seminvariant is rational and integral in the i(i = 0, 2, Â· Â· Â· ,m) (Cayley).

In order to prove this theorem we need only note that f0 is the transformed form of f under a transformation of determinant unity and that the seminvariant, as S, is invariantive under this transformation. Hence

S 0, 0,
2
0
,
3
20
, Â· Â· Â· ,
m
m−1
0 = S(a_0, a_1, a_2, Â· Â· Â· , a_m), (177)

which proves the theorem.

For illustration consider the seminvariant

$S = a_0a_4 - 4a_1a_3 + 3a_2^2$
.

This becomes

S =
1
20
(3 22
+ 4),

or

$S = a_0a_4 - 4a_1a_3 + 3a_2^2$
=
1
a_0^2 [3($a_0a_2 - a_1^2$
)2 + (a_0^3
$a_4 - 4a_0^2$
$a_1a_3 + 6a_0a_1^2$
$a_2 - 3a_1^4$
)].

This is an identity. If the coefficients appertain to the binary quartic the equation becomes (cf. (125))

1

2a20
i = 3   22
+   4.

Again if we take for S the cubic invariant J of the quartic we obtain

1
6J =
a0 0 1
a0
     2
0 1
a0
     2
1
a20
     3
1
a0
     2
1
a20
     3
1
a30
     4

,
or

1
6a30
J =   2   4 −   32
−   23
.

Combining the two results for i and J we have

     2   4 =
1
2a20
i   2 − 3   32
=
1
6a30
J +   32
+   23
.

Now 2   2 is the seminvariant leading coefficient of the Hessian H of the quartic f, and   3 is the leader of the covariant T. In view of Roberts' theorem we may expect the several covariants of f to satisfy the same identity as their seminvariant leaders. Substituting 1
2H for   2, T for   3, and f for a0, the last equation gives

H3 +
1
3f3J + 2T2 −
1
2if2H = 0,

which is the known syzygy (cf. (140)).

8.2 Ternary Seminvariants

We treat next the seminvariants of ternary forms, Let the ternary quantic of order m be

f =Xmi
m!
m1!m2!m3!am1m2m3xm1
1 xm2
2 xm3
3 , (m1 + m2 + m3 = m).
When this is transformed by ternary collineations,
V :
x1 = 1x01 + μ1x02 + 1x03,
x2 = 2x01 + μ2x02 + 2x03,
x3 = 3x01 + μ3x01 + 3x03, (μ) 6= 0
it becomes f0, where the new coefficients a0 are of order m in the 's, μ's, and 's. This form f may be represented symbolically by
f = amx
= (a1x1 + a2x2 + a3x3)m
The transformed form is then (cf. (76))
f0 = (ax01 + aμx02 + ax03)m (178)
=Xmi
m!
m1!m2!m3!am1
am2
μ am3
μ xm1
1 xm2
2 xm3
3 (m1 + m2 + m3 = m).

Then we have
a0m1m2m3 = am1
am2
μ am3

Now let
μ
@
@ = μ1
@
@1
+ μ2
@
@2
+ μ3
@
@3
,

@
@ = 1
@
@1
+ 2
@
@2
+ 3
@
@3
(cf. (58)).
Then, evidently (cf. (75) and (231))
m!

(m − m2 − m3)!a0m1m2m3 = μ
@
@m2
@
@m3
am (m1+m2+m3 = m).
(179)
This shows that the leading coefficient of the transformed form is am , i.e. the form f itself with (x) replaced by (), and that the general coefficient results from the double ternary polarization of am as indicated by (179).

Definition.
Let be a rational, integral, homogeneous function of the coefficients of f, and 0 the same function of the coefficients of f0. Then if for any operator μ @
@ ,
@
@ , Â· Â· Â· , say for @
@μ, the relation

@
@μ0 = 0
is true, is called a seminvariant of f.

The reader should compare this definition with the analytical definition of an invariant, of Chapter I, Section 2, XI.

8.2.1 Annihilators

. A consequence of this definition is that a seminvariant satisfies a linear partial differential equation, or annihilator, analogous to in the binary theory.
For,

@
@μ0 = @0
@a0m00
@a0m00
@μ + Â· Â· Â· + @0
@a0m1m2m3
@a0m1m2m3
@μ + Â· Â· Â·
+ @0
@a000m
@a000m
@μ ,
and

@a0m1m2m3
@μ
=
@
@μam1
am2
μ am3
= m2am1+1
am2−1
μ am3
= m2a0m1+1m2−1m3 .

Hence

@
@μ0 =Xmi
m2a0m1+1m2−1m3

$$\frac{\partial 0}{\partial a^0_{m_1 m_2 m_3}} = 0 \quad (m_1 + m_2 + m_3 = m) . \qquad (180)$$

Now since the operator

$$\sum m_2 a^0_{m_1+1\, m_2-1\, m_3} \frac{\partial}{\partial a^0_{m_1 m_2 m_3}}$$

annihilates 0 then the following operator, which is ordinarily indicated by

$x_2 x_1$, is an annihilator of .

$$x_2 x_1 = \sum m_2 a_{m_1+1\, m_2-1\, m_3} \frac{\partial}{\partial a_{m_1 m_2 m_3}} \quad (m_1 + m_2 + m_3 = m) \qquad (181)$$

The explicit form of a ternary cubic is

$$f = a_{300} x_1^3 + 3a_{210} x_1^2 x_2 + 3a_{120} x_1 x_2^2 + a_{030} x_2^3 + 3a_{201} x_1^2 x_3 + 6a_{111} x_1 x_2 x_3 + 3a_{021} x_2^2 x_3 + 3a_{102} x_1 x_3^2 + 3a_{012} x_2 x_3^2 + a_{003} x_3^3 .$$

In this particular case

$$x_2 x_1 = a_{300} \frac{\partial}{\partial a_{210}} + 2a_{210} \frac{\partial}{\partial a_{120}} + 3a_{120} \frac{\partial}{\partial a_{080}} + a_{201} \frac{\partial}{\partial a_{111}} + 2a_{111} \frac{\partial}{\partial a_{021}} + a_{102} \frac{\partial}{\partial a_{012}} . \qquad (182)$$

This operator is the one which is analogous to in the binary theory. From μ $\frac{\partial}{\partial}$ 0 by like processes, one obtains the analogue of O, e.g. $x_1 x_2$. Similarly

$x_1 x_3$, $x_3 x_1$, $x_2 x_3$, $x_3 x_2$ may all be derived. An independent set of these six operators characterize full invariants in the ternary theory, in the same sense that ,O characterize binary invariants. For such we may choose the cyclic set

$$x_1\frac{\partial}{\partial x_2},\quad x_2\frac{\partial}{\partial x_3},\quad x_3\frac{\partial}{\partial x_1}.$$

Now let the ternary m-ic form

$$f = a_{m00}x_1^m + ma_{m-110}x_1^{m-1}x_2 + \cdots + a_{0m0}x_2^m + m(a_{m-101}x_1^{m-1} + (m-1)a_{m-211}x_1^{m-2}x_2 + \cdots + a_{0m-11}x_2^{m-1})x_3 + \cdots\cdots\cdots$$

be transformed by the following substitutions of determinant unity:

$$x_1 = x_1' - \frac{a_{m-110}}{a_{m00}}x_2' - \frac{a_{m-101}}{a_{m00}}x_3',\quad x_2 = x_2',\quad x_3 = x_3'. \qquad (183)$$

Then the transformed form f' lacks the terms $x_1'^{m-1}x_2'$, $x_1'^{m-1}x_3'$. The coefficients of the remaining terms are seminvariants. We shall illustrate this merely.

Let $m = 2$,

$$f = a_{200}x_1^2 + 2a_{110}x_1x_2 + a_{020}x_2^2 + 2a_{101}x_1x_3 + 2a_{011}x_2x_3 + a_{002}x_3^2.$$

Then

$$a_{200}f' = a_{200}^2x_1'^2 + (a_{020}a_{200} - a_{110}^2)x_2'^2 + 2(a_{011}a_{200} - a_{101}a_{110})x_2'x_3' + (a_{002}a_{200} - a_{101}^2)x_3'^2.$$

It is easy to show that all coefficients of f' are annihilated by $x_2\frac{\partial}{\partial x_1}$.

Likewise if the ternary cubic be transformed by

$$x_1 = x_1' - \frac{a_{210}}{a_{300}}x_2' - \frac{a_{201}}{a_{300}}x_3',\quad x_2 = x_2',\ x_3 = x_3',$$

and the result indicated by

$$a_{300}^2f' = A_{300}x_1'^3 + 3A_{210}x_1'^2$$

1 x02 + . . ., we have
A300 = a3
300, (184)
A210 = 0,
A120 = a300 a300a120 − a2
210,
A030 = 2a3
210 − 3a210a120a300 + a030a2
300,
A201 = 0,
A111 = a300 (a300a111 − a210a201) ,
A021 = a2
300a021 − a300a201a120 − 2a210a111a300 + 2a2
210a201,
A102 = a300 a300a102 − a2
201,
A012 = a2
300a012 − a300a102a210 − 2a300a201a111 + 2a2
201a210,
A003 = 2a3
201 − 3a300a201a102 + a003a2
300.
These are all seminvariants of the cubic. It will be noted that the vanishing of
a complete set of seminvariants of this type gives a (redundant) set of sufficient
conditions that the form be a perfect mth power. All seminvariants of f are
expressible rationally in terms of the A's, since f0 is the transformed of f by a
transformation of determinant unity.
8.2.2 Symmetric functions of groups of letters.
If we multiply together the three linear factors of
f = (1)
1 x1 + (1)
2 x2 + (1)
3 x3(2)
1 x1 + (2)
2 x2 + (2)
3 x3(3)
1 x1 + (3)
2 x2 + (3)
3 x3

the result is a ternary cubic form (a 3-line), f = a300x31
+ . . . . The coefficients
of this quantic are
a300 =X(1)
1 (2)
1 (3)
1 = (1)
1 (2)
1 (3)
1 ,
a210 =X(1)
1 (2)
1 (3)
2 = (1)
1 (2)
1 (3)
2 + (1)
1 (2)
2 (3)

1 + (1)
2 (2)
1 (3)
1 ,
a120 =X(1)
1 (2)
2 (3)
2 = (1)
1 (2)
2 (3)
2 + (1)
2 (2)
1 (3)
2 + (1)
2 (2)
2 (3)
1 ,
a030 =X(1)
2 (2)
2 (3)
2 = (1)
2 (2)
2 (3)
2 ,
a201 =X(1)
1 (2)
1 (3)
3 = (1)
1 (2)
1 (3)
3 + (1)
1 (2)
3 (3)
1 + (1)
3 (2)
1 (3)
1 ,
a111 =X(1)
1 (2)
2 (3)
3 = (1)
1 (2)
2 (3)
3 + (1)
1 (2)
3 (3)
2 + (1)
2 (2)
1 (3)
3
+ (1)
2 (2)
3 (3)
1 + (1)
3 (2)
1 (3)
2 + (1)
3 (2)
2 (3)

1 ,
a021 = (1)
2 (2)
2 (3)
3 = (1)
1 (2)
3 (3)
3 + (1)
2 (2)
3 (3)
2 + (1)
3 (2)
2 (3)
2 ,
a102 = (1)
1 (2)
3 (3)
3 = (1)
1 (2)
3 (3)
3 + (1)
3 (2)
1 (3)
3 + (1)
3 (2)
3 (3)
1 ,
a012 = (1)
2 (2)
3 (3)
3 = (1)
2 (2)
3 (3)
3 + (1)
3 (2)
2 (3)
3 + (1)
3 (2)
3 (3)
2 ,
a003 = (1)
3 (2)
3 (3)
3 = (1)
3 (2)
3 (3)
3 .
These functions are all unaltered by those interchanges of letters which have the effect of permuting the linear factors of f among themselves. Any function of the (j)
i having this property is called a symmetric function of the three groups of three homogeneous letters,
(1)
1 , (1)
2 , (1)
3 ,
(2)
1 , (2)
2 , (2)

3 ,
(3)
1 , (3)
2 , (3)
3 ,
In general, a symmetric function of m groups of three homogeneous letters,
1, 2, 3, i.e. of the groups
1 (1)
1 , (1)
2 , (1)
3 ,
2 (2)
1 , (2)
2 , (2)
3 ,
. . .
m (m)
1 , (m)
2 , (m)
3 ,
is such a function as is left unaltered by all of the permutations of the letters
which have the effect of permuting the groups 12, . . . , m among themselves:
at least by such permutations. This is evidently such a function as is left unchanged
by all permutations of the superscripts of the ’s. A symmetric function
of m groups of the three letters 1, 2, 3, every term of which involves as a

factor one each of the symbols (1), (2), . . . , (m) is called an elementary symmetric
function. Thus the set of functions a310, a210, . . . above is the complete
set of elementary symmetric functions of three groups of three homogeneous
variables. The non-homogeneous elementary symmetric functions are obtained
from these by replacing the symbols (1)
3 (2)
3 (3)
3 each by unity.
The number N of elementary symmetric functions of m groups of two nonhomogeneous
variables am,0,0,ami − 1, 1, 0 Â· Â· Â· is, by the analogy with the coefficients
of a linearly factorable ternary form of order m,
N = m + m + (m − l) + (m − 2) + Â· Â· Â· + 2 + l =
1
2m(m + 3).
The N equations aijk = , regarded as equations in the 2m unknowns a(r)
1 , a(s)
2 (r, s =
1, Â· Â· Â· ,m), can, theoretically, be combined so as to eliminate these 2munknowns.
The result of this elimination will be a set of
1
2m(m + 3) − 2m =
1
2m(m − 1)
equations of condition connecting the quantities am00, am−110, Â· Â· Â· only. If these
a’s are considered to be coefficients of the general ternary form f of order m,
whose leading coefficient a003 is unity, the 1
2m(m − 1) equations of condition
constitute a set of necessary and sufficient conditions in order that f may be
linearly factorable.
Analogously to the circumstances in the binary case, it is true as a theorem
that any symmetric function of m groups of two non-homogeneous variables
is rationally and integrally expressible in terms of the elementary symmetric
functions. Tables giving these expressions for all functions of weights 1 to 6

inclusive were published by Junker[1] in 1897.

8.2.3 Semi-discriminants

We shall now derive a class of seminvariants whose vanishing gives a set of conditions in order that the ternary form f of order m may be the product of m linear forms.

The present method leads to a set of conditional relations containing the exact minimum number $\frac{1}{2}m(m-1)$; that is, it leads to a set of $\frac{1}{2}m(m-1)$ independent seminvariants of the form, whose simultaneous vanishing gives necessary and sufficient conditions for the factorability. We shall call these seminvariants semi-discriminants of the form. They are all of the same degree $2m - 1$; and are readily formed for any order m as simultaneous invariants of a certain set of binary quantics related to the original ternary form.

If a polynomial, f_3^m, of order m, and homogeneous in three variables (x_1, x_2, x_3) is factorable into linear factors, its terms in (x_1, x_2) must furnish the (x_1, x_2) terms of those factors. Call these terms collectively a_{m0}^x, and the terms linear in

1 Wiener Denkschriften for 1897.

x_3 collectively $x_3 a_{m-1,1}^x$. Then if the factors of the former were known, and were distinct, say

$$a_{m0}^x = a_{00} \prod_{i=1}^m (r_2^{(i)} x_1 - r_1^{(i)} x_2) \div \prod_{i=1}^m (r_2^{(i)}),$$

the second would give by rational means the terms in x_3 required to complete the several factors. For we could find rationally the numerators of the partial fractions in the decomposition of $a_{m-1,1}^x / a_{m0}^x$, viz.

$$\frac{a_{m-1,1}^x}{a_{m0}^x} \equiv \frac{\prod_{i=1}^m r_2^{(i)}}{a_{00}} \sum_{i=1}^m \frac{\alpha_i}{r_2^{(i)} x_1 - r_1^{(i)} x_2},$$

and the factors of the complete form will be, of course,

$$r_2^{(i)} x_1 - r_1^{(i)} x_2 + \alpha_i x_3 \quad (i = 1, 2, \cdots, m).$$

Further, the coefficients of all other terms in f_3^m are rational integral functions of the $r^{(i)}$ on the.one hand, and of the α_i on the other, symmetrical in the sets $(r_2^{(i)}, -r_1^{(i)}, \alpha_i)$. We shall show in general that all these coefficients in the case of any linearly factorable form are rationally expressible in terms of those occurring in a^m_{0x}, a^{m-1}_{1x}. Hence will follow the important theorem,

Theorem. If a ternary form f_3^m is decomposable into linear factors, all its coefficients, after certain $2m$, are expressible rationally in terms of those $2m$ coefficients. That is, in the space whose coördinates are all the coefficients of ternary forms of order m, the forms composed of linear factors fill a rational spread of $2m$ dimensions.

We shall thus obtain the explicit form of the general ternary quantic which is factorable into linear factors. Moreover, in case f_3^m is not factorable a similar development will give the theorem,

Theorem. Every ternary form f_3^m, for which the discriminant D of a^m_{0x} does not vanish, can be expressed as the sum of the product of m distinct linear forms, plus the square of an arbitrarily chosen linear form, multiplied by a "satellite" form of order $m - 2$ whose coefficients are, except for the factor D^{-1}, integral rational seminvariants of the original form f_3^m.

A class of ternary seminvariants

Let us write the general ternary quantic in homogeneous variables as follows:

$$f_3^m = a^m_{0x} + a^{m-1}_{1x}x_3 + a^{m-2}_{2x}x_3^2 + \cdots + a^0_{mx}x_3^m,$$

where

$$a^{m-i}_{ix} \equiv a_{i0}x_1^{m-i} + a_{i1}x_1^{m-i-1}x_2 + a_{i2}x_1^{m-i-2}x_2^2 + \cdots + a_{i\,m-i}x_1^{m-i} \quad (i = 0, 1, 2, \cdots, m).$$

Then write

$$\frac{a^{m-1}_{1x}}{a^m_{0x}} = \frac{a^{m-1}_{1x}}{\prod_{k=1}^{m}(r_2^{(k)}x_1 - r_1^{(k)}x_2)} = \sum_{k=1}^{m} \frac{a_k}{r_2^{(k)}x_1 - r_1^{(k)}x_2} \quad (a_{00} = r_2^{(1)} r_2^{(2)} \cdots r_2^{(m)});$$

and we have in consequence, assuming that $D \neq 0$, and writing

$a'^m_{0r^{(k)}} = \partial a^m_{0x}$

$\partial x_1 \big)_{x_1=r_1^{(k)},\,x_2=r_2^{(k)}}$, $a''_{m0}(r^{(k)}) = \left(\dfrac{\partial a_{m0}^x}{\partial x_2}\right)_{x_1=r_1^{(k)},\,x_2=r_2^{(k)}}$,

the results

$$a_k = r_2^{(k)}\, a_{m-1\,1}r^{(k)}/a'_{m0}r^{(k)} = -r_1^{(k)}\, a_{m-1\,1}r^{(k)}/a''_{m0}r^{(k)}. \quad (185)$$

Hence also

$$a''_{m0}r^{(k)} = -\frac{r_1^{(k)}}{r_2^{(k)}}\, a'_{m0}r^{(k)}. \quad (186)$$

The discriminant of a_{m0}^x can be expressed in the following form:

$$D = \prod_{j=1}^{m} a'_{m0}r^{(j)}/a_{00}(-1)^{\frac{1}{2}m(m-1)}, \quad (187)$$

and therefore

$$a_k = \frac{r_2^{(k)}\, a_{m-1\,1}r^{(k)}\, a'_{m0}r^{(1)}\, a'_{m0}r^{(2)} \cdots a'_{m0}r^{(k-1)}\, a'_{m0}r^{(k+1)} \cdots a'_{m0}r^{(m)}}{a_{00}(-1)^{\frac{1}{2}m(m-1)}D}, \quad (188)$$

and in like manner we get

$$\prod_{k=1}^{m} a_k = a_{m-1\,1}r^{(1)}\, a_{m-1\,1}r^{(2)} \cdots a_{m-1\,1}r^{(m)}/(-1)^{\frac{1}{2}m(m-1)}D. \quad (189)$$

The numerator of the right-hand member of this last equality is evidently the

resultant (say Rm) of am0
x and am−1
1x .
Consider next the two differential operators
1 = ma00
@
@a10
+ (m − 1)a01
@
@a11
+ Â· Â· Â· + a0m−1
@
@a1m−1
,
2 = ma0m
@
@a1m−1
+ (m − 1)a0m−1
@
@a1m−2
+ Â· Â· Â· + a01
@
@a10
;
and particularly their effect when applied to am−1
1x . We get (cf. (186))
1am−1
1r(k) = a0m
0r(k) ,2am−1
1r(k) = a00m
0r(k) = −
r(k)
1
r(k)
2
a0m
0r(k) , (190)
and from these relations we deduce the following:
1
mYk=1
ak,=
1Rm
(−1) 1
2m(m−1)1!D
= a00Xam−1
1r(1)am−1
1r(2) Â· Â· Â· am−1
1r(m−1)
a0m
0r(1)a0m
0r(2) Â· Â· Â· a0m
0r(m−1)
, (191)
or, from (185)
1Rm
(−1) 1
2m(m−1)1!D
=X12 Â· Â· Â· m−1r(m)
2 . (192)

In (191) the symmetric function P is to be read with reference to the r’s, the superscripts of the r’s replacing the subscripts usual in a symmetric function.

Let us now operate with 2 on both members of (191). This gives

12Rm
(−1) 1
2m(m−1)1!D
= a00Xam−1
1r(1)am−1
1r(2) Â· Â· Â· am−1
1r(m−2)
a0m
0r(1)a0m
0r(2) Â· Â· Â· a0m
0r(m−2) −
r(m−1)
1
r(m−1)
2
−
r(m)
1
r(m)
2 !.

Let h represent an elementary symmetric function of the two groups of homogeneous variables r1, r2 which involves h distinct letters of each group, viz.

r(m−l+l)
i (j = 1, 2, Â· Â· Â· , h). Then we have

12Rm
(−1) 1
2m(m−1)1!1!D
= h(−1)12 Â· Â· Â· m−22r(m−1)
1 r(m)
2 i. (193)

We are now in position to prove by induction the following fundamental formula:

m−s−t
1 t
2Rm
(−1)12
m(m−1)(m − s − t)!t!D
(194)
= h(−1)t12 Â· Â· Â· sm−sr(s+1)
1 r(s+2)
1 Â· Â· Â· r(s+t)
1 r(s+2) Â· Â· Â· r(s+t)
1 r(s+t+1)
2 Â· Â· Â· r(m)
2 i (s = 0, 1, Â· Â· Â· ,m; t = 0, 1, Â· Â· Â· ,m − s),

where the outer summation covers all subscripts from 1 to m, superscripts of the r’s counting as subscripts in the symmetric function. Representing by Jm−s−t,t the left-hand member of this equality we have from (190)

2Jm−s−t,t = ((−1)t+1 am−1
1r(1)am−1
1r(2) Â· Â· Â· am−1
1r(s−1)
alm
0r(1)alm
0r(2) Â· Â· Â· alm
0r(s−1)

Ã— r(1)
2 r(2)
2 Â· Â· Â· r(s)
2
r(s)
1
r(s)
2
m−sr(s+1)
1 Â· Â· Â· r(s+t)
1 r(s+t+1)
2 Â· Â· Â· r(m)
2).
This equals
(−1)t+112 Â· Â· Â· s−1S,
where S is a symmetric function each term of which involves t + 1 letters r1
and m − s − t letters r2. The number of terms in an elementary symmetric
function of any number of groups of homogeneous variables equals the number
of permutations of the letters occurring in any one term when the subscripts
(here superscripts) are removed. Hence the number of terms in m−s is
(m − s)!
(m − s − t)!t! ,
and the number of terms in S is
(m − s + 1)(m − s)!/t!(m − s − t)!.

But the number of terms in
m−s+1 r(s)
1 r(s+1)
1 Â· Â· Â· r(s+t)
1 r(s+t+1)
2 Â· Â· Â· r(m)
2
is
(m − s + 1)!/(m − s − t)!(t + 1)!.
Hence
S = (t + 1)m−s+1,
and so
2Jm−s−t,t
t + 1
=X(−1)t+122 Â· Â· Â· s−1m−s+1.
This result, with (193), completes the inductive proof of formula (194).
Now the functions Jm−s−t,t are evidently simultaneous invariants of the
binary forms am0
x, a0m
0x , a00m
0x , am−1
1x . We shall show in the next paragraph that
the expressions
Im−s−t,t Dast − DJm−s−t,t(s = 2, 3, Â· Â· Â· ,m; t = 0, 1, Â· Â· Â· ,m − s)
are, in reality, seminvariants of the form f3m as a whole.
STRUCTURE OF A TERNARY FORM
The structure of the right-hand member of the equality (194) shows at once that
the general (factorable or non-factorable) quantic f3m(D 6= 0) can be reduced
to the following form:
f3m =
mYk=1 r(k)
2 x1 − r(k)
1 x2 + k+
m Xs=2

m−s Xt=0
(ast − Jm−s−t,t)xm−s−t
1 xt
2. (195)
This gives explicitly the “satellite” form of f3m, with coefficients expressed rationally
in terms of the coefficients of f3m. It may be written
Dμm−2 =
m Xs=2
m−s Xt=0 Dast −
m−s−t
1 t
2Rm
(−1)12
m(m−1)(m − s − t)! t!xm−s−t
1 xt
2
=
m Xs=2
m−s Xt=0
Im−s−t,txm−s−t
1 xt
2.
(196)
Now the coefficients Im−s−t,t are seminvariants of f3m. To fix ideas let m = 3
and write the usual set of ternary operators,

x1x2 = 01
@
@a00
+ 202
@
@a01
+ 303
@
@a02
+ 11
@
@a10
+ 212
@
@a11
+ 313
@
@a20
,

x2x1 = 300
@
@a01
+ 201
@
@a02
+ 02
@
@a03
+ 210
@
@a11
+ 11

@
@a12
+ 20
@
@a21
,

x3x1 = 20
@
@a30
+ 210
@
@a20
+ 300
@
@a10
+ 11
@
@a21
+ 201
@
@a11
+ 02
@
@a12
,
etc.
Then I10 is annihilated by
x2x1 , but not by
x1x2 , I01 is annihilated by

x1x2 but not by
x2x1 , and I00, is annihilated by
x1x2 , but not by
x3x1 . In
general Im−s−t,t fails of annihilation when operated upon by a general operator

xixj which contains a partial derivative with respect to ast. We have now
proved the second theorem.

8.2.4 The semi-discriminants

A necessary and sufficient condition that f3m should degenerate into the product
of m distinct linear factors is that μm−2 should vanish identically. Hence, since
the number of coefficients in μm−2 is 1
2m(m − 1), these equated to zero give a
minimum set of conditions in order that f3m should be factorable in the manner
stated. As previously indicated we refer to these seminvariants as a set of semidiscriminants
of the form f3m. They are
Im−s−t,t = Dast −
m−s−t
1 t
2Rm
(−1) 1
2m(m−1)t!(m − s − t)! s = 2, 3, . . . ,m;
t = 0, 1, . . . ,m − s (197)
They are obviously independent since each one contains a coefficient (ast) not
contained in any other. They are free from adventitious factors, and each one
is of degree 2m − 1.
In the case where m = 2 we have

I00 = −a20
2a00 a01
a01 2a02 +
a00 a01 a02
a10 a11 0
0 a10 a11

.
This is also the ordinary discriminant of the ternary quadratic.
The three semi-discriminants of the ternary cubic are given in Table V. In
this table we have adopted the following simpler notation for the coefficients of
f:
f = a0x31
+ a1x21
x2 + a2x1x22
+ a3x32
+ b0x21
x3 + b1x1x2x3 + b2x22
x3
+ c0x1x23
+ c1x2x23
+ d0x33.
In the notation of (197) the seminvariants in this table are
I00 = Da30 + R3,
I10 = Da20 + 1R3,
I01 = Da21 + 2R3,

TABLE V
I10 −I01 −I00
4a1a2a3b20
a22
a3b20
a23
b20
−a32
b20
−3a1a23
b20
+a1a3b0b21
−9a23
b20
+a21
a3b21
+a21
b0b22
+3a1a3b21
−3a2a3b21
−2a2b0b22
−a22
b21
+a31
b22
−a2a3n20
b1
+3a2b22
−4a1a2b22
+a22
b20
b2

+6a1a3b0b2 +9a3b22
−2a1a3b20
b2
−2a22
b0b2 −a1a2a3b0b1 −a1a2b0b1b2
+a1a22
b0b1 +2a21
a3b0b2 −a2b31
+3a2a3b0b1 −6a2a3b0b2 −a1b1b22
−4a21
a3b0b1 +4a22
b1b2 +a2b21
b2
+a1a2b1b2 −3a1a3b1b2 +b32
−9a3b1b2 −a21
a2b1b2 +a21
a22
d0
−a21
a22
c0 +a21
a22
c1 +18a1a2a3d0
−18a1a2a3c0 +18a1a2a3c1 −4a32
d0
+4a32
c0 −4a32
c1 −4a31
a3d0
+4a31
a3c0 −4a31
a3c1 −27a23
d0
+27a23
c0 −27a23
c1
where D is the discriminant of
 a00x31
+ a01x21
x2 + Â· Â· Â· + a03x32
,
and R3 the resultant of and
 = a10x21
+ a11x1x2 + a12x22
.
Corresponding results for the case m = 4 are the following:
I00 =
1
27a40(4i31
− J2
1) − R4,
where
i1 = a2
02 − 3a01a03 + 12a00a04,
J1 = 27a2
01a04 + 27a00a2
03 + 2a3
02 − 72a00a02a04 − 9a01a02a03,
R4 =

a_{10} a_{11} a_{12} a_{13} 0 0
0 a_{10} a_{11} a_{12} a_{13} 0
0 0 a_{10} a_{11} a_{12} a_{13}
$a_{01}a_{10} - a_{00}a_{11}$ $a_{02}a_{10} - a_{00}a_{12}$ $a_{03}a_{10} - a_{00}a_{13}$ $a_{04}a_{10}$ 0 0
a_{00} a_{01} a_{02} a_{03} a_{04} 0
0 a_{00} a_{01} a_{02} a_{03} a_{04}

,

the other members of the set being obtained by operating upon R_4 with powers of 1, 2:

$$1 = 4a_{00}\frac{@}{@a_{10}} + 3a_{01}\frac{@}{@a_{11}} + 2a_{02}\frac{@}{@a_{12}} + a_{03}\frac{@}{@a_{13}},$$

$$2 = 4a_{04}\frac{@}{@a_{13}} + 3a_{03}\frac{@}{@a_{12}} + 2a_{02}\frac{@}{@a_{11}} + a_{01}\frac{@}{@a_{10}},$$

according to the formula

$$I_{4-s-t,t} = a_{st}D - \frac{{}^{4-s-t}_{1}\,{}^{t}_{2}R_4}{(4-s-t)!t!}$$

$(s = 2, 3, 4;\ t = 0, 1, ..., 4 - s)$.

8.2.5 Invariants of m-lines.

The factors of a^m_{0x} being assumed distinct we can always solve $I_{m-s-t,t} = 0$ for a_{st}, the result being obviously rational in the coefficients occurring in a^m_{0x}, a^{m-1}_{1x}. This proves the first theorem of III as far as the case $D \neq 0$ is concerned. Moreover by carrying the resulting values of $a_{st}(s = 2, 3, \ldots, m;\ t = 0, 1, \ldots, m - s)$ back into f_{3m} we get the general form of a ternary quantic which is factorable into linear forms. In the result a^m_{0x}, a^{m-1}_{1x} are perfectly general (the former, however, subject to the negative condition $D \neq 0$), whereas

(−1) 1

2m(m−1)Dam−j
jx
m−j
1 Rm
(m − j)! xm−j
1 +
m−j−1
1 2Rm
(m − j − 1)!1! xm−j−1
1 x2 + Â· Â· Â·
+
m−j
2 Rm
(m − j)! xm−j
2 (j = 2, 3, . . . ,m).

Thus the ternary form representing a group of m straight lines in the plane, or in other words the form representing an m-line is, explicitly,

f = am0
x + x3am−1
1x
+ D−1(−1) 1
2m(m−1)
m Xj=2
xj
3
m−j Xi=0
m−i−j
1 i
2Rm
(m − i − j)!i! xm−i−j
1 xi
2. (198)

This form, regarded as a linearly factorable form, possesses an invariant theory, closely analogous to the theory of binary invariants in terms of the roots.

If we write a30
x = x32
l0x1/x2 , a21
x = x22
l1x1/x2 (a00 = 1), and assume that the roots of l0 = 0 are −r1, −r2, −r3, then the factored form of the three-line will be, by the partial fraction method of III (185),

f =
3 Yi=1
(x1 + rix2 − l1−ri /l00 −ri).

Hence the invariant representing the condition that the 3-line f should be a pencil of lines is

Q =
1 r1 l1−r1 /l00 −r1
1 r2 l1−r2 /l00 −r2
1 r3 l1−r3 /l00 −r3

.

This will be symmetric in the quantities r1, r2, r3 after it is divided by pR, where R = (rl − r2)2(r2 − r3)2(r3 − r1)2 is the discriminant of the binary cubic a30
x. Expressing the symmetric function Q1 = Q/pR, in terms of the coefficients of a30
x, we have

$$Q_1 = 2a_{01}^2 a_{12} - a_{01}a_{02}a_{11} + 9a_{00}a_{03}a_{11} - 6a_{01}a_{03}a_{10} + 2a_{02}^2 a_{10} - 6a_{00}a_{02}a_{12}.$$

This is the simplest full invariant of an m-line f.

8.3 Modular Invariants and Covariants

Heretofore, in connection with illustrations of invariants and covariants under the finite modular linear group represented by T_p, we have assumed that the coefficients of the forms were arbitrary variables. We may, however, in connection with the formal modular concomitants of the linear form given in Chapter VI, or of any form f taken simultaneously with L and Q, regard the coefficients of f to be themselves parameters which represent positive residues of the prime number p. Let f be such a modular form, and quadratic,

$$f = a_0x_1^2 + 2a_1x_1x_2 + a_2x_2^2.$$

Let p = 3. In a fundamental system of formal invariants and covariants modulo 3 of f we may now reduce all exponents of the coefficients below 3 by Fermat's theorem,

$$a_i^3 \equiv a_i \pmod 3 \quad (i = 0, 1, 2).$$

The number of individuals in a fundamental system of f is, on account of these reductions, less than the number in the case where the a's are arbitrary variables. We call the invariants and covariants of f, where the a's are integral, modular concomitants (Dickson). The theory of modular invariants and covariants has been extensively developed. In particular the finiteness of the totality of this type of concomitants for any form or system of forms has been proved. The proof that the concomitants of a quantic, of the formal modular type, constitute a finite, complete system has, on the contrary, not been accomplished up to the present (December, 1914). The most advantageous method for evolving fundamental systems of modular invariants is one discovered by Dickson depending essentially upon the separation of the totality of forms f with particular integral coefficients modulo p into classes such that all forms in a class are permuted among themselves by the transformations of the modular group given by T_p

2. The presentation of the elements of this modern theory is beyond the scope of this book. We shall, however, derive by the transvection process the fundamental system of modular concomitants of the quadratic form f, modulo 3. We have by transvection the following results (cf. Appendix, 48, p. 241):

TABLE VI

Notation	Transvectant	Concomitant (Mod 3)
	$(f, f)^2$	$a_1^2 - a_0a_2$
q	$(f^3, Q)^6$	$a_0^2a_2 + a_0a_2^2 + a_0a_1^2 + a_1^2a_2 - a_0^3 - a_2^3$
L		$x_1^3x_2 - x_1x_2^3$
Q	$((L, L)^2 L)$	$x_1^6 + x_1^4x_2^2 + x_1^2x_2^4 + x_2^6$
f		$a_0x_1^2 + 2a_1x_1x_1 + a_2x_2^2$

f4 (f,Q)2 a0x41
+ a1x31
x2 + a1x1x32
+ a2x42
C1 (f3,Q)5 (a20
a1 − a31
)x21
+ (a0 − a2)(a21
+ a0a2)x1x2 + (a31
− a1a22
)x22
C2 (f2,Q)4 (a20
+ a21
− a0a2)x21
+ a1(a0 + a2)x1x2 + (a21
+ a22
− a0a2)x22

8.3.1 Fundamental system of modular quadratic form, modulo 3.

Also in q and C1 we may make the reductions a ai(mod3)(i = 0, 1, 2). We now give a proof due to Dickson, that these eight forms constitute a fundamental system of modular invariants and covariants of f.

Much use will be made, in this proof, of the reducible invariant

I = (a20
− 1)(a21
− 1)(a22
− 1) q2 + 2 − 1 (mod 3).

In fact the linearly independent invariants of f are

1,, I, q,2. (i)

Proceeding to the proposed proof, we require the seminvariants of f. These are the invariants under

x1 x01 + x02, x2 x02 (mod 3).

These transformations replace f by f0, where

a00 a0, a01 a0 + a1, a02 a0 − a1 + a2 (mod 3). (t)

Hence, as may be verified easily, the following functions are all seminvariants:

a0, a20
, a0, a02, a20
,B = (a20
− 1)a1. (s)

Theorem. Any modular seminvariant is a linear homogeneous function of the eleven linearly independent seminvariants (i), (s).

For, after subtracting constant multiples of these eleven, it remains only to consider a seminvariant

S = 1a1a22
+ 2a1a2 + 3a1 + 4a21
a22
+ 5a21
a2 + 6a21
+ a22
+ a2,

[2]Transactions American Math. Society, Vol. 10 (1909), p. 123.

in which 1, 2 are linear expressions in a20
, a0, 1; and 3, . . . , 6 are linear expressions in a0, 1; while the coefficients of these linear functions and , are constants independent of 0, 1, 2. In the increment to S under the above induced transformations (t) on the a's the coefficient of a1a22
is −a04, whence

4 = 0. Then that of a21

a2 is 1 0; then that of a1a2 is − a05, whence
  5 0; then that of a21
is −2 0; then that of a1 is − − a06, whence
  6 0. Now S = 3a1, whose increment is 3a0, whence 3 0 Hence
the theorem is proved.
Any polynomial in , I, q, a0,B is congruent to a linear function of the eleven seminvariants (i), (s) by means of the relations
I2 −I, q2 I − 2 + 1,
(A) I Iq Ia0 IB q qB a0B 0,
B B, a20
2 2 + a20
 − ,
a0q a20
2 − a20
,B2 (1 − a20
)
9>
>=>>;
(mod 3),
together with a30
 a0,3   (mod 3).
Now we may readily show that any covariant, K, of order 6t is of the form P + LC, where C is a covariant of order 6t − 4 and P is a polynomial in the eight concomitants in the above table omitting f4. For the leading coefficient of a modular covariant is a modular seminvariant. And if t is odd the covariants
if 3t, iQt,C3t
1 ,C3t
2 , (i an invariant)
have as coefficients of x6t
1
a0i, i,B, + a20
,
respectively. The linear combinations of the latter give all of the seminvariants (i), (s). Hence if we subtract from K the properly chosen linear combination
the term in x6t
1 cancels and the result has the factor x2. But the only covariants
having x2 as a factor are multiples of L. Next let t be even. Then
f3t,f3t, iQf3t−3,QC3t−3
1 , i1Qt, i = 1,,2.
i1 = I,,2, q.
have as coefficients of x6t
1
a20
, a20
, a0i,B, i1.
Lemma 6. If the order ! of a covariant C of a binary quadratic form modulo 3 is not divisible by 3, its leading coefficient S is a linear homogeneous function of the seminvariants (i), (s), other than 1, I, q.
In proof of this lemma we have under the transformation x1 x10+x20, x2 x20,
C = Sx!
1 + S1x!−1
1 x2 + . . . Sx0!
1 + (S1 + !S)x0!−1
1 x02 + ...
For a covariant C the final sum equals
Sx0!
1 + S01 x0!−1
1 x02 + . . . , S01 = S1(a00, a01, a02),

where a00, . . . are given by the above induced transformation on the a’s. Hence
S01 − S1 !S(mod 3).
Now write
S1 = ka20
a21
a22
+ t (t of degree < 6),
and apply the induced transformations. We have
S01 = ka20
(a0 + a1)2(a0 − a1 + a2)2 + t0 ka20
(a0r + a21
+ a1a2 + a21
a22
) + t0,
where r is of degree 3 and t0 of degree < 6. Hence
!S k(aor + a20
a21
+ a20
a1a2) + t0 − t (mod 3).
Since ! is prime to 3, S is of degree < 6. Hence S does not contain the term
a20
a21
a22
, which occurs in I but not in any other seminvariant (i), (s). Next if
S = 1 + , where is a function of a0, a1, a2, without a constant term, IC is a
covariant C0 with S0 = I. Finally let S = q +1 +2+32 +tB where t is
a constant and the i are functions of a0. Then by (A)
qS = I − 2 + 1 + 1q,
which has the term a20
a21
a22
(from I). The lemma is now completely proved.
Now consider covariants C of order ! = 6t + 2. For t odd, the covariants
f3t+1,Qtf,C3t+1
2 , f3tC2,C3t
1 C2,
have as coefficients of x!
1
a20
, a0,2 − a20
 + a20
, a0 + a0,B,
respectively. Linear combinations of products of these by invariants give the
seminvariants (s) and ,2. Hence, by the lemma, c P + LC0, where P is a
polynomial in the covariants of the table omitting f4. For t even the covariants
fQt, f4Qt−1,C2Qt,C1Qt
have a0, a20
,Delta + a20
,B as coefficients of x!
1 .
Taking up next covariants C of order ! = 6t + 4 coefficients of x!
1 in
f4Qt, f2Qt,C1C2Qt,C2
1Qt
are, respectively, a0, a20
,B, − a20
. Linear combinations of their products by
invariants give all seminvariants not containing 1, I, q. Hence the eight concomitants
of the table form a fundamental system of modular concomitants of

f (modulo 3). They are connected by the following syzygies:
fC1 2(2 +)L, fC2 (1 +)f4
C2
2 − C2
1 (+ 1)2f2,C3
2 − ff4 Q (mod 3).
No one of these eight concomitants is a rational integral function of the remaining seven. To prove this we find their expressions for five special sets of values of a0, a1, a2 (in fact, those giving the non-equivalent fs under the group of transformations of determinant unity modulo 3):

f q C1 C2 f4
(1) 0 0 0 0 0 0
(2) x21
0 -1 0 x21
x41
(3) −x21
0 1 0 x21
−x41
(4) x21
+ x22
-1 0 0 0 x41
+ x42
(5) 2x1x2 1 0 −x21
+ x22
x21
+ x22
x31
x2 + x1x32
To show that L and Q are not functions of the remaining concomitants we use case (1). For f4, use case (4). No linear relation holds between f,C1,C2 in which C1 is present, since C1 is of index 1, while f,C2 are absolute covariants. Now f 6= kC2 by case (4); C2 6= kf by case (5). Next q 6= F() by (2) and (3);
 6= F(q) by (4) and (5).
# Chapter 9
# INVARIANTS OF TERNARY FORMS
In this chapter we shall discuss the invariant theory of the general ternary form
f = amx
= bmx
= Â· Â· Â· .
Contrary to what is a leading characteristic of binary forms, the ternary f is not linearly factorable, unless indeed it is the quantic (198) of the preceding chapter. Thus f represents a plane curve and not a collection of linear forms. This fact adds both richness and complexity to the invariant theory of f. The symbolical theory is in some ways less adequate for the ternary case. Nevertheless this method has enabled investigators to develop an extensive theory of plane curves with remarkable freedom from formal difficulties.1
9.1 Symbolical Theory
As in Section 2 of Chapter VIII, let
f(x) = amx
= (a1x1 + a2x2 + a3x3)m = bmx
= Â· Â· Â· .
Then the transformed of f under the collineations V (Chap. VIII) is
f0 = (ax01 + aµx02 + ax03)m. (199)
9.1.1 Polars and transvectants.
If (y1, y2, y3) is a set cogredient to the set (x1, x2, x3), then the (y) polars of f are (cf. (61))
fyk = am−k

$$a_x^{m-k}a_y^{k} \quad (k = 0, 0, \cdots, m). \quad (200)$$

[1]Clebsch, Lindemann, Vorlesungen über Geometrie.

If the point (y) is on the curve $f = 0$, the equation of the tangent at (y) is

$$a_x a_y^{m-1} = 0. \quad (201)$$

The expression

$$\frac{(m-1)!(n-1)!(p-1)!}{m!n!p!}\begin{vmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial y_1} & \frac{\partial}{\partial y_2} & \frac{\partial}{\partial y_3} \\ \frac{\partial}{\partial z_1} & \frac{\partial}{\partial z_2} & \frac{\partial}{\partial z_3} \end{vmatrix} f(x)\phi(y)\psi(z)\Bigg\}_{y=z=x} \quad (202)$$

is sometimes called the first transvectant of $f(x)$, $\phi(x)$, $\psi(x)$, and is abbreviated (f, ϕ, ψ). If

$$f(x) = a_x^m = a_x'^m = \cdots,\ \phi(x) = b_x^n = b_x'^n = \cdots,\ \psi(x) = c_x^p = c_x'^p = \cdots$$

then, as is easily verified,

$$(f, \phi, \psi) = (abc)a_x^{m-1}b_x^{n-1}c_x^{p-1}.$$

This is the Jacobian of the three forms. The rth transvectant is

$$(f, \phi, \psi)^r = (abc)^r a_x^{m-r}b_x^{n-r}c_x^{p-r} \quad (r = 0, 1, \cdots). \quad (203)$$

For $r = 2$ and $f = \phi = \psi$ this is called the Hessian curve. Thus

$$(f, f, f)^2 = (abc)^2 a_x^{m-2}b_x^{n-2}c_x^{p-2} = 0$$

is the equation of the Hessian. It was proved in Chapter I that Jacobians are

concomitants. A repetition of that proof under the present notation shows that transvectants are likewise concomitants. In fact the determinant in (202) is itself an invariant operator, and
0 = (μ).
Illustration.
As an example of the brevity of proof which the symbolical notation affords for some theorems we may prove that the Hessian curve of f = 0 is the locus of all points whose polar conics are degenerate into two straight lines.
If g = 2x
= 2x
= Â· Â· Â· = a200x21
+ Â· Â· Â· is a conic, its second transvectant is its discriminant, and equals
()2 = (XÂ±123)2 = 6
a200 a110 a101
a110 a020 a011
a101 a011 a002

,
since 2
1 = 2
1 = Â· Â· Â· = a200 etc. If ()2 = 0 the conic is a 2-line.

Now the polar conic of f is
P = a2
xam−2
y = a02
x a0m−2
y = Â· Â· Â· ,
and the second transvectant of this is
(P, P, P)2 = (aa0a00)2am−2
y a0m−2
y a00m−2
y , (204)
But this is the Hessian of f in (y) variables. Hence if (y) is on the Hessian the polar conic degenerates, and conversely.
Every symbolical monomial expression consisting of factors of the two types (abc), ax is a concomitant. In fact if
= (abc)p(abd)q Â· Â· Â· ar
xbs
x Â· Â· Â· ,
then
0 =
a b c
aμ bμ cμ
a b c

p
a b d
aμ bμ dμ
a b d

q
Â· Â· Â· ar
xbs
x Â· Â· Â· ,
since, by virtue of the equations of transformation a0x = ax, Â· Â· Â· . Hence by the formula for the product of two determinants, or by (14), we have at once
0 = (μ)p+q+Â·Â·Â·(abc)p(abd)q Â· Â· Â· ar

xbs
x Â· Â· Â· = (μ)p+q+Â·Â·Â·.

The ternary polar of the product of two ternary forms is given by the same formula as that for the polar of a product in the binary case. That is, formula (77) holds when the forms and operators are ternary.

Thus, the formula for the rth transvectant of three quantics, e.g.

$$T = (f,\ ,\ )^r = (abc)^r a_x^{m-r} b_x^{n-r} c_x^{p-r},$$

may be obtained by polarization: That is, by a process analogous to that employed in the standard method of transvection in the binary case. Let

$$(bc)_1 = b_2c_3 - b_3c_2,\quad (bc)_2 = b_3c_1 - b_1c_3,\quad (bc)_3 = b_1c_2 - b_2c_1. \quad (205)$$

Then

$$a_{(bc)} = (abc). \quad (206)$$

Hence T may be obtained by polarizing a_x^m r times, changing y_i into $(bc)_i$ and multiplying the result by $b_x^{n-r} c_x^{p-r}$. Thus

$$(a_x^2 b_x, c_x^3, x_x^3) = \frac{1}{3}\left[\binom{2}{1}\binom{1}{1} a_x a_y b_y + \binom{2}{2}\binom{1}{0} a_y^2 b_x\right]_{y=(cd)} c_x = \frac{2}{3}(acd)(bcd)a_x c_x + \frac{1}{3}(acd)^2 b_x c_x.$$

Before proceeding to further illustrations we need to show that there exists for all ternary collineations a universal covariant. It will follow from this that a complete fundamental system for a single ternary form is in reality a simultaneous system of the form itself and a definite universal covariant. We introduce these facts in the next paragraph.

9.1.2 Contragrediency.

Two sets of variables (x_1, x_2, x_3), (u_1, u_2, u_3) are said to be contragredient when they are subject to the following schemes of transformation respectively:

$$V: \begin{aligned} x_1 &= \lambda_1 x'_1 + \mu_1 x'_2 + \nu_1 x'_3 \\ x_2 &= \lambda_2 x'_1 + \mu_2 x'_2 + \nu_2 x'_3 \\ x_3 &= \lambda_3 x'_1 + \mu_3 x'_2 + \nu_3 x'_3 \end{aligned}$$

$$\Lambda: \begin{aligned} u'_1 &= \lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 \\ u'_2 &= \mu_1 u_1 + \mu_2 u_2 + \mu_3 u_3 \\ u'_3 &= \nu_1 u_1 + \nu_2 u_2 + \nu_3 u_3. \end{aligned}$$

Theorem. A necessary and sufficient condition in order that (x) may be contragredient to (u) is that

$$u_x \equiv u_1x_1 + u_2x_2 + u_3x_3$$

should be a universal covariant.

If we transform u_x by V and use Λ this theorem is at once evident.

It follows, as stated above, that the fundamental system of a form f under
V , is a simultaneous system of f and ux (cf. Chap. VI, Â§4).
The reason that ux = u1x1 +u2x2 does not figure in the corresponding way
in the binary theory is that cogrediency is equivalent to contragrediency in the
binary case and ux is equivalent to (xy) = x1y2 − x2y1 which does figure very
prominently in the binary theory. To show that cogrediency and contragrediency
are here equivalent we may solve
u01 = 1u1 + 2u2
u02 = µ1u1 + µ2u2,
we find
−(µ)u1 = 2u02 + µ2(−u01),
(µ)u2 = 1u02 + µ1(−u01),
which proves that y1 = +u2, y2 = −u1 are cogredient to x1, x2. Then ux
becomes (yx) (cf. Chap. 1, Â§3, V).
We now prove the principal theorem of the symbolic theory which shows
that the present symbolical notation is sufficient to represent completely the
totality of ternary concomitants.
9.1.3 Fundamental theorem of symbolical theory.
Theorem. Every invariant formation of the ordinary rational integral type, of
a ternary quantic
f = amx
= Â· Â· Â· =Xmj
m!
m1!m2!m3!am1m2m3xm1
1 xm2
2 xm3
3 Xmj = m,

can be represented symbolically by three types of factors, viz.
(abc), (abu), ax,
together with the universal covariant ux.
We first prove two lemmas.
Lemma 7. The following formula is true:
nDn
@
@1
@
@2
@
@3
@
@µ1
@
@µ2
@
@µ3
@
@1
@
@2
@
@3

n
1 2 3
µ1 µ2 µ3
1 2 3

n

= C, (207)
where C 6= 0 is a numerical constant.
In proof of this we note that Dn, expanded by the multinomial theorem,
gives
Dn =Xij
n!
i1!i2!i3!i1
1 i2
2 i3
3 (µ23−µ32)i1 (µ31−µ13)i2 (µ12−µ21)i3 . Xij = n.
Also the expansion of n is given by the same formula where now (rµst) is
replaced by @
@r
@
@µs
@
@t . We may call the term given by a definite set i1, i2, i3
of the exponents in Dn, the correspondent of the term given by the same set of
exponents in n. Then, in nDn, the only term of Dn which gives a non-zero
result when operated upon by a definite term of n is the correspondent of that
definite term. But Dn may be written
Dn =Xij
n!
i1!i2!i3!i1
1 i2
2 i3
3 (µ)i1
1 (µ)i2
2 (µ)i3
3 .
An easy differentiation gives
 @
@µ
@
@3
(µ)i1
1 (µ)i2
2 (µ)i3
3 = i3(i1 + i2 + i3 + 1)(µ)i1
1 (µ)i2
2 (µ)i3−1
3 ,
and two corresponding formulas may be written from symmetry. These formulas
hold true for zero exponents. Employing them as recursion formulas we have
immediately for nDn,
nDn =
n Xij=0 n!
i1!i2!i3!2
(i1!i2!i3!)2(i1 + i2 + i3 + 1)!
=
n Xij=0
(n!)2(n + 1)! =
1
2
(n!)3(n + 1)2(n + 2). (208)
This is evidently a numerical constant C 6= 0, which was to be proved (cf. (91)).

Lemma 8. If P is a product of m factors of type , n of type µ, and p of
type , then kP is a sum of a number of monomials, each monomial of which

contains k factors of type (),m−k factors of type , n−k of type μ, and
p − k of type .
This is easily proved. Let P = ABC, where
A = (1)
(2)
Â· Â· Â· (m)
,
B = (1)
μ (2)
μ Â· Â· Â· (n)
μ ,
C = (1)
(2)
Â· Â· Â· (p)
.
Then
3P
1μ23
=Xr,i,t
(r)
1 (s)
2 (t)
3
ABC
(r)
(s)
μ (t)
0@
r = 1, Â· Â· Â· ,m
s = 1, Â· Â· Â· , n
t = 1, Â· Â· Â· , p1A .
Writing down the six such terms from P and taking the sum we have
P =Xr,s,t (r)(s)(t) ABC
(r)
(s)
μ (t)

, (209)
which proves the lemma for k = 1, inasmuch as A

(r)

has m − 1 factors; and
so forth. The result for kP now follows by induction, by operating on both
members of equation (209) by , and noting that (r)(s)(t) is a constant
as far as operations by are concerned.
Let us now represent a concomitant of f by (a, x), and suppose that it does
not contain the variables (u), and that the corresponding invariant relation is
(a0, x0, Â· Â· Â·) = (μ)!(a, x, Â· Â· Â·). (210)
The inverse of the transformation V is
x01 = (μ)−1[(μ)1x1 + (μ)2x2 + (μ)3x3]
etc. Or, if we consider (x) to be the point of intersection of two lines
vx = v1x1 + v2x2 + v3x3,
wx = w1x1 + w2x2 + w3x3,
we have
x1 : x2 : x3 = (vw)1 : (vw)2 : (vw)3.
Substitution with these in x01, Â· Â· Â· and rearrangement of the terms gives for the
inverse of V
V −1 :8>

<>
:
x01 = vµw−vwµ
(µ) ,
x02 = vw−vw
(µ) ,
x03 = vwµ−vµw
(µ) .

We now proceed as if we were verifying the invariancy of , substituting
from V −1 for x01, x03, x03 on the left-hand side of (210), and replacing a0m1m2m3
by its symbolical equivalent am1
 am2
µ am3
 (cf. (199)). Suppose that the order of
 is !. Then after performing these substitutions and multiplying both sides of
(210) by (µ)! we have
(am1
 am2
µ am3
 , vwµ − vµw, Â· Â· Â·) = (µ)w+!(a, x, Â· Â· Â·),
and every term of the left-hand member of this must contain w+! factors with
each suffix, since the terms of the right-hand member do. Now operate on both
sides by . Each term of the result on the left contains one determinant factor
by lemma 2, and in addition w + ! − 1 factors with each suffix. There will be
three types of these determinant factors e.g.
(abc), (avw) = ax, (abv).
The first two of these are of the form required by the theorem. The determinant
(abv) must have resulted by operating upon a term containing a, bµ, v and
evidently such a term will also contain the factor wµ or else w. Let the term
in question be
Rabµvwµ.
Then the left-hand side of the equation must also contain the term
−Rabµvµw,
and operation of upon this gives
−R(abw)vµ,
and upon the sum gives
R[(abv)wµ − (abw)vµ] .
Now the first identity of (212) gives
(abv)wµ − (abw)vµ = (bvw)aµ − (avw)bµ = bxaµ − bµax.
Hence the sum of the two terms under consideration is
R(bxaµ − bµax),
and this contains in addition to factors with a suffix µ only factors of the required
type ax. Thus only the two required types of symbolical factors occur in the
result of operating by .
Suppose now that we operate by w+! upon both members of the invariant
equation. The result upon the right-hand side is a constant times the concomitant
(a, x) by lemma 1. On the left there will be no terms with , µ, suffixes,
since there are none on the right. Hence by dividing through by a constant we

have (a, x) expressed as a sum of terms each of which consists of symbolical
factors of only two types viz.
(abc), ax,
which was to be proved. Also evidently there are precisely ! factors ax in each
term, and w of type (abc), and ! = 0 if is an invariant.
The complete theorem now follows from the fact that any invariant formation
of f is a simultaneous concomitant of f and ux. That is, the only new type of
factor which can be introduced by adjoining ux is the third required type (abu).
9.1.4 Reduction identities.

We now give a set of identities which may be used in performing reductions.
These may all be derived from

ax bx cx
ay by cy
az bz cz

= (abc)(xyz), (211)
as a fundamental identity (cf. Chap. III, Â§3, II). We let u1, u2, u3 be the
coÂ¨ordinates of the line joining the points (x) = (x1, x2, x3), (y) = (y1, y2, y3).
Then
u1 : u2 : u3 = (xy)1 : (xy)2 : (xy)3.
Elementary changes in (211) give
(bcd)ax − (cda)bx + (dab)cx − (abc)dx = 0,
(bcu)ax − (cua)bx + (uab)cx − (abc)ux = 0, (212)
(abc)(def) − (dab)(cef) + (cda)(bef) − (bcd)(aef) = 0.
Also we have
axby − aybx = (abu), (213)
vawb − vbwa = (abx).
In the latter case (x) is the intersection of the lines v,w.
To illustrate the use of these we can show that if f = a2
x = Â· Â· Â· is a quadratic,
and D its discriminant, then
(abc)(abd)cxdx =
1
3Df.
In fact, by squaring the first identity of (212) and interchanging the symbols,
which are now all equivalent, this result follows immediately since (abc)2 = D.

9.2 Transvectant Systems
9.2.1 Transvectants from polars.
We now develop a standard transvection process for ternary forms.
Theorem. Every monomial ternary concomitant of f = amx
= Â· Â· Â· ,
 = (abc)p(abd)q Â· Â· Â· (bcd)r Â· Â· Â· (abu)s(bcu)t Â· Â· Â· a Â· Â· Â· ,
is a term of a generalized transvectant obtained by polarization from a concomitant
1 of lower degree than .
Let us delete from the factor a
x , and in the result change a into v, where v
is cogredient to u. This result will contain factors of the three types (bcv), (bcd),
(buv), together with factors of type bx. But (uv) is cogredient to x. Hence the
operation of changing (uv) into x is invariantive and (buv) becomes bx. Next
change v into u. Then we have a product 1 of three and only three types, i.e.
(bcu), (bcd), bx,
1 = (bcd) Â· Â· Â· (bcu) Â· Â· Â· b
xc
x Â· Â· Â· .
Now 1 does not contain the symbol a. Hence it is of lower degree than . Let
the order of be !, and its class µ. Suppose that in there are i determinant
factors containing both a and u, and k which contain a but not u. Then
 + i + k = m.
Also the order of 1 is
!1 = ! + 2i + k − m,
and its class
µ1 = µ − i + k.
We now polarize 1, by operating (v @
@u)k(y @
@x)i upon it and dividing out the
appropriate constants. If in the resulting polar we substitute v = a, y = (au)

and multiply by am−i−k
x we obtain the transvectant (generalized)
 = (1, amx
, ui
x)k,i
. (214)
The concomitant is a term of .
For the transvectant thus defined k + i is called the index. In any ternary
concomitant of order ! and class µ the number ! + µ is called the grade.

DEFINITION.
The mechanical rule by which one obtains from a concomitant
C = Aa1xa2x Â· Â· Â· arx1u2u Â· Â· Â· su,
any one of the three types of concomitants
C1 = A(a1a2a3)a4x Â· Â· Â· arx1u2u Â· Â· Â· su,
C2 = Aa11a2x Â· Â· Â· arx2u3u Â· Â· Â· su,
C3 = A(123)a1x Â· Â· Â· arx4u Â· Â· Â· su,
is called convolution. In this a11 indicates the expression
a1111 + a1212 + a1313.
Note the possibility that one might be x, or one a might be u.
9.2.2 The difference between two terms of a transvectant.
Theorem. . The difference between any two terms of a transvectant equals
reducible terms whose factors are concomitants of lower grade than , plus a
sum of terms each term of which is a term of a transvectant Â¯ of index 5 k+i,
Â¯ = (Â¯ 1, amx
, ui
x)k,i
.
In this Â¯1 is of lower grade than 1 and is obtainable from the latter by convolution.
Let 1 be the concomitant C above, where A involves neither u nor x. Then,
with numerical, we have the polar
P = v
@
@uky
@
@xi
1
= AXa1ya2y Â· Â· Â· aiyai+1x Â· Â· Â· arx1v Â· Â· Â· kvk+1u Â· Â· Â· su. (215)
Now in the ith polar of a simple product like
p = 1x2x . . . tx,
two terms are said to be adjacent when they differ only in that one has a factor
of type hyjx whereas in the other this factor is replaced by hxjy. Consider
two terms, t1,t2 of P. Suppose that these differ only in that nvkuahyajx in
t1 is replaced in t2 by uvahxajy. Then t1 − t2 is of the form
t1 − t2 = B(vuahyajx − uvahxajy).

We now add and subtract a term and obtain
t1 − t2 = B[vu(ahyajx − ahxajy) + ahxajy(vu − uv)]. (216)
Each parenthesis in (216) represents the difference between two adjacent terms
of a polar of a simple product, and we have by (213)
t1 − t2 = B(yx(ahaj))vu + B((uv))hxjy. (217)
The corresponding terms in are obtained by the replacements v = a, y = (au).
They are the terms of
S = −B0((au)(ahaj)x)au − B0((au))(ajau)ahx,
or, since
((au)(ahaj)x) = (aahaj)ux − (ahaju)ax,
of
S = B0(ahaju)auax − B0(ahaja)auux
+ B0((au))(ajau)ahx,

where B becomes B0 under the replacements v = a, y = (au). The middle
term of this form of S is evidently reducible, and each factor is of lower grade
than . By the method given under Theorem I the first and last terms of S are
respectively terms of the transvectants
Â¯ 1 = (B1(ahaju)uu, amx
, ui−1
x)k,i−1
,
Â¯ 2 = (B1(x)ajxahx, amx
, ui+1
x)k−1,i+1
.
The middle term is a term of
Â¯ 3 = (−B1(ahaju)uu, amx
, ui−1
x)k−1,i+1
Â· ux.
In each of these BI is what B becomes when v = u, y = x; and the first form in
each transvectant is evidently obtained from ux1 Cux by convolution. Also
each is of lower grade than 1.
Again if the terms in the parentheses in form (216) of any difference t1−t2 are
not adjacent, we can by adding and subtracting terms reduce these parentheses
each to the form 2
[(1 − 2) + (2 − 3) + Â· Â· Â· (I−1 − I)] , (218)
2Isserlis. On the ordering of terms of polars etc. Proc. London Math. Society, ser. 2, Vol.
6 (1908).

where every difference is a difference between adjacent terms, of a simple polar.
Applying the results above to these differences i − i+1 the complete theorem
follows.
As a corollary it follows that the difference between the whole transvectant
and any one of its terms equals a sum of terms each of which is a term
of a transvectant of amx
with a form Â¯1 of lower grade than 1 obtained by
convolution from the latter. For if
= 11 + 22 + Â· Â· Â· + rr + Â· Â· Â·
where the 's are numerical, then r is a term of . Also since our transvectant
is obtained by polarization, Pi = 1. Hence
− r = 1(1 − r) + 2(2 − r) + Â· Â· Â· ,
and each parenthesis is a difference between two terms of . The corollary is
therefore proved.
Since the power of ux entering is determinate from the indices k, i we may
write in the shorter form
= (1, amx
)k,i.
The theorem and corollary just proved furnish a method of deriving the
fundamental system of invariant formations of a single form f = amx
by passing
from the full set of a given degree i − 1, assumed known, to all those of the
fundamental system, of degree i. For suppose that all of those members of the
fundamental system of degrees i − 1 have been previously determined. Then
by forming products of their powers we can build all invariant formations of
degree i − 1. Let the latter be arranged in an ordered succession
0, 00, 000, Â· Â· Â·
in order of ascending grade. Form the transvectants of these with am, j =
((j), amx
)k,i. If j contains a single term which is reducible in terms of forms of
lower degree or in terms of transvectants j , j0 < j, then j may, by the theorem
and corollary, be neglected in constructing the members of the fundamental

system of degree i. That is, in this construction we need only retain one term from each transvectant which contains no reducible terms. This process of constructing a fundamental system by passing from degree to degree is tedious for all systems excepting that for a single ternary quadratic form. A method which is equivalent but makes no use of the transvectant operation above described, and the resulting simplifications, has been applied by Gordan in the derivation of the fundamental system of a ternary cubic form. The method of Gordan was also successfully applied by Baker to the system of two and of three conics. We give below a derivation of the system for a single conic and a summary of Gordan's system for a ternary cubic (Table VII).

9.2.3 Fundamental systems for ternary quadratic and cubic.

Let $f = a_x^2 = b_x^2 = \cdots$. The only form of degree one is f itself. It leads to the transvectants

$$(a_x^2, b_x^2)^{0,1} = (abu)a_xb_x = 0, \quad (a_x^2, b_x^2)^{0,2} = (abu)^2 = L.$$

Thus the only irreducible formation of degree 2 is L. The totality of degree 2 is, in ascending order as to grade,

$$(abu)^2, \quad a_x^2b_x^2.$$

All terms of $(f^2, f)^{k,i}$ are evidently reducible, i.e. contain terms reducible by means of powers of f and L. Also

$$((abu)^2, c_x^2)^{1,0} = (abc)(abu)c_x = \frac{1}{3}(abc)\left[(abu)c_x + (bcu)a_x + (cau)b_x\right] = \frac{1}{3}(abc)^2u_x,$$

$$((abu)^2, c_x^2)^{2,0} = (abc)^2 = D.$$

Hence the only irreducible formation of the third degree is D. Passing to degree four, we need only consider transvectants of fL with f. Moreover the only possibility for an irreducible case is evidently

$$(fL, f)^{1,1} = (abd)(abu)(cdu)c_x = \frac{1}{4}(abu)(cdu)\left[(abd)c_x + (bcd)a_x + (dca)b_x + (acb)d_x\right] \equiv 0.$$

All transvectants of degree > 4 are therefore of the form

$$(f^hL^j, f)^{k,i} \quad (i + k < 3),$$

and hence reducible. Thus the fundamental system of f is

$$u_x, f, L, D.$$

The explicit form of D was given in section 1. A symmetrical form of L in terms of the actual coefficients of the conic is the bordered discriminant

$$L: \begin{vmatrix} a_{200} & a_{110} & a_{101} & u_1 \\ a_{110} & a_{020} & a_{011} & u_2 \\ a_{101} & a_{011} & a_{002} & u_3 \\ u_1 & u_2 & u_3 & 0 \end{vmatrix}$$

.
To verify that L equals this determinant we may expand (abu)2 and express the symbols in terms of the coefficients.
We next give a table showing Gordan's fundamental system for the ternary cubic. There are thirty-four individuals in this system. In the table, i indicates the degrees.
The reader will find it instructive to derive by the methods just shown in the case of the quadratic, the forms in this table of the first three or four degrees.

TABLE VII
i INVARIANT FORMATION
0 ux
1 a3
x
2 (abu)2axbx
3 (abu)2(bcu)axc2
x, a3
x = (abc)2axbxcx, s3
u = (abc)(abu)(acu)(bcu)
4 (au)a2
x2x
, ass2
ua2
x, S = a3s
, p6
u = (abu)2(cdu)2(bcu)(adu)
5 ass2
u(abu)axb2
x, asbssua2
xb2
x, as(abu)2s2
ubx, t3
u = asbssu(abu)2
6 asbssu(bcu)a2
xbxc2
x, ass2
u(abu)2(bcu)c2
x, att2
ua2
x, T = a3t
7 s2
up5
u(spx), att2
u(abu)axb2
x, atbttua2
xb2
x, att2
u(abu)2bx
8 atbttu(bcu)a2
xbxc2
x, q6x
= atbtcta2
xb2
xc2
x, att2
u(abu)2(bcu)c2
x, s2
ut2
u(stx)

9 (aqu)a2
xq5x
, p5
ut2
u(ptx), ats2
utua2
x(stx)
10 atbts2
ua2
xb2
x(stx), ats2
utu(abu)2bx(stx)
11 (qu)a2
xq5x
12 (aq)a2
x2x
q5x
, p5
us2
ut2
u(pst)

9.2.4 Fundamental system of two ternary quadrics.

We shall next define a ternary transvectant operation which will include as special cases all of the operations of transvection which have been employed in this chapter. It will have been observed that a large class of the invariant formations of ternary quantics, namely the mixed concomitants, involve both the (x) and the (u) variables. We now assume, quite arbitrarily, two forms involving both sets of variables e.g.

= Aa1xa2x Â· Â· Â· arx1u2u Â· Â· Â· su,

= Bb1xb2x Â· Â· Â· bx1u2u Â· Â· Â· u,

in which A,B are free from (x) and (u). A transvectant of , and of four indices, the most general possible, may be defined as follows: Polarize by the following operator,

Xy(1)
1
@
@x1 y(1)
2
@
@x2
Â· Â· Â·y(1)

@
@x y(2)
1
@
@x1 y(2)
2
@
@x2
Â· Â· Â·y(2)

@
@x
Ã— v(1)
1
@
@u1 v(1)
2

@
@u2
Â· Â· Â·v(1)

@
@u v(2)
1
@
@u1 v(2)
2
@
@u2
Â· Â· Â·v(2)

@
@u
,
wherein i, i, i, vi = 0 or 1, and
X = i,X = j,X = k,X = l; i + j 5 r, k + l 5 s.

Substitute in the resulting polar
(a) y(1)
p = p (p = 1, 2, . . . , i),
(b) y(2)
p = (bpu) (p = 1, 2, . . . , j),
(c) v(1)
p = bp (p = 1, 2, . . . , k),
(d) v(2)
p = (px) (p = 1, 2, . . . , l),
and multiply each term of the result by the bx, u factors not affected in it. The
resulting concomitant we call the transvectant of and of index i, j
k, l,
and write
= (,)i,j
k,l.
An example is
(a1xa2xu, b1xb2xu)1,1
1,0 = a1b2 (a2b1u) + a1b1 (a2b2u)
+ a2b2 (a1b1u) + a2b1 (a1b2u).
If, now, we introduce in place of successively products of forms of the fundamental
system of a conic, i.e. of
f = a2
x,L = 2u
= (a0a00u)2,D = (aa0a00)2,
and for products of forms of the fundamental system of a second conic,
g = b2
x,L0 = 2u= (b0b00u)2,D0 = (bb0b00)2,
we will obtain all concomitants of f and g. The fundamental simultaneous
system of f, g will be included in the set of transvectants which contain no
reducible terms, and these we may readily select by inspection. They are 17 in
number and are as follows:

= (a2
x, b2
x)0,2
0,0 = (abu)2,
C1 = (a2
x, b2
x)0,1

0,0 = (abu)axbx,
A122 = (a2
x, 2u
)2,0
0,0 = a2
,
C2 = (a2
x, 2u
)1,0
0,0 = aaxu,
C3 = (a2
u, b2
x)0,0
1,0 = bubx,
A112 = (2u
, b2
x)0,0
2,0 = 2
b ,
C4 = (2u
, 2u
)0,0
0,1 = (x)uu,
F = (2u
, 2u
)0,0
0,2 = (x)2,
C5 = (2u
, b2
x2u
)0,0
1,1 = b(x)bxu,
C6 = (a2
x, b2
x2u
)1,1
0,0 = a(abu)bxu,
C7 = (a2
x2u
, b2
x)0,1
1,0 = ab(abu)axu,
C8 = (a2
x2u
, 2u
)1,0
0,1 = a(x)axu,
G = (fL, gL0)1,0
1,1 = ab(x)axbx,
= (fL, gL0)1,1
1,0 = ab(abu)uu,
K1 = (fL, gL0)1,1
0,1 = a(abu)u(x)bx,
K2 = (fL, gL0)0,1
1,1 = ab(abu)u(x)ax,
K3 = (fL, gL0)1,1
1,1 = ab(abu)(x).
The last three of these are evidently reducible by the simple identity
(abu)(x) =

aa ba u
a b u
ax bx ux

.
The remaining 14 are irreducible. Thus the fundamental system for two ternary quadrics consists of 20 forms. They are, four invariants D, D0, A112, A122; four covariants f, g, F, G; four contravariants L, L0, , ; eight mixed concomitants Ci, (i = I, . . . , 8).

9.3 Clebsch's Translation Principle

Suppose that (y), (z) are any two points on an arbitrary line which intersects the curve f = amx= 0. Then
u1 : u2 : u3 = (yz)1 : (yz)2 : (yz)3
are contragredient to the x's. If (x) is an arbitrary point on the line we may write
x1 = 1y1 + 2z1, x2 = 1y2 + 2z2, x3 = 1y3 + 2z3,

and then (1, 2) may be regarded as the coordinates of a representative point (x) on the line with (y), (z) as the two reference points. Then ax becomes
ax = a1x1 + a2x2 + a3x3 = 1ay + 2az,
and the () coordinates of the m points in which the line intersects the curve f = 0 are the m roots of
g = gm
= (ay1 + az2)m = (by1 + bz2)m = Â· Â· Â· .
Now this is a binary form in symbolical notation, and the notation differs from the notation of a binary form h = amx
= (a1x1 +a2x2)m = . . . only in this, that
a1, a2 are replaced by ay, az, respectively. Any invariant,
I1 =Xk(ab)p(ac)q Â· Â· Â· ,
of h has corresponding to it an invariant I of g,
I =Xk(aybz − azby)p(aycz − azcy)q Â· Â· Â· .
If I = 0 then the line cuts the curve f = amx
= 0 in m points which have the
protective property given by I1 = 0. But (cf. (213)),
(aybz − azby) = (abu).
Hence,
Theorem. If in any invariant. I1 = Pk(ap)p(ac)q Â· Â· Â· of a binary form h = amx
= (a1x1 + a2x2)m = Â· Â· Â· we replace each second order determinant (ab) by the third order determinant (abu), and so on, the resulting line equation represents the envelope of the line ux when it moves so as to intersect the curve f = amx
= (a1x1 +a2x2 +a3x3)m = 0 in m points having the protective property
I1 = 0.
By making the corresponding changes in the symbolical form of a simultaneous invariant I of any number of binary forms we obtain the envelope of ux when the latter moves so as to cut the corresponding number of curves in a point range which constantly possesses the projective property I = 0. Also this translation principle is applicable in the same way to covariants of the binary forms.
For illustration the discriminant of a binary quadratic h = a2
x = b2
x = Â· Â· Â· is
D = (ab)2. Hence the line equation of the conic f = a2
x = (a1x1+a2x2+a3x3)2 =
Â· Â· Â· = 0 is
L = (abu)2 = 0.
For this is the envelope of ux when the latter moves so as to touch f = 0, i.e. so that D = 0 for the range in which ux cuts f = 0.

The discriminant of the binary cubic h = (a1x1 + a2x2)3 = b3
x = Â· Â· Â· is
R = (ab)2(ac)(bd)(cd)2.

Hence the line equation of the general cubic curve f = a3
x = Â· Â· Â· is (cf. Table
VII)
p6
u = L = (abu)2(acu)(bdu)(cdu)2 = 0.
We have shown in Chapter I that the degree i of the discriminant of a binary form of order m is 2(m− 1). Hence its index, and so the number of symbolical determinants of type (ab) in each term of its symbolical representation, is
k =
1
2im = m(m − 1).
It follows immediately that the degree of the line equation, i.e. the class of a plane curve of order m is, in general, m(m − 1).
Two binary forms h1 = amx
= a0m
x = . . . , h2 = bmx
= . . ., of the same order
have the bilinear invariant
I = (ab)m
If I = 0 the forms are said to be apolar (cf. Chap. III, (71)); in the case m = 2, harmonic. Hence (abu)m = 0 is the envelope of ux = 0 when the latter moves so as to intersect two curves f = amx
= 0, g = bmx
= 0, in apolar point ranges.

APPENDIX
EXERCISES AND THEOREMS
1. Verify that I = a0a4 − 4a1a3 + 3a22
is an invariant of the binary quartic
f = a0x41
+ 4a1x31
x2 + 6a2x21
x32
+ 4a3x1x32
,+a4x42
for which
I0 = (µ)4I.
2. Show the invariancy of
1(a0x1 + a1x2) − 0(a1x1 + a2x2),
for the simultaneous transformation of the forms
f = 0x1 + 1x2,
g = a0x21
+ 2a1x1x2 + a2x22
.
Give also a verification for the covariant C of Chap. I, Â§1, V, and for J, of Chap. II, Â§3.
3. Compute the Hessian of the binary quintic form
f = a0x51
+ 5a1x41
x2 + ...
The result is
1
2H = (a0a2 − a21
)x61
+ 3(a0a3 − a1a2)x51

x2 + 3(a0a4 + a1a3 − 2a22)x41
x32
+ (a0a5 + 7a1a4 − 8a2a3)x31
x32
+ 3(a1a5 + a2a44
− 2a23
)x21x42
+ 3(a2a5 − a3a4)x1x52
+ (a3a5 − a24
)x62
.

4. Prove that the infinitesimal transformation of 3-space which leaves the differential element,
 = dx2 + dy2 + dz2
invariant, is an infinitesimal twist or screw motion around a determinate invariant line in space. (A solution of this problem is given in Lie's Geometrie der BerÂ¨uhrungstransformationen Â§3, p. 206.)

5. The function
q = a20
a2 + a0a22
+ a0a21
+ a21
a2 − a30
− a32
,
is a formal invariant modulo 3 of the binary quadratic
f = a0x21
+ 2a1x1x2 + a2x22
(Dickson)

6. The function a0a3 + a1a2 is a formal invariant modulo 2 of the binary cubic form.

7. Prove that a necessary and sufficient condition in order that a binary form f of order m may be the mth power of a linear form is that the Hessian covariant of f should vanish identically.

8. Show that the set of conditions obtained by equating to zero the 2m − 3 coefficients of the Hessian of exercise 7 is redundant, and that only m − 1 of these conditions are independent.

9. Prove that the discriminant of the product of two binary forms equals the product of their discriminants times the square of their resultant.

10. Assuming (y) not cogredient to (x), show that the bilinear form
f =Xaikxiyk = a11x1y1 + a12x1y2 + a21x2y1 + a22x2y2,
has an invariant under the transformations
 : x1 = 11 + 12, x2 = 11 + 12,
y1 = 21 + 22, y2 = 21 + 22,
in the extended sense indicated by the invariant relation
[a011 a021
a012 a022
] = [1 1
1 1
][2 2
2 2
][a11 a21
a12 a22
].

11. Verify the invariancy of the bilinear expression
Hfg = a11b22 + a22b11 − a12b21 − a21b12,
for the transformation by of the two bilinear forms
f =Xaikxiyk, g =Xbikxiyk.

12. As the most general empirical definition of a concomitant of a single binary form f we may enunciate the following: Any rational, integral function of the coefficients and variables of f which needs, at most, to be multiplied by a function   of the coefficients in the transformations T, in order to be made equal to the same function of the coefficients and variables of f0, is a concomitant of f.
Show in the case where is homogeneous that   must reduce to a power of the modulus, and hence the above definition is equivalent to the one of Chap. I, Section 2. (A proof of this theorem is given in Grace and Young, Algebra of Invariants, Chapter II.)
13. Prove by means of a particular case of the general linear transformation on p variables that any p-ary form of order m, whose term in xm1
is lacking, can
always have this term restored by a suitably chosen linear transformation.
14. An invariant of a set of binary quantics
f1 = a0xm1
+ Â· Â· Â· , f2 = b0xn1
+ Â· Â· Â· , f3 = c0xp
1 + Â· Â· Â· ,

satisfies the differential equations
X
 = (a0
@
@a1
+ 2a1
@
@a2
+ Â· Â· Â· + mam−1
@
@am
+ b0
@
@b1
+ 2b1
@
@b2
+ Â· Â· Â· + c0
@
@c1
+ Â· Â· Â·) = 0
XO = (ma1
@
@a0
+ (m − 1)a2
@
@a1
+ Â· Â· Â· + am
@
@am−1
+ nb1
@
@b0
+ (n − 1)b2
@
@b1
+ Â· Â· Â· + pc1
@
@c0

+ Â· Â· Â·) = 0
The covariants of the set satisfy
$X - x_2$
@
$@x_1 = 0$,
$XO - x_1$
@
$@x_2 = 0$.
15. Verify the fact of annihilation of the invariant
J = 6
a_0 a_1 a_2
a_1 a_2 a_3
a_2 a_3 a_4

of the binary quartic, by the operators and O.
16. Prove by the annihilators that every invariant of degree 3 of the binary quartic is a constant times J.
(SUGGESTION. Assume the invariant with literal coefficients and operate by and O.)
17. Show that the covariant J, of Chap. II, Section 3 is annihilated by the operators
$X - x_2$
@
$@x_1$
$,XO - x_1$
@
$@x_2$
.
18. Find an invariant of respective partial degrees 1 and 2, in the coefficients of a binary quadratic and a binary cubic.
The result is
$I = a_0(b_1b_3 - b_2^2$
$) - a_1(b_0b_3 - b_1b_2) + a_2(b_6b_2 - b_1^2$
).
19. Determine the index of I in the preceding exercise. State the circumstances concerning the symmetry of a simultaneous invariant.
20. No covariant of degree 2 has a leading coefficient of odd weight.
21. Find the third polar of the product f Â· g, where f is a binary quadratic and g is a cubic.
The result is
$(fg)_{y^3}$ =
1
10
$(fg_{y^3} + 6f_yg_{y^2} + 3f_{y^2}g_y)$.

22. Compute the fourth transvectant of the binary quintic f with itself.
The result is
$(f, f)_4 = 2(a_0a_4-4a_1a_3+3a_2^2$
$)x_1^2$
$+2(a_0a_6-3a_1a_4+2a_2a_3)a_1a_2+2(a_1a_5-4a_2a_4+3a_3^2$
$)x_2^2$
.
23. If $F = a_x^3$
b_x^2
c_x, prove
F_{y^3} =
1
6
31

02
03
3a3
yb2
xcx + 1
02
13
2a2
yaxbybxcz
+ 1
02
23
1aya2
xb2
ycx + 1
12
13
1aya2
xbybxcy
+1
12
23
0a3
xb2
ycy.

24. Express the covariant
Q = (ab)2(cb)c2
xax
of the binary cubic in terms of the coefficients of the cubic by expanding the symbolical Q and expressing the symbol combinations in terms of the actual coefficients. (Cf. Table I.)

25. Express the covariant +j = ((f, f)4, f)2 of a binary quintic in terms of the symbols.

The result is
−j = (ab)2(bc)2(ca)2axbxcx = −(ab)4(ac)(bc)c3
x.

26. Let be any symbolical concomitant of a single form f, of degree i in the coefficients and therefore involving i equivalent symbols. To fix ideas, let be a monomial. Suppose that the i symbols are temporarily assumed non-equivalent. Then , when expressed in terms of the coefficients, will become a simultaneous concomitant 1 of i forms of the same degree as f, e.g.
f = a0xm1
+ ma1xm−1
1 x2 + Â· Â· Â· ,
f1 = b0xm1
+ mb1xm−1
1 x2 + Â· Â· Â· ,
....
fi−1 = l0xm1+ ml1xm−1
1 x2 + Â· Â· Â· .

Also 1 will be linear in the coefficients of each f, and will reduce to again when we set bi = Â· Â· Â· = l1 = a1, that is, when the symbols are again made equivalent. Let us consider the result of operating with
 = p0
@
@a0
+ p1
@
@a1

+ Â· Â· Â· + pm
@
@am
= p
@
@a,
upon . This will equal the result of operating upon 1, the equivalent of ,
and then making the changes
bj = Â· Â· Â· = lj = aj(j = 0, Â· Â· Â· ,m).

Now owing to the law for differentiating a product the result of operating @
@aj
upon is the same as operating
@
@aj
+ @
@bj
+ Â· Â· Â· + @
@lj
upon 1 and then making the changes b = Â· Â· Â· = l = a. Hence the operator
which is equivalent to in the above sense is
1 = p
@
@a+ p
@
@b+ Â· Â· Â· + p
@
@l.
When 1 is operated upon 1 it produces i concomitants the first of which is 1
with the a's replaced by the p's, the second is 1 with the b's replaced by the
p's, and so on. It follows that if we write
m
x = p0xm1
+ mp1xm−1
1 x2 + Â· Â· Â· ,
and
 = (ab)r(ac)s Â· Â· Â· a
xb
x Â· Â· Â· ,
we have for the sum of i symbolical concomitants in the first of which the
symbol a is replaced by , in the second the symbol b by and so forth.
For illustration if is the covariant Q of the cubic,
Q = (ab)2(cb)c2
xax,
then
Q = (b)2(cb)c2
xx + (a)2(c)c2
xax + (ab)2(b)2x
ax.
Again the operator and the transvectant operator are evidently permutable.
Let g, h be two covariants of f and show from this fact that
(g, h)r = (g, h)r + (g, h)r.
27. Assume
f = a3
x,
 = (f, f)2 = (ab)2axbx = 2
x,
Q = (f, (f, f)2) = (c)c2
xx = (ab)2(cb)c2

xax = Q3
x,
R = (,)2 = (ab)2(cd)2(ac)(bd),
and write
Q = Q3
x = Q0x31
+ 3Q1x21x2 + 3Q2x1x22
+ Q3x32
.
Then from the results in the last paragraph (26) and those in Table I of Chapter III, prove the following for the Aronhold polar operator delta = (Q @
@a):
f = Q,
 = 2(aQ)2axQx = 2(f,Q)2 = 0,
Q = 2(f, (f,Q)2) + (Q,) = −
1
2Rf,
R = 4(, (f,Q)2)2 = 0.

28. Demonstrate by means of Hermite's reciprocity theorem that there is a single invariant or no invariant of degree 3 of a binary quantic of order m according as m is or is not a multiple of 4 (Cayley).
29. If f is a quartic, prove by Gordan's series that the Hessian of the Hessian of the Hessian is reducible as follows:
((H,H)2, (H,H)2)2 = −
1
108i2Jf +
1
9H H2 −
1
24i2.
Adduce general conclusions concerning the reducibility of the Hessian of the Hessian of a form of order m.
30. Prove by Gordan's series,
((f, i)2, f)2 =
1
6i2 +
1
15
(f, i)4f,
where i = (f, f)4, and f is a sextic. Deduce corresponding facts for other values of the order m.
31. If f is the binary quartic
f = a4
x = b4
x = c4
x = Â· Â· Â· ,
show by means of the elementary symbolical identities alone that
(ab)2(ac)2b2
xc2
x =
1
2f Â· (ab)4.
(SUGGESTION. Square the identity
2(ab)(ac)bxcx = (ab)2c2
x + (ac)2b2
x − (bc)2a2
x.)
32. Derive the fundamental system of concomitants of the canonical quartic

$X^4 + Y^4 + 6mX^2Y^2$,
by particularizing the a coefficients in Table II.

33. Derive the syzygy of the concomitants of a quartic by means of the canonical form and its invariants and covariants.

34. Obtain the typical representation and the associated forms of a binary quartic, and derive by means of these the syzygy for the quartic.

The result for the typical representation is

f3 Â· f(y) = 4 + 3H22 + 4T3 + (
1
2if 2 −
3
4H2)4.

To find the syzygy, employ the invariant J.

35. Demonstrate that the Jacobian of three ternary forms of order m is a combinant.

36. Prove with the aid of exercise 26 above that

(f,)2r+1 = (a)2r+1an−2r−1
x n−2r−1
x

is a combinant of f = anx
and = n
x.

37. Prove that Q = (ab)(bc)(ca)axbxcx, and all covariants of Q are combinants of the three cubics a3
x, b3
x, c3
x (Gordan).

38. Let f and g be two binary forms of order m. Suppose that is any invariant of degree i of a quantic of order m. Then the invariant constructed for the form v1f + v2g will be a binary form Fi of order i in the variables? v1, v2. Prove that any invariant of Fi is a combinant of f, g. (Cf. Salmon, Lessons Introductory to Modern Higher Algebra, Fourth edition, p. 211.)

39. Prove that the Cartesian equation of the rational plane cubic curve

xi = ai03
1 + ai12
12 + Â· Â· Â· + ai33
2(i = 1, 2, 3),

is

(x1, x2, x3) =
|a0a1x| |a0a2x| |a0a3x|
|a0a2x| |a0a3x| + |a1a2x| |a1a3x|
|a0a3x| |a1a3x| |a2a3x|

= 0.

40. Show that a binary quintic has two and only two linearly independent seminvariants of degree five and weight five.

The result, obtained by the annihilator theory, is

(a40
a5−5a30
a1a4+10a20
a21
a3−10a0a31
a2+4a51
)+µ(a0a2−a21
)(a20
a3−3a0a1a2+2a31
).

41. Demonstrate that the number of linearly independent seminvariants of

weight w and degree i of a binary form of order m is equal to

$$(w; i, m) - (w - 1; i, m),$$

where $(w; i, m)$ denotes the number of different partitions of the number w into i or fewer numbers, none exceeding m. (A proof of this theorem is given in Chapter VII of Elliotts' Algebra of Quantics.)

42. If $f = a_x^m = b_x^m = \cdots$ is a ternary form of order m, show that

$$(f, f)^{0,2k} = (abu)^{2k} a_x^{m-2k} b_x^{m-2k}.$$

Prove also

$$((f, f)^{0,2k}, f)^{r,s} = \frac{1}{\binom{2m-4k}{s}} \sum_{i=0}^{s} \binom{m-2k}{i} \binom{m-2k}{s-i} (abc)^r \times (abu)^{2k-r}(bcu)^{s-i}(acu)^i a_x^{m-i-2k} b_x^{m-s+i-2k} c_x^{m-r-s}.$$

43. Derive all of the invariant formations of degrees 1, 2, 3, 4 of the ternary cubic, as given in Table VII, by the process of passing by transvection from those of one degree to those of the next higher degree.

44. We have shown that the seminvariant leading coefficient of the binary covariant of $f = a_x^m$,

$$= (ab)^p(ac)^q \cdots a_x b_x \cdots,$$

is

$$0 = (ab)^p(ac)^q \cdots a_1 b_1 \cdots,$$

If we replace a_1 by a_x, b_1 by b_z etc. in 0 and leave $a_2, b_2, \cdots$ unchanged, the factor (ab) becomes

$$(a_1x_1 + a_2x_2)b_2 - (b_1x_1 + b_2x_2)a_2 = (ab)x_1.$$

At the same time the actual coefficient $a_r = a_1^{m-r} a_2^r$ of f becomes

$$a_x^{m-r} a_2^r = \frac{(m-r)!}{m!} \frac{\partial^r f}{\partial x_2^r}.$$

Hence, except for a multiplier which is a power of x_1 a binary covariant may be derived from its leading coefficient 0 by replacing in 0, $a_0, a_1, \cdots, a_m$ respectively by

$$f, \frac{1}{m}\frac{\partial f}{\partial x_2}$$

,
1
m(m − 1)
@2f
@x22
, Â· Â· Â· ,
(m − r)!
m!
@rf
@xr
2
, Â· Â· Â· ,
1
m!
@mf
@xm2
.
Illustrate this by the covariant Hessian of a quartic.
45. Prove that any ternary concomitant of f = amx
can be deduced from its
leading coefficient (save for a power of ux) by replacing, in the coefficient, apqr,
by
p!
m! y
@
@xq z
@
@xr
amx
.
(Cf. Forsyth, Amer. Journal of Math., 1889.)
46. Derive a syzygy between the simultaneous concomitants of two binary
quadratic forms f, g (Chap. VI).
The result is
= 2J2
12 = D1g2 + D2f2 − 2hfg,
where J12 is the Jacobian of the two forms, h their bilinear invariant, and D1,
D2 the respective discriminants of f and g.
47. Compute the transvectant
(f, f)0,2 = (abu)2axbx,
of the ternary cubic
f = a3
x = b3
x =X 3!
p!q!r!apqrxp
1xq
2xr
3,
in terms of its coefficients apqr(P + q + r = 3).
The result for 1
2 (f, f)0,2 is given in the table below. Note that this mixed
concomitant may also be obtained by applying Clebsch's translation principle
to the Hessian of a binary cubic.

2
1u21
x21
u1u2 x21
u22

x21
u1u2 x21
u2u1 x21
u23
a120a102 2a111a201 a102a300 2a210a111 2a201a210 a300a120
−a2
111 −2a210a102 −a2
201 −2a120a201 −2a111a300 −a2
210
x1x2u21
x1x2u1u2 x1x2u22
x1x2u1u3 x1x2u2u23
x1x2u23
a120a012 2a2
111 a102a210 2a210a021 2a201a120 a300a030
+a102a030 −2a102a120 +a300a012
+2a201a021
x22
u21
x22
u1u2 x22
u22
x22
u1u3 x22
u2u3 x22
u33
a030a012 2a021a111 a012a210 2a120a021 2a111a120 a210a030
−a2
021 −2a120a012 −a2
111 −2a030a111 −2a021a210 −a2
120
x1x3u21
x1x2u1u2 x1x3u22
x1x3u1u3 x1x3u2u3 x1x3u23
a120a003 2a201a012 a300a003 2a2
111 2a210a102 a120a021
−2a111a012 −2a210a002 −a201a102 −2a201a021 −2a300a012 −2a111a210
+a102a021 −2a120a201 +a300a021
+a210a012
x2x3u21
x2x3u1u2 x2x3u22
x2x3u1u3 x2x3u2u3 x2x3u23
a030a003 2a021a102 a012a201 2a120a012 2a2
111 a210a021
−a021a012 −2a120a003 −2a111a102 −2a030a102 −2a021a201 −2a120a111
+a210a003 −2a210a012 +a030a201
+2a120a102
x2
xu21
x23
u1u2 x23
u33
x23
u1u3 x23
u2u3 x23u23
a021a003 2a012a102 a003a201 2a111a01 2a102a111 a201a021
−a2
012 −2a111a003 −a2
102 −2a021a102 −2a012a201 −a2

111

48. Prove that a modular binary form of even order, the modulus being $p > 2$, has no covariant of odd order.

(Suggestion. Compare Chap. II, Section 2, II. If is chosen as a primitive root, equation (48) becomes a congruence modulo $p - 1$.)

Index

Chapter 10

PROJECT GUTENBERG "SMALL PRINT"

Baixar livros de Literatura
Baixar livros de Literatura de Cordel
Baixar livros de Literatura Infantil
Baixar livros de Matemática
Baixar livros de Medicina
Baixar livros de Medicina Veterinária
Baixar livros de Meio Ambiente
Baixar livros de Meteorologia
Baixar Monografias e TCC
Baixar livros Multidisciplinar
Baixar livros de Música
Baixar livros de Psicologia
Baixar livros de Química
Baixar livros de Saúde Coletiva
Baixar livros de Serviço Social
Baixar livros de Sociologia
Baixar livros de Teologia
Baixar livros de Trabalho
Baixar livros de Turismo