

UNIVERSIDADE FEDERAL DE MINAS GERAIS
INSTITUTO DE CIÊNCIAS EXATAS
DEPARTAMENTO DE MATEMÁTICA

**Integrabilidade algébrica de folheações
holomorfas
e o problema de Poincaré**

Maurício Barros Corrêa Júnior

Tese apresentada como requisito à obtenção do título de Doutor junto ao
Programa de Pós-Graduação em Matemática da UFMG.

Maio de 2010

Livros Grátis

<http://www.livrosgratis.com.br>

Milhares de livros grátis para download.

Algebraic integrability of holomorphic foliations and the Poincaré problem

Maurício Barros Corrêa Júnior

Advisor: Marcio Gomes Soares

May 2010

À memória de minha mãe.

Agradecimentos

À Deus.

À minha mãe por me ensinar a perseverar. Aos meus irmãos Nádia e Douglas, meus sobrinhos Nayara e João Marcos, minhas tias, todos pelo incentivo e principalmente pelo apoio nos momentos difíceis.

Ao meu amor Catia pela paciência e companheirismo.

Ao meu orientador Marcio Soares pelos ensinamentos e por acreditar em meu trabalho.

Aos membros da banca Severino Coutinho, Marcos Jardim, Bruno Scárdua e Israel Vainsencher pelos comentários e sugestões.

Ao professor Rogério Mol por contribuir efetivamente em minha formação.

Aos colegas da pós-graduação Alexandre, Ezequiel, Flávio, Gilberto, Gustavo, Justino, Luis Guillermo, Rafael e Rutyele.

À Rodrigo e Patrícia pela disponibilidade e amizade.

Ao CNPq pelo apoio.

Contents

Introduction	9
1 The number of invariant divisors and Poincaré's problem	15
1.1 The degree of foliations with respect to a polarization	15
1.2 The extatic divisor	17
1.3 Optimal examples on projective spaces	25
2 Foliations on simplicial toric varieties	28
2.1 Toric Varieties	28
2.1.1 The toric homogeneous coordinates	30
2.1.2 Existence of rational first integrals	32
2.2 Multiprojective foliations	38
2.2.1 Riccati foliations	51
2.2.2 Totally invariant hypersurfaces	52
2.2.3 The polar divisor	53
2.3 Weighted projective foliations	56
2.3.1 Poincaré's problem for quasi-homogeneous foliations	61
3 Bound for the sectional genus of a variety invariant by Pfaff fields	65
3.1 Sectional genus of a polarized variety	66
3.2 Pfaff fields	66
3.3 Bound for invariant varieties with stable tangent bundle	70

Resumo

G. Darboux apresentou, em [32], uma teoria sobre a existência de integrais primeiras para equações diferenciais polinomiais baseado na existência de um número suficientemente grande de soluções algébricas. Concomitantemente H. Poincaré, em [69], considerou o problema da integrabilidade algébrica para equações diferenciais polinomiais no plano. Ele observou que, neste caso, seria suficiente limitar o grau das soluções algébricas. Nesta mesma direção, P. Painlevé, em [65], enunciou o problema de integrabilidade como:

“É possível reconhecer o gênero de uma solução geral de uma equação diferencial polinomial em duas variáveis com uma integral primeira racional?”

Hoje em dia esses problemas são conhecidos como *Problema de Poincaré e Problema de Painlevé*. Em [53] A. Lins Neto construiu uma família de folheações em $\mathbb{P}^2_{\mathbb{C}}$, com grau e tipo analítico das singularidades fixados, com integrais primeiras de grau arbitrariamente grande, gerando assim contra-exemplos para os problemas de Poincaré e de Painlevé. Entretanto, podemos obter uma resposta afirmativa para tais problemas se forem impostas algumas condições sobre o tipo analítico das singularidades ou sobre as possíveis curvas invariantes.

O atual interesse no Problema de Poincaré foi estimulado por vários trabalhos, como os trabalhos de D. Cerveau e A. Lins Neto [21] e M. Carnicer [18]. Muitos autores vem trabalhando nestes problemas e em algumas de suas generalizações, veja por exemplo os artigos de M. Soares [75], J.V. Pereira [67], M. Brunella & L.G. Mendes [14], E. Esteves & S. Kleiman [35], Cavalier & Lehmann [19] e Zamora [81].

O problema de limitar o gênero de uma curva invariante em termos do grau de uma folheação unidimensional em $\mathbb{P}^n_{\mathbb{C}}$ foi considerado por exemplo por Campillo, Carnicer e de la Fuente em [16]. Eles mostraram que, se C é uma curva, com singularidades nodais, invariante por uma folheação unidimensional \mathcal{F} em $\mathbb{P}^n_{\mathbb{C}}$, então

$$\frac{2p_a(C) - 2}{\deg(C)} \leq \deg(\mathcal{F}) - 1, \quad (1)$$

onde $p_a(C)$ é o gênero aritmético de C .

Em [34], Esteves e Kleiman estenderam o trabalho de Jouanolou sobre equações de Pfaff algébricas sobre um esquema suave V . Eles introduziram o conceito de

campos de Pfaff em V , que é um mapa de feixes $\eta : \Omega_V^s \rightarrow L$, onde L é um feixe inversível em V , e o número inteiro $1 \leq s \leq n - 1$ é chamado o posto de η . Uma subvariedade $X \subset V$ é dita invariante por η se o mapa η fatora a um mapa natural $\Omega_V^s|_X \rightarrow \Omega_X^s$.

Nesta tese obtemos cotas para o gênero seccional de variedades Gorenstein que são invariantes por um campo de Pfaff em \mathbb{P}_k^n , onde k é um corpo algebricamente fechado e de característica zero. Mais precisamente, nosso resultado é o seguinte.

Teorema. *Seja $X \subset \mathbb{P}_k^n$ uma variedade projetiva Gorenstein invariante por um campo de Pfaff holomorfo \mathcal{F} em \mathbb{P}_k^n cujo o posto é igual a dimensão de X , e tal que $\text{codim}(\text{Sing}(X), X) \geq 2$. Então*

$$\frac{2g(X, \mathcal{O}_X(1)) - 2}{\text{deg}(X)} \leq \text{deg}(\mathcal{F}) - 1, \quad (2)$$

onde $g(X, \mathcal{O}_X(1))$ é o gênero seccional de X com respeito ao fibrado em retas $\mathcal{O}_X(1)$ associado a uma seção hiperplana.

Este resultado generaliza a cota obida por Campillo, Carnicer e de la Fuente em [16, Theorem 4.1 (a)].

Retornando ao problema de integrabilidade, lembramos que o trabalho de J.P. Jouanolou em [49] também dá um melhoramento e generalização à teoria de Darboux, caracterizando a existência de integrais primeiras racionais para uma equação de Pfaff em \mathbb{P}_k^n , onde k é algebricamente fechado e de característica zero. Mais precisamente, seja ω uma 1-forma torcida $\omega \in H^0(\mathbb{P}_k^n, \Omega_{\mathbb{P}_k^n}^1 \otimes \mathcal{O}(d+2))$, onde d é o que chamamos de grau de ω . Segue de [49] Teorema 3.3, p.g 102, que ω admite uma integral primeira racional se, e somente se, possui infinitas hipersuperfícies algébricas irredutíveis invariantes. Mais geralmente, Jouanolou provou em [50] que sobre uma variedade complexa compacta X , satisfazendo algumas condições cohomológicas, uma equação de Pfaff $\omega \in H^0(X, \Omega_X^1 \otimes \mathcal{L})$, onde \mathcal{L} é um fibrado em retas, admite uma integral primeira meromorfa se, e somente se, possui um número infinito de divisores irredutíveis invariantes. Além disso, se ω não admite integral primeira meromorfa, então o número de divisores irredutíveis invariantes é no máximo

$$\dim_{\mathbb{C}}(H^0(X, \Omega_X^2 \otimes \mathcal{L})/\omega \wedge H^0(X, \Omega_X^1)) + \rho(X) + 1,$$

onde $\rho(X)$ é o número de Picard de X .

Em [42] E. Ghys retirou todas as hipótese dadas por Jouanolou mostrando que este resultado é válido para toda variedade complexa compacta. M. Brunella e M. Nicolau em [15] provou este mesmo resultado para equações de Pfaff em característica positiva e para folheações não-singulares de codimensão um sobre variedades compactas e com estrutura transversal holomorfa. Recentemente, S. Cantat em [17] mostrou uma versão dinâmica discreta deste resultado provando

que se um endomorfismo sobrejetivo f de uma variedade complexa X possuir um certo número de hipersuperfícies analíticas invariantes, então f preserva uma fibração meromorfa.

Mostramos uma versão destes resultados para campos de vetores sobre variedades tóricas completas e singulares. Seja \mathbb{P}_Δ uma variedade tórica simplicial completa associada a um *fan* Δ e $\mathcal{T}\mathbb{P}_\Delta$ seu feixe de Zarisk. Uma folheação holomorfa singular \mathcal{F} em \mathbb{P}_Δ é uma seção global de $\mathcal{T}\mathbb{P}_\Delta \otimes K_{\mathcal{F}}$, onde $K_{\mathcal{F}}$ é um feixe inversível em \mathbb{P}_Δ . Denotamos por \mathbb{T}^n o toro agindo em \mathbb{P}_Δ e chamamos um \mathbb{T}^n -invariante divisor de Weil por \mathbb{T}^n -*divisor*. Usamos a existência de coordenadas homogêneas para variedades tóricas simpliciais para provar o seguinte resultado.

Teorema. *Seja \mathcal{F} uma folheação unidimensional sobre uma variedade tórica simplicial completa \mathbb{P}_Δ de dimensão n e número de Picard $\rho(\mathbb{P}_\Delta)$. Se \mathcal{F} admite*

$$h^0(\mathbb{P}_\Delta, \mathcal{O}(K_{\mathcal{F}})) + \rho(\mathbb{P}_\Delta) + n$$

\mathbb{T}^n -divisores irredutíveis invariantes, então \mathcal{F} admite uma integral primeira racional.

Observe que, em geral \mathbb{P}_Δ é uma variedade singular com singularidades quocientes. Portanto em dimensão dois este resultado mostra que o teorema de Darboux-Jouanolou-Ghys é válido para uma classe de superfícies tóricas singulares.

A versão afim e não-singular deste resultado foi provada por J. LLibre e X. Zhang em [57]. Eles mostraram que se o número de hipersuperfícies algébricas invariantes por um campo polinomial Z em \mathbb{C}^n , de grau d , é pelo menos

$$\binom{d+n-1}{n} + n$$

então Z admite uma integral primeira racional.

Além disso, estudamos folheações unidimensionais em duas classes de variedades tóricas, os espaços multiprojetivos e espaços projetivos com pesos. Com hipóteses convenientes obtemos cotas para o problema de Poincaré nestas variedades.

Finalmente, estendemos alguns resultados devidos a J. V. Pereira para integrabilidade de folheações holomorfas \mathcal{F} sobre uma variedade complexa M , usando o conceito de *seção extática* com respeito a um sistema linear de dimensão finita $V \subset H^0(M, \mathcal{O}(D))$, onde D é um divisor efetivo sobre M . O lugar de zeros da seção extática é o lugar de inflexão do sistema linear com respeito a um campo de vetores que induz \mathcal{F} .

Denote por $\varepsilon(V, \mathcal{F})$ a seção extática de \mathcal{F} com respeito a V . Se \mathcal{F} é uma folheação unidimensional sobre uma variedade complexa M , então uma integral primeira holomorfa (ou meromorfa) para \mathcal{F} é um mapa holomorfo (resp. meromorfo) $\Theta : M \rightarrow Y$, onde Y é uma variedade complexa, tal que as fibras de Θ

são invariantes por \mathcal{F} . J. V. Pereira em [67] mostrou o seguinte teorema:

Teorema. *Seja \mathcal{F} uma folheação holomorfa unidimensional sobre uma variedade complexa M . Se V é um sistema linear de dimensão finita tal que $\varepsilon(V, \mathcal{F})$ é identicamente nulo então existe um conjunto aberto denso U de M onde $\mathcal{F}|_U$ admite uma integral primeira holomorfa. Além disso, se M é uma variedade projetiva, então \mathcal{F} admite uma integral primeira meromorfa.*

Nos casos não-algébricos e não-compactos o resultado acima não garante que o anulamento da seção extática $\varepsilon(V, \mathcal{F})$ implica na existência de uma integral primeira meromorfa para \mathcal{F} . Forneceremos o seguinte adendo para o teorema de J. V. Pereira.

Teorema. *Sejam \mathcal{F} uma folheação holomorfa unidimensional sobre uma variedade complexa M e V sistema linear de dimensão finita. Se $\varepsilon(V, \mathcal{F})$ é identicamente nula, então \mathcal{F} admite uma integral primeira meromorfa $\Theta : M \rightarrow \mathbb{P}^1$.*

J. V. Pereira em [67] mostrou que uma folheação em $\mathbb{P}_{\mathbb{C}}^2$, de grau $d > 1$, que não admite uma integral primeira racional de grau $\leq k$, possui no máximo

$$\binom{k+2}{k} + \frac{(d-1)}{k} \cdot \binom{k+2}{2}.$$

curvas invariantes de grau k .

Seja (M, L) uma variedade projetiva polarizada e denote por $\mathcal{N}(\mathcal{F}, V)$ o número de divisores \mathcal{F} -invariantes contidos no sistema linear $V \subset H^0(M, \mathcal{O}(D))$. Usamos o conceito de grau de folheações e divisores com respeito à polarização L e divisor extático para o seguinte resultado.

Teorema. *Seja \mathcal{F} uma folheação unidimensional sobre uma variedade projetiva polarizada (M, L) e D um divisor efetivo. Suponha que \mathcal{F} admite integral primeira racional. Então*

$$\deg_L(D) \cdot (\mathcal{N}(\mathcal{F}, V) - h^0(V)) \leq (\deg_L(\mathcal{F}) - \deg_L(M)) \cdot \binom{h^0(V)}{2},$$

onde $h^0(V) = \dim_{\mathbb{C}} V$. Em particular, temos que :

- i) o número de divisores \mathcal{F} -invariantes contidos no sistema linear $V \subset |\mathcal{O}(D)|$ é no máximo

$$\frac{(\deg_L(\mathcal{F}) - \deg_L(M))}{\deg_L(D)} \cdot \binom{h^0(V)}{2} + h^0(V),$$

onde $h^0(V) = \dim_{\mathbb{C}} V$.

ii) se $\mathcal{H} \subset |\mathcal{O}(D)|$ é um pencil e $\mathcal{N}(\mathcal{F}, \mathcal{H}) > 2$, então

$$\deg_L(D) \leq \deg_L(\mathcal{F}) - \deg_L(M)$$

A parte ii) deste teorema nos dá um critério numérico para decidir se uma folheação holomorfa \mathcal{F} sobre uma variedade polarizada (M, L) admite uma integral primeira racional. Isto é, se supormos que o número de divisores \mathcal{F} -invariantes contidos em um pencil \mathcal{H} é maior que 2 e que \mathcal{F} possui um divisor invariante $C \in \mathcal{H}$ satisfazendo a condição

$$\deg_L(C) > \deg_L(\mathcal{F}) - \deg_L(M),$$

então \mathcal{F} admite uma integral primeira racional. Este resultado está relacionado a uma conjecture de Alcides Lins Neto. Em [53] ele levantou a seguinte questão:

“Dado $d \geq 2$, existe $M(d) \in \mathbb{N}$ tal que se uma folheação em \mathbb{P}^2 , de grau d , tem uma solução algébrica invariante de grau $k \geq M(d)$, então ela tem uma integral primeira racional?”

Seja \mathcal{F} uma folheação unidimensional em \mathbb{P}^n de grau $d \geq 2$. Segue do critério mencionado acima que se o número de hipersuperfícies de grau k invariantes por \mathcal{F} contidas em um pencil de mesmo grau é maior que 2 e $k > M(d) = d - 1$, então \mathcal{F} tem uma integral primeira racional.

Introduction

G. Darboux presented , in [32], a theory on the existence of first integrals for polynomial differential equations based on the existence of sufficiently many invariant algebraic hypersurfaces. Concomitantly H. Poincaré, in [69], considered the problem of algebraic integration of polynomial differential equations in the plane. He observed that, in this case, it would be sufficient to bound the degree of algebraic solutions. In the same vein P. Painlevé, in [65], stated an integrability problem as follows:

“Is it possible to recognize the genus of the general solution of an algebraic differential equation in two variables which has a rational first integral?”

Nowadays these problems are known as *Poincaré’s type Problems* and *Painlevé’s type Problems*. In [53] A. Lins Neto constructed families of foliations on $\mathbb{P}_{\mathbb{C}}^2$, with fixed degree and local analytic type of the singularities, where foliations with rational first integrals of arbitrarily large degree appear. In other words, such families show that the questions of Poincaré and Painlevé have a negative answer in general. However, one can obtain an affirmative answer provided some additional hypotheses are assumed.

The current interest in Poincaré’s problem was stimulated by several works, like D. Cerveau and A. Lins Neto [21] and M. Carnicer [18]. Many authors have been working on these problems and on some of its generalizations, see for instance the papers M. Soares [75], J.V. Pereira [67], M. Brunella & L.G. Mendes [14], E. Esteves & S. Kleiman [35], Cavalier & Lehmann [19], and Zamora [81].

The problem of bounding the genus of an invariant curve in terms of the degree of a one-dimensional foliation on $\mathbb{P}_{\mathbb{C}}^n$ was considered for instance by Campillo, Carnicer and de la Fuente [16]. They showed that, if C is a reduced curve which is invariant by a one-dimensional foliation \mathcal{F} on $\mathbb{P}_{\mathbb{C}}^n$, then

$$\frac{2p_a(C) - 2}{\deg(C)} \leq \deg(\mathcal{F}) - 1 + a, \quad (3)$$

where $p_a(C)$ is the arithmetic genus of C and a is an integer obtained from the concrete problem of imposing singularities to projective hypersurfaces. For instance, if C has only nodal singularities then $a = 0$, and thus formula (3.1) follows from [38].

In [34], Esteves and Kleiman had extended Jouanolou's work on algebraic Pfaff equations on a smooth scheme V . They introduced the notion of a *Pfaff field* in V , which is a nontrivial sheaf map $\eta : \Omega_V^s \rightarrow L$, where L is an invertible sheaf on V , and the integer $1 \leq s \leq n - 1$ is called the rank of η . A subvariety $X \subset V$ is said to be invariant under η if the map η factors through the natural map $\Omega_V^s|_X \rightarrow \Omega_X^s$. A Pfaff system on V induces, via exterior powers and the perfect pairing of differential forms, a Pfaff field on V .

In this thesis we establish upper bounds for the sectional genus of Gorenstein varieties which are invariant under Pfaff fields on \mathbb{P}_k^n , where k is an algebraically closed field of characteristic zero. More precisely, our result is the following.

Theorem. *Let $X \subset \mathbb{P}_k^n$ be a Gorenstein projective variety which is invariant under a holomorphic Pfaff field \mathcal{F} on \mathbb{P}_k^n whose rank is equal to the dimension of X , and such that $\text{codim}(\text{Sing}(X), X) \geq 2$. Then*

$$\frac{2g(X, \mathcal{O}_X(1)) - 2}{\deg(X)} \leq \deg(\mathcal{F}) - 1, \quad (4)$$

where $g(X, \mathcal{O}_X(1))$ is the sectional genus of X with respect to the line bundle $\mathcal{O}_X(1)$ associated to the hyperplane section.

This generalizes a bound obtained by Campillo, Carnicer and de la Fuente in [16, Theorem 4.1 (a)].

Let us return to the integrability problem. The work of J.P. Jouanolou in [49] also gives an improvement and generalization of the Darboux theory of integrability characterizing the existence of rational first integrals for Pfaff equations on \mathbb{P}_k^n , where k is an algebraically closed field of characteristic zero. Namely, let ω be a twisted 1-form $\omega \in H^0(\mathbb{P}_k^n, \Omega_{\mathbb{P}_k^n}^1 \otimes \mathcal{O}(m+1))$, where m was called by Jouanolou the degree of ω ¹. Then follows from [49] Theorem 3.3, p.g 102, that ω admits a rational first integral if and only if possesses an infinite number of irreducible hypersurfaces. More generally, Jouanolou proved in [50] that on a complex compact manifold X satisfying certain conditions on its Hodge-to-de Rham spectral sequence, a Pfaff equation $\omega \in H^0(X, \Omega_X^1 \otimes \mathcal{L})$, where \mathcal{L} is a line bundle, admits a meromorphic first integral if and only if possesses an infinite number of invariant irreducible divisors. Moreover, if ω does not admit a meromorphic first integral, then the number of invariant irreducible divisors is at most

$$\dim_{\mathbb{C}}(H^0(X, \Omega_X^2 \otimes \mathcal{L})/\omega \wedge H^0(X, \Omega_X^1)) + \rho(X) + 1,$$

where $\rho(X)$ is the Picard number of X .

¹Nowadays, a Pfaff equation ω on \mathbb{P}_k^n is usually given by a global section of $\Omega_{\mathbb{P}_k^n}^1 \otimes \mathcal{O}(d+2)$, where d is the number of tangency points of a generic line with the distribution induced by ω . Thus in the Jouanolou's notation $m = d + 1$.

E. Ghys in [42] drops all hypotheses given by Jouanolou showing that this result is valid for all compact complex manifold. M. Brunella and M. Nicolau in [15] proved this same result for Pfaff equations in positive characteristic and for non-singular codimension one transversal holomorphic foliations on compact manifolds. A discrete dynamical version of Jouanolou's theorem was recently proved by S. Cantat. He proved In [17] that if there exist N invariant irreducible hypersurfaces with

$$N \geq \dim(M) + h^{1,1}(M)$$

then f preserves a nontrivial meromorphic fibration.

We show a version of this results for vector fields on complete singular toric varieties. Let \mathbb{P}_Δ be a simplicial toric variety associated by a fan Δ and $\mathcal{T}\mathbb{P}_\Delta$ its Zarisk's sheaf. A singular holomorphic foliation \mathcal{F} on \mathbb{P}_Δ is a global section of $\mathcal{T}\mathbb{P}_\Delta \otimes K_{\mathcal{F}}$, where $K_{\mathcal{F}}$ is a invertible sheaf on \mathbb{P}_Δ . We denote \mathbb{T}^n the torus acting on \mathbb{P}_Δ and we call a \mathbb{T}^n -invariant Weil divisor as \mathbb{T}^n -divisor. We use the existence of homogeneous coordinate for simplicial toric varieties to prove the following result.

Theorem. *Let \mathcal{F} be an one-dimensional foliation on a complete simplicial toric variety \mathbb{P}_Δ of dimension n and picard number $\rho(\mathbb{P}_\Delta)$. If \mathcal{F} admits*

$$h^0(\mathbb{P}_\Delta, \mathcal{O}(K_{\mathcal{F}})) + \rho(\mathbb{P}_\Delta) + n$$

invariants irreducible \mathbb{T}^n -divisors, then \mathcal{F} admit a rational first integral.

Observe that, in general \mathbb{P}_Δ is a singular variety with quotient singularities. Therefore, in two dimension this result show that the Darboux-Jouanolou-Ghys's theorem is valid for a class of singular toric variety .

The affine and non-singular version of this result was proved by J. LLibre and X. Zhang in [57]. They showed that if the number of invariant algebraic hypersurfaces of a polynomial vector field Z in \mathbb{C}^n , of degree d , is at least

$$\binom{d+n-1}{n} + n$$

then Z admits a rational first integral.

Moreover, we study one-dimensional foliations in two classes of toric varieties, the multiprojective spaces and weighted projective planes. Under suitable hypotheses we obtain bounds for Poincaré's problem in this varieties.

Finally, we extend some results due to J. V. Pereira for integrability of a one-dimensional foliation \mathcal{F} on a complex manifold M , using the concept of *extatic section* with respect to the a finite dimensional linear system $V \subset H^0(M, \mathcal{O}(D))$, where D is an effective divisor on M . The zero locus of extatic section is the inflection locus of linear systems with respect to the vector field inducing \mathcal{F} .

Denote by $\varepsilon(V, \mathcal{F})$ the extatic section of \mathcal{F} with respect V . if \mathcal{F} is a holomorphic one-dimensional foliation on a complex manifold M , then a holomorphic (or meromorphic) first integral for \mathcal{F} is a holomorphic (resp. meromorphic) map $\Theta : M \rightarrow Y$, where Y is a complex manifold, such that the fibers of Θ are invariants by \mathcal{F} . J. V. Pereira showed in [67] the following theorem:

Theorem. *Let \mathcal{F} be a one-dimensional holomorphic foliation on a complex manifold M . If V is a finite dimensional linear system such that $\varepsilon(V, \mathcal{F})$ vanishes identically, then there exists an open and dense set U where $\mathcal{F}|_U$ admits a holomorphic first integral. Moreover, if M is a projective variety, then \mathcal{F} admits a meromorphic first integral.*

In the non-algebraic and non-compact cases the result above does not guarantee that the vanishing of extatic section $\varepsilon(V, \mathcal{F})$ implies in the existence of a meromorphic integral first for \mathcal{F} . We provided the following addendum for J. V. Pereira's theorem.

Theorem. *Let \mathcal{F} be a one-dimensional holomorphic foliation on a complex manifold M and V a finite dimensional linear system. If $\varepsilon(V, \mathcal{F})$ vanishes identically then \mathcal{F} admits a meromorphic first integral $\Theta : M \rightarrow \mathbb{P}^1$.*

J. V. Pereira in [67] showed that a foliation on $\mathbb{P}_{\mathbb{C}}^2$, of degree $d > 1$, that does not admit rational first integral of degree $\leq k$, it has at most

$$\binom{k+2}{k} + \frac{(d-1)}{k} \cdot \binom{k+2}{2}.$$

invariant curves of degree k .

Let (M, L) be a polarized projective variety and denote by $\mathcal{N}(\mathcal{F}, V)$ the number of \mathcal{F} -invariant divisors contained in the linear system $V \subset H^0(M, \mathcal{O}(D))$. We use the concept of degree of foliations and divisors with respect to polarization L and extatic divisor to show the following result.

Theorem. *Let \mathcal{F} be a one-dimensional foliation on a polarized projective algebraic manifold (M, L) and D an effective \mathcal{F} -invariant divisor. Suppose that \mathcal{F} does not admit a rational first integral. Then*

$$\deg_L(D) \cdot (\mathcal{N}(\mathcal{F}, V) - h^0(V)) \leq (\deg_L(\mathcal{F}) - \deg_L(M)) \cdot \binom{h^0(V)}{2},$$

where $h^0(V) = \dim_{\mathbb{C}} V$. In particular, we have that:

- i) *the number of divisors \mathcal{F} -invariant contained on the linear system $V \subset \mathcal{O}(D)$ is at most*

$$\frac{(\deg_L(\mathcal{F}) - \deg_L(M))}{\deg_L(D)} \cdot \binom{h^0(V)}{2} + h^0(V),$$

where $h^0(V) = \dim_{\mathbb{C}} V$.

ii) if $\mathcal{H} \subset |\mathcal{O}(D)|$ is a pencil and suppose that $\mathcal{N}(\mathcal{F}, \mathcal{H}) > 2$, then

$$\deg_L(D) \leq \deg_L(\mathcal{F}) - \deg_L(M)$$

The part ii) of this theorem give us a numerical criteria to decide if a holomorphic foliation \mathcal{F} on the polarized variety (M, L) admits a rational first integral. That is, if we suppose that the number of \mathcal{F} -invariant divisors contained on the pencil \mathcal{H} is > 2 and \mathcal{F} possesses a invariant effective divisor $C \in \mathcal{H}$ satisfying the condition

$$\deg_L(C) > \deg_L(\mathcal{F}) - \deg_L(M),$$

then \mathcal{F} admit a rational first integral. This result is related to a Lins Neto conjecture. In [53] he stated the following problem:

“Given $d \geq 2$, is there $M(d) \in \mathbb{N}$ such that if a foliation on \mathbb{P}^2 , of degree d , has an algebraic solution of degree greater than or equal to $M(d)$, then it has a rational first integral?”

Let \mathcal{F} be a one-dimensional foliation on \mathbb{P}^n of degree $d \geq 2$. It follows that if the number of \mathcal{F} -invariant hypersurfaces of degree k contained on a pencil of the same degree is > 2 and $k > M(d) = d - 1$, then \mathcal{F} has a rational first integral.

Chapter 1

The number of invariant divisors and Poincaré's problem

1.1 The degree of foliations with respect to a polarization

Let (M, L) be a n -dimensional polarized projective variety, i.e, M is smooth and L is a very ample line bundle on M . The degree of a holomorphic vector bundle E on M related to the polarization L is defined by

$$\deg_L(E) = \int_M c_1(E) \cdot L^{n-1},$$

where \int_M denote the degree of cycle.

Proposition 1.1.1. *Let H be a line bundle on M such that $H^0(M, H) \neq \{0\}$. Then $\deg_L(H) \geq 0$.*

Proof. See [52, Theorem 1.24]. □

Remark 1.1.1. Let D be an effective divisor on M . The degree of D is defined by $\deg_L(\mathcal{O}(D))$. Since D is effective we have that $H^0(M, \mathcal{O}(D)) \neq \{0\}$, thus $\deg_L(\mathcal{O}(D)) \geq 0$.

Definition 1.1.1. *A one-dimensional foliation on M is a global holomorphic section of $TM \otimes K_{\mathcal{F}}$, where $K_{\mathcal{F}}$ is a line bundle on M .*

Let D be an analytic hypersurface on M defined locally by functions $\{f_\alpha \in \mathcal{O}(\mathcal{U}_\alpha)\}_{\alpha \in \Lambda}$, where $\{\mathcal{U}_\alpha\}_{\alpha \in \Lambda}$ is an open covering of M . If $\mathcal{U}_{\alpha\beta} := \mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \emptyset$ then there exist $f_{\alpha\beta} \in \mathcal{O}^*(\mathcal{U}_{\alpha\beta})$, such that $f_\alpha = f_{\alpha\beta} f_\beta$. Let \mathcal{F} be a holomorphic

foliation given by collections $(\{\vartheta_\alpha\}; \{\mathcal{U}_\alpha\}; \{g_{\alpha\beta} \in \mathcal{O}^*(\mathcal{U}_\alpha)\})_{\alpha \in \Lambda}$ on M , where $g_{\alpha\beta}$ is the cocycle inducing $K_{\mathcal{F}}$. Consider the following functions

$$\zeta_\alpha^{(\mathcal{F}, D)} = \vartheta_\alpha(f_\alpha)|_D \in \mathcal{O}(\mathcal{U}_\alpha \cap D).$$

If $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap D \neq \emptyset$, using Leibniz's rule we get $\zeta_\alpha^{(\mathcal{F}, D)} = f_{\alpha\beta} g_{\alpha\beta} \zeta_\beta^{(\mathcal{F}, D)}$. With this we obtain a global section $\zeta^{(\mathcal{F}, D)}$ of the line bundle $(K_{\mathcal{F}} \otimes \mathcal{O}(D))|_D$. The *tangency variety* of \mathcal{F} with D is given by

$$\mathcal{T}(\mathcal{F}, D) = \{p \in D; \zeta^{(\mathcal{F}, D)}(p) = 0\}.$$

Definition 1.1.2. Let (M, L) be a polarized variety and \mathcal{F} an foliation on M of dimension one. The degree of \mathcal{F} with respect to the polarization L is the intersection number

$$\deg_L(\mathcal{F}) := \int_L \mathcal{T}(\mathcal{F}, L) \cdot L^{n-2}.$$

Proposition 1.1.2. Let \mathcal{F} be a foliation on a polarized variety (M, L) . Then

$$\deg_L(\mathcal{F}) = \deg_L(K_{\mathcal{F}}) + \deg_L(M),$$

where $\deg_L(M) = \deg_L(L)$ is the degree of M with respect to L .

Proof. We have the adjunction formula $\mathcal{T}(\mathcal{F}, L) = (K_{\mathcal{F}} + L)|_L$ and by definition

$$\begin{aligned} \deg_L(\mathcal{F}) &= \int_L \mathcal{T}(\mathcal{F}, L) \cdot L^{n-2} = \int_L (K_{\mathcal{F}} + L) \cdot L^{n-2} \\ &= \int_M K_{\mathcal{F}} \cdot L^{n-1} + \int_M L^n \\ &= \deg_L(K_{\mathcal{F}}) + \deg_L(M). \end{aligned}$$

□

We shall assume $\deg_L(K_{\mathcal{F}}) \geq 0$, or equivalently $\deg_L(\mathcal{F}) - \deg_L(M) \geq 0$.

Example 1.1.1. Let \mathcal{F} be a foliation on M , where $\text{Pic}(M) \simeq \mathbb{Z}$. We can take a hyperplane section $\mathcal{H} = H \cap M$ to be a positive generator of $\text{Pic}(M)$, so we denote by $\mathcal{O}_M(k) := \mathcal{H}^{\otimes k}$ the k -th tensor power of \mathcal{H} . If we write $K_{\mathcal{F}} = \mathcal{O}_M(d-1)$, then $\deg(K_{\mathcal{F}}) = (d-1) \deg(M)$. Hence

$$\deg(\mathcal{F}) = \deg(K_{\mathcal{F}}) + \deg(M) = (d-1) \deg(M) + \deg(M) = d \cdot \deg(M).$$

In the case where $M = \mathbb{P}^n$ we will have, as is known, that $\deg(\mathcal{F}) = d$.

1.2 The extatic divisor

The method adopted here stems from the work of J. V. Pereira [67], where the notion of extactic variety is exploited. In this section we digress briefly on extactic varieties and their main properties.

Let H be a holomorphic line bundle on M . Consider the linear system $V \subset H^0(M, H)$ and take an open covering $\{\mathcal{U}_\alpha\}_{\alpha \in \Lambda}$ of M which trivializes H and $K_{\mathcal{F}}$. In the open set \mathcal{U}_α we can consider the morphism

$$T_\alpha^{(k)} : V \otimes \mathcal{O}_{\mathcal{U}_\alpha} \rightarrow \mathcal{O}_{\mathcal{U}_\alpha}^k$$

defined by

$$T_\alpha^{(k)}(s_\alpha) = s_\alpha + X_\alpha(s_\alpha) \cdot t + X_\alpha^2(s_\alpha) \cdot \frac{t^2}{2!} + \cdots + X_\alpha^{(k-1)}(s_\alpha) \cdot \frac{t^{(k-1)}}{(k-1)!},$$

where s_α and X_α are local representations, respectively, of a section $s \in V \subset H^0(M, H)$ and the section $X_{\mathcal{F}} \in H^0(M, TM \otimes K_{\mathcal{F}})$ inducing \mathcal{F} . If $\mathcal{U}_\alpha \cap \mathcal{U}_\gamma \neq \emptyset$ then $s_\alpha = g_{\alpha\gamma} s_\gamma$ and $X_\alpha = f_{\alpha\gamma} X_\gamma$, where $g_{\alpha\gamma}, f_{\alpha\gamma} \in \mathcal{O}^*(\mathcal{U}_\alpha)$ are the cocycles which define, respectively, the line bundles H and $K_{\mathcal{F}}$. Using the compatibility described above and Leibniz's rule we get

$$X_\alpha(s_\alpha) = f_{\alpha\beta} X_\beta(g_{\alpha\gamma}) \cdot s_\beta + g_{\alpha\beta} f_{\alpha\beta} \cdot X_\beta(s_\beta)$$

Following this process up to order $k = \dim_{\mathbb{C}} V$, we obtain

$$\begin{bmatrix} s_\alpha \\ X_\alpha^2(s_\alpha) \\ X_\alpha^3(s_\alpha) \\ \vdots \\ X_\alpha^{k-1}(s_\alpha) \end{bmatrix} = \begin{bmatrix} g_{\alpha\beta} & 0 & 0 & 0 & 0 \\ X_\beta(g_{\alpha\beta}) \cdot f_{\alpha\beta} & g_{\alpha\beta} \cdot f_{\alpha\beta} & 0 & 0 & 0 \\ \ddots & \ddots & g_{\alpha\beta} \cdot f_{\alpha\beta}^2 & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & 0 \\ \ddots & \ddots & \ddots & \ddots & g_{\alpha\beta} \cdot f_{\alpha\beta}^{k-1} \end{bmatrix} \cdot \begin{bmatrix} s_\beta \\ X_\beta^2(s_\beta) \\ X_\beta^3(s_\beta) \\ \vdots \\ X_\beta^{k-1}(s_\beta) \end{bmatrix}$$

Denoting the $k \times k$ matrix above by $\Theta_{\alpha\beta}(\mathcal{F}, V) \in GL(k, \mathcal{O}_{\mathcal{U}_{\alpha\beta}})$, we see that

$$\begin{cases} \Theta_{\alpha\beta}(\mathcal{F}, V)(p) \cdot \Theta_{\beta\alpha}(\mathcal{F}, V)(p) = I, & \text{for all } p \in \mathcal{U}_\alpha \cap \mathcal{U}_\beta \\ \Theta_{\alpha\beta}(\mathcal{F}, V)(p) \cdot \Theta_{\beta\lambda}(\mathcal{F}, V)(p) \cdot \Theta_{\lambda\alpha}(\mathcal{F}, V)(p) = I, & \text{for all } p \in \mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \mathcal{U}_\lambda. \end{cases}$$

That is, the family of matrices $\{\Theta_{\alpha\gamma}(\mathcal{F}, V)\}_{\alpha\gamma}$ defines a cocycle of a vector bundle of rank k on M that we denote by $J_{\mathcal{F}}^{k-1}H$. Now, using the trivializations $\{\Theta_{\alpha\gamma}(\mathcal{F}, V)\}_{\alpha, \beta \in \Lambda}$ we get the morphisms

$$T^{(k)} : V \otimes \mathcal{O}_M \rightarrow J_{\mathcal{F}}^{k-1}H.$$

Taking the determinant of $T^{(k)}$ we have the morphism

$$\det(T^{(k)}) : \bigwedge^k V \otimes \mathcal{O}_M \rightarrow \bigwedge^k J_{\mathcal{F}}^{k-1} H,$$

and tensorizing by $(\bigwedge^k V)^*$ we obtain a global section of $\bigwedge^k J_{\mathcal{F}}^{k-1} H \otimes (\bigwedge^k V)^*$ given by

$$\varepsilon(\mathcal{F}, V) : \mathcal{O}_M \rightarrow \bigwedge^k J_{\mathcal{F}}^{k-1} H \otimes (\bigwedge^k V)^*.$$

Remark 1.2.1. Note that the cocycle of $\bigwedge^k J_{\mathcal{F}}^{k-1} H$ is given by

$$\det(\Theta_{\alpha\gamma}(\mathcal{F}, V)) = g_{\alpha\beta}^k \cdot f_{\alpha\beta}^{\binom{k}{2}},$$

where $g_{\alpha\beta}$ and $f_{\alpha\beta}$ are respectively the trivializations of H and $K_{\mathcal{F}}$. Therefore, we obtain the isomorphism $\bigwedge^k J_{\mathcal{F}}^{k-1} H \simeq H^{\otimes k} \otimes (K_{\mathcal{F}})^{\otimes \binom{k}{2}}$.

Definition 1.2.1. *The extatic divisor of \mathcal{F} with respect to the linear system $V \subset H^0(M, H)$ is the divisor $\mathcal{E}(\mathcal{F}, V) = (\varepsilon(\mathcal{F}, V))$ given by the zeros of the section*

$$\varepsilon(\mathcal{F}, V) \in H^0\left(M, \bigwedge^k J_{\mathcal{F}}^{k-1} H \otimes (\bigwedge^k V)^*\right).$$

The section $\varepsilon(\mathcal{F}, V)$ is called **extatic section** of \mathcal{F} with respect V .

J. V. Pereira [67] obtained the following results, which elucidate the role of the extatic divisor :

Proposition 1.2.1. *([67], Proposition 5) Let \mathcal{F} be a one-dimensional holomorphic foliation on a complex manifold M . If V is a finite dimensional linear system, then every \mathcal{F} -invariant hypersurface which is contained in the zero locus of some element of V must be contained in the zero locus of $\mathcal{E}(V, \mathcal{F})$.*

Proof. Let $\{s_1, \dots, s_k\}$ be a basis for $V \subset H^0(M, H)$. On the open \mathcal{U}_α the extatic section is given by

$$\varepsilon(V, \mathcal{F})_\alpha = \det \begin{pmatrix} s_1^\alpha & s_2^\alpha & \cdots & s_k^\alpha \\ X_\alpha(s_1^\alpha) & X_\alpha(s_2^\alpha) & \cdots & X_\alpha(s_k^\alpha) \\ \vdots & \vdots & \ddots & \vdots \\ X_\alpha^{k-1}(s_1^\alpha) & X_\alpha^{k-1}(s_2^\alpha) & \cdots & X_\alpha^{k-1}(s_k^\alpha) \end{pmatrix},$$

where X_α is a vector field that induces \mathcal{F} on \mathcal{U}_α and s_i^α is local representation of the section s_i , $i = 1, \dots, k$. Let f_α be the local equation defining an element on V and suppose that $(f_\alpha = 0)$ is \mathcal{F} -invariant. Change basis so that V is generated by $f_\alpha, v_2, \dots, v_\ell$. It follows that $X_\alpha^j(f_\alpha) = h_\alpha^j f_\alpha$, $1 \leq j \leq k-1$, where h_α^j is an analytic function. □

If \mathcal{F} is a holomorphic one-dimensional foliation on a complex manifold M , then a holomorphic (or meromorphic) first integral for \mathcal{F} is a holomorphic (resp. meromorphic) map $\Theta : M \rightarrow Y$, where Y is a complex manifold with $\dim(M) > \dim(Y)$, such that the fibers of Θ are invariant by \mathcal{F} . J. V. Pereira showed in [67] the following theorem:

Theorem 1.2.1. ([67], Theorem 3). *Let \mathcal{F} be a one-dimensional holomorphic foliation on a complex manifold M . If V is a finite dimensional linear system such that $\varepsilon(V, \mathcal{F})$ vanishes identically, then there exists an open and dense set U where $\mathcal{F}|_U$ admits a first integral. Moreover, if M is a projective variety, then \mathcal{F} admits a meromorphic first integral.*

In the non-algebraic and non-compact cases Theorem 1.2.1 does not guarantee that the vanishing of the extatic section $\varepsilon(V, \mathcal{F})$ implies the existence of a meromorphic first integral for \mathcal{F} . We show that if $\varepsilon(V, \mathcal{F})$ vanishes identically, then \mathcal{F} admits a meromorphic first integral with values in \mathbb{P}^1 .

Theorem 1.2.2. *Let \mathcal{F} be a one-dimensional holomorphic foliation on a complex manifold M and V a finite dimensional linear system. If $\varepsilon(V, \mathcal{F})$ vanishes identically then \mathcal{F} admits a meromorphic first integral $\Theta : M \rightarrow \mathbb{P}^1$.*

Proof. Suppose that the foliation \mathcal{F} is given by the collections

$$(\{\mathcal{U}_\alpha\}, \{X_\alpha\}, \{g_{\alpha\beta} \in \mathcal{O}^*(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)\})_{\alpha\beta \in \Lambda}.$$

We will show the existence of a local meromorphic first integral on each open \mathcal{U}_α . That is, there exists a meromorphic function θ^α such that $X_\alpha(\theta^\alpha) = 0$, where X_α is the vector field defining \mathcal{F} on \mathcal{U}_α . After this, we must prove that $\theta^\alpha = \theta^\beta$ on $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \emptyset$, thus we shall obtain a global meromorphic function defining a first integral for \mathcal{F} . For the existence of θ^β on \mathcal{U}_α , we will use the same arguments given in the proof of Theorem 4.3 of [24] for the case of polynomial vector fields on \mathbb{C}^2 .

Let $\{s_1, \dots, s_k\}$ be a \mathbb{C} -base for V . Suppose that $\varepsilon(V, \mathcal{F})$ vanishes identically. Then on the open \mathcal{U}_α we have that

$$\varepsilon(V, \mathcal{F})_\alpha = \det \begin{pmatrix} s_1^\alpha & s_2^\alpha & \cdots & s_k^\alpha \\ X_\alpha(s_1^\alpha) & X_\alpha(s_2^\alpha) & \cdots & X_\alpha(s_k^\alpha) \\ \vdots & \vdots & \ddots & \vdots \\ X_\alpha^{k-1}(s_1^\alpha) & X_\alpha^{k-1}(s_2^\alpha) & \cdots & X_\alpha^{k-1}(s_k^\alpha) \end{pmatrix} \equiv 0,$$

where X_α is a vector field that induces \mathcal{F} on \mathcal{U}_α and s_i^α is the local representation of the section s_i , $i = 1, \dots, k$.

To say that $\varepsilon(V, \mathcal{F})_\alpha \equiv 0$ means that the columns of the above matrix are dependent over the field of meromorphic functions $\mathcal{M}(\mathcal{U}_\alpha)$. Hence, there are meromorphic functions $\theta_1^\alpha, \dots, \theta_k^\alpha$ on \mathcal{U}_α , such that

$$M_i^\alpha = \sum_{j=1}^k \theta_j^\alpha X_\alpha^i(s_j^\alpha) = 0, \quad 0 \leq i \leq k-1. \quad (1.1)$$

Now, let $r(\alpha)$ be the smallest integer with the property that there exist meromorphic functions $\theta_1^\alpha, \dots, \theta_{r(\alpha)}^\alpha$ and $s_1^\alpha, \dots, s_{r(\alpha)}^\alpha \in V$, linearly independent over \mathbb{C} , such that (1.1) holds. We clearly have $1 < r(\alpha) \leq k$ and we may assume $\theta_{r(\alpha)}^\alpha = 1$. Applying the derivation X_α to both sides of (1.1) we get

$$X_\alpha(M_i^\alpha) = X_\alpha(\theta_1^\alpha)X_\alpha^i(s_1^\alpha) + \dots + \underbrace{X_\alpha(\theta_{r(\alpha)}^\alpha)}_0 X_\alpha^i(s_{r(\alpha)}^\alpha) + \underbrace{\theta_{r(\alpha)}^\alpha}_{1} X_\alpha^{i+1}(s_{r(\alpha)}^\alpha) = 0 \quad (1.2)$$

for all $0 \leq i \leq r(\alpha) - 2$. Subtracting (1.2) from M_{i+1}^α we obtain

$$X_\alpha(M_i^\alpha) - M_{i+1}^\alpha = X_\alpha(\theta_1^\alpha)X_\alpha^i(s_1^\alpha) + \dots + X_\alpha(\theta_{r(\alpha)-1}^\alpha)X_\alpha^i(s_{r(\alpha)-1}^\alpha) = 0, \quad 0 \leq i \leq r(\alpha) - 2.$$

By the minimality of $r(\alpha)$ we must have $X_\alpha(\theta_1^\alpha) = \dots = X_\alpha(\theta_{r(\alpha)-1}^\alpha) = 0$ and hence, provided these are not all constants, we have a first integral for X_α on \mathcal{U}_α . This in fact occurs because, since M_0^α is

$$M_0^\alpha = \theta_1^\alpha s_1^\alpha + \dots + \theta_{r(\alpha)-1}^\alpha s_{r(\alpha)-1}^\alpha + s_{r(\alpha)}^\alpha = 0,$$

we conclude that not all the θ^α 's could be constant since $s_1^\alpha, \dots, s_{r(\alpha)}^\alpha \in V$ are linearly independent over \mathbb{C} . Now we will show that $r = r(\alpha) = r(\beta)$, for all $\alpha, \beta \in \Lambda$. Suppose that $r(\alpha) < r(\beta)$. In $\mathcal{U}_{\alpha\beta}$ we have that $s_i^\alpha = f_{\alpha\beta} s_i^\beta$, $i = 1, \dots, r(\alpha)$, and $X_\alpha = g_{\alpha\beta} X_\beta$, with $f_{\alpha\beta}, g_{\alpha\beta} \in \mathcal{O}^*(\mathcal{U}_{\alpha\beta})$. Using this we conclude that $X_\beta(\theta_i^\alpha) = 0$ on $\mathcal{U}_{\alpha\beta}$, for all $i = 1, \dots, r(\alpha) - 1$, and

$$\theta_1^\alpha s_1^\beta + \dots + \theta_{r(\alpha)-1}^\alpha s_{r(\alpha)-1}^\beta + s_{r(\alpha)}^\beta = 0.$$

Applying the derivation X_β in this equation and using that $X_\beta(\theta_i^\alpha) = 0$, for all $i = 1, \dots, r(\alpha) - 1$, we get

$$\sum_{j=1}^{r(\alpha)} \theta_j^\alpha X_\beta^i(s_j^\beta) = 0, \quad 0 \leq i \leq r(\alpha) - 1,$$

by the minimality of $r(\beta)$ we can conclude that $\theta_1^\alpha = \dots = \theta_{r(\alpha)}^\alpha = 0$, but this implies that $M_0^\alpha = s_{r(\alpha)}^\alpha = 0$, and this is a contradiction. The case $r(\beta) < r(\alpha)$ is similar.

Now, consider the equations

$$\theta_1^\alpha s_1^\alpha + \dots + \theta_{r-1}^\alpha s_{r-1}^\alpha + s_r^\alpha = 0$$

$$\theta_1^\beta s_1^\alpha + \cdots + \theta_{r-1}^\beta s_{r-1}^\alpha + s_r^\alpha = 0.$$

Subtracting these equations we obtain

$$(\theta_1^\alpha - \theta_1^\beta) s_1^\alpha + \cdots + (\theta_{r-1}^\alpha - \theta_{r-1}^\beta) s_{r-1}^\alpha = 0.$$

Define $h_i^{\alpha\beta} = (\theta_i^\alpha - \theta_i^\beta) \in \mathcal{M}(\mathcal{U}_{\alpha\beta})$, $i = 1, \dots, r-1$. Applying the derivation X_α to the last equation and using $X_\alpha(\theta_i^\alpha) = X_\alpha(\theta_i^\beta) = 0$ we get

$$\sum_{j=1}^{r-1} h_j^{\alpha\beta} X_\alpha^i(s_j^\alpha) = 0, \quad 0 \leq i \leq r-2.$$

Again, by the minimality of r we have that $h_1^{\alpha\beta} = \cdots = h_{r-1}^{\alpha\beta} = 0$, i.e. $\theta_i^\alpha = \theta_i^\beta$ on $\mathcal{U}_{\alpha\beta}$, for all $i = 1, \dots, r-1$. Therefore, we obtain a meromorphic first integral Θ^i locally given by $\Theta_{\mathcal{U}_\alpha}^i = \theta_i^\alpha$, for some $i = 1, \dots, r-1$. □

Let $D = \sum_\gamma a_\gamma D_\gamma$ be an effective divisor and \mathcal{F} a one-dimensional foliation on the complex manifold M . We say that D is \mathcal{F} -invariant if D_γ is invariant by \mathcal{F} for all γ .

Theorem 1.2.3. *Let \mathcal{F} be a one-dimensional foliation on a polarized projective algebraic manifold (M, L) and D an effective divisor. Suppose that \mathcal{F} does not admit a rational first integral. Then*

$$\deg_L(D) \cdot (\mathcal{N}(\mathcal{F}, V) - h^0(V)) \leq (\deg_L(\mathcal{F}) - \deg_L(M)) \cdot \binom{h^0(V)}{2},$$

where $\mathcal{N}(\mathcal{F}, V)$ is the number of \mathcal{F} -invariant divisors contained on the linear system $V \subset |D|$ and $h^0(V) = \dim_{\mathbb{C}} V$. In particular, we have that:

- i) the number of \mathcal{F} -invariant divisors contained on the linear system $V \subset |\mathcal{O}(D)|$ is at most

$$\frac{(\deg_L(\mathcal{F}) - \deg_L(M))}{\deg_L(D)} \cdot \binom{h^0(V)}{2} + h^0(V),$$

where $h^0(V) = \dim_{\mathbb{C}} V$.

- ii) if $V \subset |\mathcal{O}(D)|$ is a pencil and $\mathcal{N}(\mathcal{F}, V) > 2$, then

$$\deg_L(D) \leq \deg_L(\mathcal{F}) - \deg_L(M).$$

Proof. It follows from theorem 1.2.1 that if \mathcal{F} does not have a rational first integral, then $\varepsilon(\mathcal{F}, V) \neq 0$. Thus, the extatic section $\varepsilon_{(\mathcal{F}, V)}$ defines an effective divisor $\mathcal{E}(\mathcal{F}, V)$ whose associated line bundle is $\bigwedge^k J_{\mathcal{F}}^{k-1} \mathcal{O}(D) \otimes (\bigwedge^k V)^*$, where $k =$

$\dim_{\mathbb{C}} V \leq h^0(D)$. Let $\mathcal{N}(\mathcal{F}, V)$ be the number of divisors of $V \subset H^0(M, \mathcal{O}(D))$ invariant by \mathcal{F} . It follows from proposition 1.2.1 that every divisor $C \in V$ invariant by \mathcal{F} is contained in the extatic divisor $\mathcal{E}(\mathcal{F}, V)$. Using this fact we can claim that

$$\deg_L(D) \cdot \mathcal{N}(\mathcal{F}, V) \leq \deg_L(\mathcal{E}(\mathcal{F}, V)).$$

Indeed, it is enough to group the \mathcal{F} -invariant divisors of the following form

$$\mathcal{E}(\mathcal{F}, V) = \sum_{j=1}^{\mathcal{N}(\mathcal{F}, V)} C_j + R$$

where $C_i \in V$ is a divisor invariant by \mathcal{F} and R is a divisor without \mathcal{F} -invariant divisor contained in V . Since $\deg_L(C_j) = \deg_L(D)$, for all $j = 1, \dots, \mathcal{N}(\mathcal{F}, V)$, we get

$$\deg_L(D) \cdot \mathcal{N}(\mathcal{F}, V) = \sum_{j=1}^{\mathcal{N}(\mathcal{F}, V)} \deg_L(C_j) \leq \deg_L(\mathcal{E}(\mathcal{F}, V)).$$

This shows the claim above. However, the line bundle associated to the extatic divisor $\mathcal{E}(\mathcal{F}, V)$ is given by $\bigwedge^k J_{\mathcal{F}}^{k-1} \mathcal{O}(D) \otimes (\bigwedge^k V)^*$. This implies that

$$[\mathcal{E}(\mathcal{F}, V)] = \bigwedge^k J_{\mathcal{F}}^{k-1} \mathcal{O}(D) \otimes (\bigwedge^k V)^*.$$

It follows from remark 1.2.1 that $\bigwedge^k J_{\mathcal{F}}^{k-1} \mathcal{O}(D) \simeq \mathcal{O}(D)^{\otimes k} \otimes (K_{\mathcal{F}})^{\otimes \binom{k}{2}}$, thus

$$[\mathcal{E}(\mathcal{F}, V)] = \mathcal{O}(D)^{\otimes k} \otimes (K_{\mathcal{F}})^{\otimes \binom{k}{2}} \otimes (\bigwedge^k V)^*.$$

Calculating the degree $\deg_L(\mathcal{E}(\mathcal{F}, V))$, we obtain

$$\begin{aligned} \deg_L(\mathcal{E}(\mathcal{F}, V)) &= \deg_L \left(\mathcal{O}(D)^{\otimes k} \otimes (K_{\mathcal{F}})^{\otimes \binom{k}{2}} \right) + \underbrace{\deg_L \left(\bigwedge^k V^* \right)}_0 \\ &= k \cdot \deg_L(D) + \deg_L(K_{\mathcal{F}}) \binom{k}{2}. \end{aligned}$$

Finally, the result it follows from $\deg_L(D) \cdot \mathcal{N}(\mathcal{F}, V) \leq \deg_L(\mathcal{E}(\mathcal{F}, V))$ and proposition 1.1.2. \square

Proposition 1.2.2. *Let \mathcal{F} be a foliation without rational first integral. If $\mathcal{E}(\mathcal{F}, V)$ is irreducible then \mathcal{F} does not admit invariant divisors contained in the linear system $V \subset |\mathcal{O}(D)|$.*

Proof. Suppose that \mathcal{F} possesses an invariant divisor $C \in V$. Since all divisors $C \in V$ invariant by \mathcal{F} are contained in the extatic divisor and by hypothesis $\mathcal{E}(\mathcal{F}, V)$ is irreducible, we have that $C = \mathcal{E}(\mathcal{F}, V)$. But

$$\deg_L(C) = \deg_L(D) < k \cdot \deg_L(D) + \deg_L(K_{\mathcal{F}}) \binom{k}{2} = \deg_L(\mathcal{E}(\mathcal{F}, V)),$$

which is an absurd. □

Let \mathcal{F} be a foliation and $\mathcal{H} \subset \mathbb{P}H^0(M, \mathcal{O}(D))$ a pencil. Suppose that $\mathcal{N}(\mathcal{F}, \mathcal{H}) > 2$. It follows from Theorem 1.2.3 part *ii*) that if \mathcal{F} possesses an invariant effective divisor C , contained in the pencil \mathcal{H} , satisfying the condition

$$\deg_L(C) = \deg_L(D) > \deg_L(\mathcal{F}) - \deg_L(M)$$

then \mathcal{F} admits a rational first integral. This result is related to a conjecture of Lins Neto. In [53] he stated the following problem:

“Given $d \geq 2$, is there $M(d) \in \mathbb{N}$ such that if a foliation on \mathbb{P}^2 , of degree d , has an algebraic solution of degree greater than or equal to $M(d)$, then it has a rational first integral?”

Let \mathcal{F} be a one-dimensional foliation on \mathbb{P}^n of degree $d > 2$. It follows that if the number of \mathcal{F} -invariant hypersurfaces of degree k contained on a pencil of the same degree is greater than 2 and $k > M(d) = d - 1$, then \mathcal{F} has a rational first integral.

J. Moulin Ollagnier showed in [64] that when $d = 2$ this question has a negative answer. He exhibited a countable family of Lotka-Volterra foliations given by

$$SLV(\ell) = x(y/2 + z) \frac{\partial}{\partial x} + y(2z + x) \frac{\partial}{\partial y} + z \left(y - \frac{2\ell + 1}{2\ell - 1} x \right) \frac{\partial}{\partial z}$$

without rational first integrals which has an irreducible algebraic solution of degree 2ℓ . C. Christopher and J. LLibre in [23] also exhibit a family of foliations of degree $d = 2$ without rational first integral which contains irreducible algebraic solutions of arbitrarily high degree. But, it follows from Theorem 1.2.3 part *i*) that, for a foliation of degree $d = 2$ the number of invariant curves of degree k contained on a pencil of the same degree is ≤ 2 .

Bounding invariant hyperplane sections

Using Zak’s bound for $h^0(M, \mathcal{O}_M(1))$ we get the following.

Corollary 1.2.1. *Let \mathcal{F} be a one-dimensional foliation on a smooth algebraic variety $M^n \subset \mathbb{P}^N$. Suppose that \mathcal{F} does not admit a rational first integral, then the number of \mathcal{F} -invariant hyperplane sections is at most*

$$\left(\frac{\deg(\mathcal{F})}{\deg(M)} - 1 \right) \cdot \binom{\left\lceil \frac{(4n-N+3)^2}{8(2n-N+1)} \right\rceil}{2} + \left\lceil \frac{(4n-N+3)^2}{8(2n-N+1)} \right\rceil,$$

where $\lceil x \rceil$ denote the largest integer not exceeding x .

Proof. It follows from theorem 1.2.3, and the fact that $\deg_{\mathcal{O}_M(1)}(\mathcal{O}_M(1)) = \deg(M)$, that the number of \mathcal{F} -invariant hyperplane sections is at most

$$\left(\frac{\deg(\mathcal{F})}{\deg(M)} - 1 \right) \cdot \binom{h^0(M, \mathcal{O}_M(1))}{2} + h^0(M, \mathcal{O}_M(1)).$$

Now, the result follows from

$$h^0(M, \mathcal{O}_M(1)) \leq \left\lceil \frac{(4n-N+3)^2}{8(2n-N+1)} \right\rceil,$$

see [79] pg. 117, Theorem 2.10. □

Example 1.2.1. Let \mathcal{F} be a one-dimensional foliation on a smooth algebraic variety $M^n \subset \mathbb{P}_{\mathbb{C}}^N$. Suppose that \mathcal{F} does not admit a rational first integral. Then, if $N \leq 2n$, the number of \mathcal{F} -invariant hyperplane sections is at most

$$\left(\frac{\deg(\mathcal{F})}{\deg(M)} - 1 \right) \cdot \binom{\binom{n+2}{2}}{2} + \binom{n+2}{2}$$

This is a consequence of corollary 1.2.1 and of the following result (see [79] corollary 2.9): if $N \leq 2n$, then $h^0(M, \mathcal{O}_M(1)) \leq \binom{n+2}{2}$.

Example 1.2.2. We recall that a nonsingular algebraic variety $M^n \subset \mathbb{P}_{\mathbb{C}}^N$ is called **linearly normal** if $h^0(M, \mathcal{O}_M(1)) = N + 1$. Zak's Linear Normality theorem say that if $N < \frac{3}{2}n + 1$ then M^n is linearly normal, see [80]. Let \mathcal{F} be a one-dimensional foliation on a linearly normal smooth algebraic variety $M \subset \mathbb{P}^N$. Suppose that \mathcal{F} does not admit a rational first integral. Then it follows from Corollary 1.2.1 that the number of \mathcal{F} -invariant hyperplane sections is at most

$$\left(\frac{\deg(\mathcal{F})}{\deg(M)} - 1 \right) \cdot \binom{N+1}{2} + N + 1.$$

1.3 Optimal examples on projective spaces

In this section we will consider foliations on $\mathbb{P}_{\mathbb{C}}^n$. We will construct some examples of foliations with the maximum number of invariant hyperplanes. Let \mathcal{F} be a one-dimensional holomorphic foliation on $\mathbb{P}_{\mathbb{C}}^n$, of degree $d > 0$, and suppose that \mathcal{F} does not admit a rational first integral. It follows from example 1.2.2 that number of \mathcal{F} -invariant hyperplanes is bounded by

$$n + 1 + \binom{n+1}{2}(d-1).$$

The next result gives us the number of invariant hyperplanes by a foliation on $\mathbb{P}_{\mathbb{C}}^n$ which contain a fixed ℓ -plane, particularly the number of invariant hyperplanes through a point and the number of invariant hyperplanes containing an invariant line.

Corollary 1.3.1. *Let \mathcal{F} be a one-dimensional holomorphic foliation on $\mathbb{P}_{\mathbb{C}}^n$ of degree $d > 0$ and suppose \mathcal{F} does not admit a rational first integral. Then, the number of \mathcal{F} -invariant hyperplanes which contain a fixed ℓ -plane, $0 \leq \ell \leq n-1$, is bounded by*

$$n - \ell + \binom{n-\ell}{2}(d-1).$$

Proof. We may assume the ℓ -plane \mathbb{L}^{ℓ} is the base locus of the linear subsystem $V_{n-\ell} \subset |\mathcal{O}_{\mathbb{P}_{\mathbb{C}}^n}(1)|$ generated by $z_{\ell+1}, \dots, z_n$. Any hyperplane containing \mathbb{L}^{ℓ} belongs to $V_{n-\ell}$. The result follows by observing that $h^0(V_{n-\ell}) = n - \ell$. \square

Consider the vector fields, defined in affine coordinates $z_0 = 1$, by

$$X_d^0 = \sum_{i=1}^n (z_i^{d-1} - 1) z_i \frac{\partial}{\partial z_i}.$$

$$X_d^1 = \frac{\partial}{\partial z_1} + \sum_{i=2}^n (z_i^{d-1} - 1) z_i \frac{\partial}{\partial z_i}.$$

$$X_d^{\ell} = \sum_{i=1}^{\ell} (z_1^d + \dots + \widehat{z_i^d} + \dots + z_{\ell}^d) \frac{\partial}{\partial z_i} + \sum_{i=\ell+1}^n (z_i^{d-1} - 1) z_i \frac{\partial}{\partial z_i}, \quad 2 \leq \ell \leq n-1.$$

Remark that the foliations $\mathcal{F}_{X_d^{\ell}}$ on $\mathbb{P}_{\mathbb{C}}^n$ induced by X_d^{ℓ} , $0 \leq \ell \leq n-1$, do all leave the hyperplane at infinity invariant.

X_d^0 is a n -dimensional version of a member of the so called ‘‘family of degree four’’ in $\mathbb{P}_{\mathbb{C}}^2$, one of the examples given by A.Lins Neto in [53]. A straightforward calculation shows that the $n+1 + \binom{n+1}{2}(d-1)$ hyperplanes listed below are invariant by $\mathcal{F}_{X_d^0}$:

$$(z_0 \dots z_n) \prod_{0 \leq i, j \leq n} (z_i^{d-1} - z_j^{d-1}) = 0.$$

It's worth remarking that all the singularities of X_d^0 have the same analytic type and are determined precisely by the intersections of these hyperplanes.

X_1 leaves invariant the line $\mathbb{L}_1 = \{z_2 = \cdots = z_n = 0\}$, which is the base locus of the linear system $\sum_{j=2}^n \lambda_j z_j$. Moreover, $n - 1 + \binom{n-1}{2}(d-1)$ hyperplanes listed below are X_1 -invariant and contain \mathbb{L}^1 :

$$(z_2 \dots z_n) \prod_{2 \leq i, j \leq n} (z_i^{d-1} - z_j^{d-1}) = 0.$$

As for X_ℓ , $2 \leq \ell \leq n - 1$, the ℓ -plane $\mathbb{L}_\ell = \{z_{\ell+1} = \cdots = z_n = 0\}$ is left invariant, as are the $n - \ell + \binom{n-\ell}{2}(d-1)$ hyperplanes, which do all contain \mathbb{L}^ℓ ,

$$(z_{\ell+1} \dots z_n) \prod_{\ell+1 \leq i, j \leq n} (z_i^{d-1} - z_j^{d-1}) = 0.$$

In this case the $(n - \ell)$ -plane $\mathbb{L}_\ell^\perp = \{z_1 = \cdots = z_\ell = 0\}$ is \mathcal{F}_{X_ℓ} -invariant whereas the hyperplane $\{z_i = 0\}$, $1 \leq i \leq \ell$ are not.

Remark 1.3.1. The foliation $\mathcal{F}_{X_d^0}$ on $\mathbb{P}_\mathbb{C}^n$ induced by the vector field X_d^0 is the unique foliation of degree d that leaves invariant the following arrangement of hyperplanes

$$\mathcal{A}_d = \left\{ (z_0 \dots z_n) \prod_{0 \leq i, j \leq n} (z_i^{d-1} - z_j^{d-1}) = 0 \right\}.$$

Indeed, the singular set $\text{Sing}(\mathcal{F})$ of \mathcal{F} is isolated and non-degenerated. On the other hand, we can see that $\text{Sing}(\mathcal{F})$ is determined by intersection of the hyperplanes of \mathcal{A}_d . It follows from [43] that \mathcal{F} is unique.

The linear extatic

Consider the extatic divisor of a foliation \mathcal{F} on $\mathbb{P}_\mathbb{C}^n$, associated to the linear system

$$|\mathcal{O}(1)| = H^0(\mathbb{P}_\mathbb{C}^n, \mathcal{O}(1)) \simeq \langle z_0, \dots, z_n \rangle_\mathbb{C},$$

given by

$$E(|\mathcal{O}(1)|, \mathcal{F}_X) = \det \begin{pmatrix} z_0 & z_1 & \cdots & z_n \\ X(z_0) & X(z_1) & \cdots & X(z_n) \\ \vdots & \vdots & \ddots & \vdots \\ X^n(z_0) & X^n(z_1) & \cdots & X^n(z_n) \end{pmatrix},$$

Let $Z(E(|\mathcal{O}(1)|, \mathcal{X}))$ be the extatic variety. This variety will be called *linear extatic*.

Lemma 1.3.1. *Let \mathcal{X} be a polynomial vector field on \mathbb{C}^n and $\mathcal{V} = Z(f_1, \dots, f_k)$ an irreducible complete intersection. Then \mathcal{V} is \mathcal{X} -invariant if and only if $\mathcal{X}(f_i) \in \mathcal{I}(f_1, \dots, f_k)$, for all $i = 1, \dots, k$.*

Proof. Consider the polynomial map $F = (f_1, \dots, f_k) : \mathbb{C}^n \longrightarrow \mathbb{C}^k$. Suppose that $\mathcal{V} = F^{-1}(0)$ is \mathcal{X} -invariant. Then

$$DF_p \cdot \mathcal{X}(p) = (\mathcal{X}(f_1)(p), \dots, \mathcal{X}(f_k)(p)) = 0,$$

for all $p \in \mathcal{V}$. This implies that $\mathcal{X}(f_i) \in I(Z(\mathcal{I}(f_1, \dots, f_k)))$. Therefore, from Hilbert's zeros theorem and using that $\mathcal{V} = Z(\mathcal{I}(f_1, \dots, f_k))$ is irreducible, we get

$$\mathcal{X}(f_i) \in \mathcal{I}(f_1, \dots, f_k),$$

for all $i = 1, \dots, k$. The converse is immediate. \square

Proposition 1.3.1. *Let \mathcal{F} be a foliation on $\mathbb{P}_{\mathbb{C}}^n$ that does not admit a rational first integral. Then all the \mathcal{F} -invariant linear subspaces are contained in the linear extatic $Z(E(|\mathcal{O}(1)|, \mathcal{F}_X))$, where X is a vector field which induces \mathcal{F} in homogeneous coordinates.*

Proof. If \mathcal{F} admits no rational first integral then $E(|\mathcal{O}(1)|, X) \neq 0$. Every linear k -codimensional subspace on $\mathbb{P}_{\mathbb{C}}^n$ is the intersection of the zeros of k homogeneous polynomials of degree one, linearly independent, let us say $f_1, \dots, f_k \in |\mathcal{O}(1)|$. Then we can take

$$\{f_1, \dots, f_k, h_{k+1}, \dots, h_{n+1}\}$$

to form a basis for $|\mathcal{O}(1)|$. Now, if $Z(f_1, \dots, f_k)$ is \mathcal{F} -invariant, it follows from proposition 1.3.1 that $X(f_i) \in \mathcal{I}(f_1, \dots, f_k)$, for all $i = 1, \dots, k$, and so we get $X^j(f_i) \in \mathcal{I}(f_1, \dots, f_k)$. Expanding the determinant

$$E(|\mathcal{O}(1)|, X) = \det \begin{pmatrix} f_1 & \cdots & f_k & \cdots & h_{n+1} \\ X(f_1) & \cdots & X(f_k) & \cdots & X(h_{n+1}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ X^n(f_1) & \cdots & X^n(f_k) & \cdots & X^n(h_{n+1}) \end{pmatrix}$$

in any of the k -th first columns we see that $E(|\mathcal{O}(1)|, X) \in \mathcal{I}(f_1, \dots, f_k)$. Therefore

$$Z(E(|\mathcal{O}(1)|, \mathcal{X})) \supset Z(f_1, \dots, f_k).$$

\square

Chapter 2

Foliations on simplicial toric varieties

We use the existence of homogeneous coordinates for simplicial toric varieties to prove a result analogous to the Darboux-Jouanolou-Ghys integrability theorem for the existence of rational first integrals for one-dimensional foliations. We study one-dimensional foliations in two classes of toric varieties, the multiprojective spaces and weighted projective planes. Under suitable hypotheses we obtain bounds for Poincaré's problem in these varieties.

2.1 Toric Varieties

Firstly, we recall some basic definitions and results about simplicial complete toric varieties emphasizing Cox's quotient construction and homogeneous coordinates. For more details, we refer the reader to the literature (e.g., to [32], [28], [40], [62]).

Let N be a free \mathbb{Z} -module of rank n and $M = \text{Hom}(N, \mathbb{Z})$ be its dual. A subset $\sigma \subset N \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^n$ is called a strongly convex rational polyhedral cone if there exists a finite number of elements $\vartheta_1, \dots, \vartheta_k \in \mathbb{Z}^d$ in the lattice N such that

$$\sigma = \{a_1\vartheta_1 + \dots + a_k\vartheta_k; a_i \in \mathbb{R}, a_i \geq 0.\}$$

We say that a subset τ of σ given by some a_i being equal to zero is a proper face of σ , and we write $\tau \prec \sigma$. A cone σ is called *simplicial* if its generators can be chosen to be linearly independent over \mathbb{R} . The dimension of a cone σ is, by definition, the dimension of a minimal subspace of \mathbb{R}^n containing σ .

Definition 2.1.1. *A non-empty collection $\Delta = \{\sigma_1, \dots, \sigma_s\}$ of k -dimensional strongly convex rational polyhedral cones in $N \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^n$ is called a complete fan if it satisfies:*

- i) if $\sigma \in \Delta$ and $\tau \prec \sigma$, then $\tau \in \Delta$;*

ii) if $\sigma_i, \sigma_j \in \Delta$, then $\sigma_i \cap \sigma_j \prec \sigma_i$ and $\sigma_i \cap \sigma_j \prec \sigma_j$;

iii) $N \otimes_{\mathbb{Z}} \mathbb{R} = \sigma_1 \cup \cdots \cup \sigma_s$.

The dimension of a fan is the maximal dimension of its cones. An n -dimensional complete fan is simplicial if all its n -dimensional cones are simplicial.

Let Δ be a fan in $N \otimes_{\mathbb{Z}} \mathbb{R}$. It follows from Gordan's Lemma (see [41]) that each k -dimensional cone σ^k in Δ (let us say generated by v_{ij}) defines a finitely generated semigroup $\sigma \cap N$. The dual $(n - k)$ -dimensional cone

$$\check{\sigma} = \{m \in M \otimes_{\mathbb{Z}} \mathbb{R}, \langle m, v_{ij} \rangle \geq 0\}$$

is then a rational polyhedral cone in $M \otimes_{\mathbb{Z}} \mathbb{R}$ and $\check{\sigma} \cap M$ is also a finitely generated semigroup. An affine n -dimensional toric variety corresponding to σ^k is the variety

$$\mathcal{U}_{\sigma} := \text{Spec} \mathbb{C}[\check{\sigma} \cap M].$$

If a cone τ is a face of σ then $\check{\tau} \cap M$ is a subsemigroup of $\check{\sigma} \cap M$, hence \mathcal{U}_{τ} is embedded into \mathcal{U}_{σ} as an open subset. The affine varieties corresponding to all cones of the fan Δ are glued together according to this rule into the toric variety \mathbb{P}_{Δ} associated with Δ . It is possible to show that a toric variety \mathbb{P}_{Δ} contains a complex torus $\mathbb{T}^n = (\mathbb{C}^*)^n$ as a Zariski open subset such that the action of \mathbb{T}^n on itself extends to an action of \mathbb{T}^n on \mathbb{P}_{Δ} .

Theorem 2.1.1. [41] *Let \mathbb{P}_{Δ} be the toric variety determined by a simplicial complete fan Δ . Then \mathbb{P}_{Δ} is projective and has quotient singularities.*

For more details see [41].

Example 2.1.1. \mathbb{T}^n , \mathbb{C}^n and \mathbb{P}^n are toric varieties.

Example 2.1.2 (Weighted projective spaces). Let $\varpi = \{\varpi_0, \dots, \varpi_n\}$ be the set of positive integers satisfying the condition $\gcd(\varpi_0, \dots, \varpi_n) = 1$. Choose $n + 1$ vectors e_0, \dots, e_n in \mathbb{R}^n such that \mathbb{R}^n is spanned by e_0, \dots, e_n and satisfies the linear relation

$$\varpi_0 e_0 + \cdots + \varpi_n e_n = 0.$$

Define N to be the lattice in \mathbb{R}^n consisting of all integral linear combinations of e_0, \dots, e_n . Let $\Delta(w)$ be the set of all possible simplicial cones in \mathbb{R}^n generated by proper subsets of $\{e_0, \dots, e_n\}$. Then $\Delta(w)$ is a rational simplicial complete n -dimensional fan. The corresponding variety $\mathbb{P}_{\Delta(w)}$ is the n -dimensional weighted projective space $\mathbb{P}(\varpi_0, \dots, \varpi_n)$. We will see in the next section that $\mathbb{P}(\varpi_0, \dots, \varpi_n)$ is a quotient of $\mathbb{C}^{n+1} \setminus \{0\}$ by the diagonal action of the torus \mathbb{C}^*

$$(z_0, \dots, z_n) \longmapsto (\lambda^{\varpi_0} z_0, \dots, \lambda^{\varpi_n} z_n), \lambda \in \mathbb{C}^*.$$

In particular, if $(\varpi_0, \dots, \varpi_n) = (1, \dots, 1)$, then $\mathbb{P}(1, \dots, 1) = \mathbb{P}^n$.

Example 2.1.3 (Multiprojective spaces). If X and Y are toric varieties then $X \times Y$ so is. Thus, the multiprojective spaces $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ are examples of toric varieties.

2.1.1 The toric homogeneous coordinates

Let \mathbb{P}_Δ be the toric variety determined by a fan Δ in $N \simeq \mathbb{Z}^n$. As usual, M will denote the \mathbb{Z} -dual of N , and cones in Δ will be denoted by σ . The one-dimensional cones of Δ form the set $\Delta(1) = \{\vartheta_1, \dots, \vartheta_{n+r}\}$, where ϑ_i denotes the unique generator of the one-dimensional cone. If σ is any cone in Δ , then $\sigma(1) = \{\vartheta_i \in \Delta(1); \rho \subset \sigma\}$ is the set of one-dimensional faces of σ . We will assume that $\Delta(1)$ spans $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$.

Each $\vartheta_i \in \Delta(1)$ corresponds to an irreducible \mathbb{T} -invariant Weil divisor D_i in \mathbb{P}_Δ , where $\mathbb{T} = N \otimes_{\mathbb{Z}} \mathbb{C}^*$ is the torus acting on \mathbb{P}_Δ , see [41, chapter 3]. It follows from [41, chapter 5] that the \mathbb{T} -invariant Weil divisors on \mathbb{P}_Δ form a free abelian group of rank $n+r$, that will be denoted \mathbb{Z}^{n+r} . Thus an element $D \in \mathbb{Z}^{n+r}$ is a sum $\sum_{i=1}^{n+r} a_i D_i$. The \mathbb{T} -invariant Cartier divisors form a subgroup $\text{Div}_{\mathbb{T}}(\mathbb{P}_\Delta) \subset \mathbb{Z}^{n+r}$.

Each $m \in M$ gives a character $\chi^m : \mathbb{T} \rightarrow \mathbb{C}^*$, and hence χ^m is a rational function on \mathbb{P}_Δ . As is well-known, χ^m gives the Cartier divisor

$$\sum_{i=1}^{n+r} -\langle m, \vartheta_i \rangle D_i,$$

see [41, section 3.3]. We will consider the map

$$\begin{aligned} M &\longrightarrow \mathbb{Z}^{n+r} \\ m &\longmapsto \sum_{i=1}^{n+r} -\langle m, \vartheta_i \rangle D_i. \end{aligned}$$

This map is injective since $\Delta(1)$ spans $N_{\mathbb{R}}$. By [41], we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & M & \rightarrow & \text{Div}_{\mathbb{T}}(\mathbb{P}_\Delta) & \rightarrow & \text{Pic}(\mathbb{P}_\Delta) & \rightarrow & 0 \\ & & \parallel & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & M & \rightarrow & \mathbb{Z}^{n+r} & \rightarrow & \mathcal{A}_{n-1}(\mathbb{P}_\Delta) & \rightarrow & 0, \end{array} \tag{2.1}$$

where $\mathcal{A}_{n-1}(\mathbb{P}_\Delta)$ is the Chow group of $(n-1)$ -cycles. For each $\vartheta_i \in \Delta(1)$, introduce a variable z_i , and consider the polynomial ring

$$S = \mathbb{C}[z_i; \vartheta_i \in \Delta(1)].$$

Note that a monomial $\prod_{i=1}^{n+r} z_i^{a_i}$ determines a divisor $\sum_{i=1}^{n+r} a_i D_i$ and to emphasize this relationship, we will write the monomial as z^D . We will grade S as follows, the degree of a monomial z^D is $\deg(z^D) = [D] \in \mathcal{A}_{n-1}(\mathbb{P}_\Delta)$.

Using the exact sequence (2.1), it follows that two monomials $\prod_{i=1}^{n+r} z_i^{a_i}$ and $\prod_{i=1}^{n+r} z_i^{b_i}$ in S have the same degree if and only if there is some $m \in M$ such that $a_i = \langle m, \vartheta_i \rangle + b_i$ for each $i = 1, \dots, n+r$. Then

$$S = \bigoplus_{\alpha \in \mathcal{A}_{n-1}(\mathbb{P}_\Delta)} S_\alpha,$$

where $S_\alpha = \bigoplus_{\deg(z^D)=\alpha} \mathbb{C} \cdot z^D$. S is called *Cox's homogeneous coordinate ring* of the toric variety \mathbb{P}_Δ .

Let $\mathcal{O}(D)$ be the coherent sheaf on X determined by a Weil divisor D , then

$$S_{\deg(D)} \simeq H^0(X, \mathcal{O}(D)),$$

moreover there is a commutative diagram

$$\begin{array}{ccc} S_{\deg(D)} \otimes S_{\deg(E)} & \longrightarrow & S_{\deg(D+E)} \\ \downarrow & & \downarrow \\ H^0(X, \mathcal{O}(D)) \otimes H^0(X, \mathcal{O}(E)) & \longrightarrow & H^0(X, \mathcal{O}(D+E)) \end{array}$$

where the top arrow is polynomial multiplication. If \mathbb{P}_Δ is a complete toric variety, then:

- i) S_α is finite dimensional for every α , and in particular, $S_0 = \mathbb{C}$.
- ii) If $\alpha = [D]$ for an effective divisor $D = \sum_{i=1}^{n+r} a_i D_i$, it follows from [27] that $\dim_{\mathbb{C}} S_\alpha = \#(\mathcal{P}_D \cap M)$, where

$$\mathcal{P}_D = \{m \in M_{\mathbb{R}}; \langle m, \vartheta_i \rangle \geq -a_i \text{ for all } i = 1, \dots, n+r\}.$$

We get the monomial

$$z^{\hat{\sigma}} = \prod_{\vartheta_i \notin \sigma} z_i$$

which is the product of all variables not coming from edges of σ . Then define $\mathcal{Z}(\Delta) = V(z^{\hat{\sigma}}; \sigma \in \Delta) \subset \mathbb{C}^{n+r}$. Now consider the group $G(\Delta) \subset \mathbb{T}^r$ given by

$$G(\Delta) = \left\{ (t_1, \dots, t_r) \in \mathbb{T}^r; \prod_{i=1}^r t_i^{\langle e_j, \vartheta_i \rangle} = 1, j = 1, \dots, r \right\}$$

Define an action of $G(\Delta)$ on $\mathbb{C}^{n+r} - \mathcal{Z}(\Delta)$ by

$$\begin{array}{ccc} G(\Delta) \times (\mathbb{C}^{n+r} - \mathcal{Z}(\Delta)) & \longrightarrow & \mathbb{C}^{n+r} - \mathcal{Z}(\Delta) \\ (g, (z_1, \dots, z_{n+r})) & \longmapsto & (g(D_1)z_1, \dots, g(D_{n+r})z_{n+r}). \end{array}$$

Theorem 2.1.2 (D. Cox, [27]). *If \mathbb{P}_Δ is a n -dimensional toric variety where $\vartheta_1, \dots, \vartheta_{n+r}$ span \mathbb{R}^n , then:*

- i) \mathbb{P}_Δ is a universal categorical quotient $(\mathbb{C}^{n+r} - \mathcal{Z})/G(\Delta)$
- ii) \mathbb{P}_Δ is an orbifold $(\mathbb{C}^{n+r} - \mathcal{Z})/G(\Delta)$ if, and only if, \mathbb{P}_Δ is simplicial.

Remark 2.1.1. We have that $\text{cod}(\text{Sing}(\mathbb{P}_\Delta)) \geq 2$. See [27].

To describe the action of $G(\Delta)$ when it has no torsion we consider the lattice of relations between generators of Δ , i.e., r linearly independent relations over \mathbb{Z} between $\vartheta_1, \dots, \vartheta_{n+r}$

$$\begin{cases} a_{11}\vartheta_1 + \dots + a_{1(n+r)}\vartheta_{n+r} = 0 \\ \vdots \\ a_{r1}\vartheta_1 + \dots + a_{r(n+r)}\vartheta_{n+r} = 0 \end{cases} \quad (2.2)$$

Thus by (2.1) the factor of $G(\Delta)$ isomorphic to \mathbb{T}^r defines an equivalence relation on $(\mathbb{C}^{n+r} - \mathcal{Z})/G(\Delta)$: let $u, v \in \mathbb{C}^{n+r} - \mathcal{Z}$, with $v = (v_1, \dots, v_{n+r})$, then $u \sim v$ if, and only if,

$$\exists (\lambda_1, \dots, \lambda_r) \in \mathbb{T}^r; u = (\lambda_1^{a_{11}} \dots \lambda_r^{a_{r1}} v_1, \dots, \lambda_1^{a_{1(n+r)}} \dots \lambda_r^{a_{r(n+r)}} v_{n+r}), \quad (2.3)$$

Therefore, when $G(\Delta)$ has no torsion, the equivalence relation on $(\mathbb{C}^{n+r} - \mathcal{Z})$ is given by this formula. If $f \in S_\alpha$, it follows from [6, Lemma 3.8] the Euler's formula

$$i_{R_i} df = \theta_i(\alpha) f,$$

where $\theta_i \in \mathbb{C}$ and $R_i = \sum_{j=1}^{n+r} a_{ij} z_{ij} \frac{\partial}{\partial z_{ij}}$, $i = 1, \dots, r$. Moreover, $\text{Lie}(G) = \langle R_1, \dots, R_r \rangle$, see [27].

An element $\alpha \in \mathcal{A}_{n-1}(\mathbb{P}_\Delta)$ gives the character $\chi^\alpha : G(\Delta) \rightarrow \mathbb{T}$. The action of $G(\Delta)$ on \mathbb{C}^{n+r} induces an action on S with the property that given $f \in S$, we have

$$f \in S_\alpha \Leftrightarrow f(g \cdot z) = \chi^\alpha(g) f(z), \quad \text{for all } g \in G(\Delta), z \in \mathbb{C}^{n+r}.$$

The graded pieces of S are the eigenspaces of the action of $G(\Delta)$ on S . We say that $f \in S_\alpha$ is homogeneous of degree α . It follows that the equation $\{f(z) = 0\}$ is well-defined in \mathbb{P}_Δ and it defines a hypersurface.

We shall consider the subfield of $\mathbb{C}(z_1, \dots, z_{n+r})$ given by

$$\tilde{K}(\mathbb{P}_\Delta) = \left\{ \frac{P}{Q} \in \mathbb{C}(z_1, \dots, z_{n+r}); \deg(P) = \alpha, \deg(Q) = \beta, \alpha, \beta \in \mathcal{A}_{n-1}(\mathbb{P}_\Delta) \right\}.$$

Thus, the field of rational functions on \mathbb{P}_Δ , denoted by $K(\mathbb{P}_\Delta)$, is the subfield of $\tilde{K}(\mathbb{P}_\Delta)$ such that $\deg(P) = \deg(Q)$.

2.1.2 Existence of rational first integrals

In this section we shall use the homogeneous coordinates for toric varieties to prove the following result.

Theorem 2.1.3. *Let \mathcal{F} be a one-dimensional foliation on a complete simplicial toric variety \mathbb{P}_Δ of dimension n and Picard number $\rho(\mathbb{P}_\Delta)$. If \mathcal{F} admits*

$$\mathcal{N}(\mathbb{P}_\Delta, K_{\mathcal{F}}, n) := h^0(\mathbb{P}_\Delta, \mathcal{O}(K_{\mathcal{F}})) + \rho(\mathbb{P}_\Delta) + n$$

invariant irreducible \mathbb{T}^n -divisors, then \mathcal{F} admits a rational first integral.

It follows from the Hirzebruch-Riemann-Roch theorem for toric varieties (see [41]) that

$$h^0(\mathbb{P}_\Delta, \mathcal{O}(K_{\mathcal{F}})) = \sum_{k=0}^n \frac{1}{k!} \deg([K_{\mathcal{F}}]^k \cap Td_k(\mathbb{P}_\Delta)),$$

where $Td_k(\mathbb{P}_\Delta)$ is the k -th homology Todd class. Therefore, we have that

$$\mathcal{N}(\mathbb{P}_\Delta, K_{\mathcal{F}}, n) = \sum_{k=0}^n \frac{1}{k!} \deg([K_{\mathcal{F}}]^k \cap Td_k(\mathbb{P}_\Delta)) + \rho(\mathbb{P}_\Delta) + n.$$

Observe that, in general, \mathbb{P}_Δ is a singular variety with quotient singularities. Therefore, in two dimensions this result shows that the theorem of Darboux-Joanoulou-Ghys is valid for a class of singular toric varieties.

One-dimensional foliations

We use the generalized Euler exact sequence for simplicial toric varieties in order to consider a holomorphic foliation as a polynomial vector field in homogeneous coordinates.

Let \mathbb{P}_Δ be a complete simplicial toric variety of dimension n , and denote $\mathcal{O}_{\mathbb{P}_\Delta} := \mathcal{O}$. There exists an exact sequence known as *the generalized Euler sequence* [27]

$$0 \rightarrow \mathcal{O}^{\oplus r} \rightarrow \bigoplus_{i=1}^{n+r} \mathcal{O}(D_i) \rightarrow \mathcal{T}\mathbb{P}_\Delta \rightarrow 0,$$

where $\mathcal{T}\mathbb{P}_\Delta = \mathcal{H}om(\Omega_{\mathbb{P}_\Delta}^1, \mathcal{O})$ is the so-called Zariski sheaf of \mathbb{P}_Δ . Let $\mathcal{O}(d_1, \dots, d_{n+r}) = \mathcal{O}(\sum_{i=1}^{n+r} d_i D_i)$, where $\sum_{i=1}^{n+r} d_i D_i$ is a Weil divisor. Tensorizing Euler's sequence by $\mathcal{O}(d_1, \dots, d_{n+r})$ we get

$$0 \rightarrow \mathcal{O}(d_1, \dots, d_{n+r})^{\oplus r} \rightarrow \bigoplus_{i=1}^{n+r} \mathcal{O}(d_1, \dots, d_i+1, \dots, d_{n+r}) \rightarrow \mathcal{T}\mathbb{P}_\Delta(d_1, \dots, d_{n+r}) \rightarrow 0$$

Definition 2.1.2. A holomorphic foliation \mathcal{F} on \mathbb{P}_Δ of multidegree (d_1, \dots, d_{n+r}) is a global section of $\mathcal{T}\mathbb{P}_\Delta \otimes \mathcal{O}(d_1, \dots, d_{n+r})$.

Proposition 2.1.1. Let $\text{Fol}((d_1, \dots, d_{n+r}), \mathbb{P}_\Delta)$ be the space of foliations of multidegree (d_1, \dots, d_{n+r}) . Let $D^j = (d_j + 1)D_j + \sum_{i \neq j}^{n+r} d_i D_i$ and $D = \sum_{i=1}^{n+r} d_i D_i$. Then $\text{Fol}((d_1, \dots, d_{n+r}), \mathbb{P}_\Delta)$ is isomorphic to a complex projective space \mathbb{P}^{N-1} , where

$$N = \sum_{j=1}^{n+r} \#(\mathcal{P}_{D^j} \cap M) - r \cdot [\#(\mathcal{P}_D \cap M)].$$

From the above exact sequence, we conclude that a foliation on \mathbb{P}_Δ of multidegree (d_1, \dots, d_r) is given by a polynomial vector field in homogeneous coordinates of the form

$$X = \sum_{i=1}^{n+r} P_i \frac{\partial}{\partial z_i},$$

where P_i is a polynomial of multidegree $(d_1, \dots, d_i + 1, \dots, d_r)$ for all $i = 1, \dots, r$, modulo addition of a vector field of the form $\sum_{i=1}^{n+r} g_i R_i$. Therefore

$$\text{Sing}(\mathcal{F}) = \{p \in \mathbb{C}^{n+r}; R_1 \wedge \dots \wedge R_{n+r} \wedge X(p) = 0\}.$$

Example 2.1.4 (Rational scroll). Let a_1, \dots, a_n be integers. Consider the \mathbb{T}^2 -action on $(\mathbb{C}^2 - \{0\}) \times (\mathbb{C}^n - \{0\})$ given as follows:

$$\begin{aligned} \mathbb{T}^2 \times (\mathbb{C}^2 - \{0\}) \times (\mathbb{C}^n - \{0\}) &\longrightarrow (\mathbb{C}^2 - \{0\}) \times (\mathbb{C}^n - \{0\}) \\ ((\lambda, \mu), ((x_1, x_2), (z_1, \dots, z_n))) &\longrightarrow ((\lambda x_1, \lambda x_2), (\mu \lambda^{-a_1} z_1, \dots, \mu \lambda^{-a_n} z_n)). \end{aligned}$$

The *rational scroll* $\mathbb{F}(a_1, \dots, a_n)$ is the quotient variety of $(\mathbb{C}^2 - \{0\}) \times (\mathbb{C}^n - \{0\})$ by this action.

Let $E = \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(a_i)$ be the vector bundle over \mathbb{P}^1 . Write $\mathbb{P}(E)$ for the projectivized vector bundle

$$\mathbb{P}(E) \rightarrow \mathbb{P}^1$$

and let $\mathcal{O}_{\mathbb{P}(E)}(1)$ be the tautological line bundle. It is possible to show that $\mathbb{F}(a_1, \dots, a_n)$ is the image of $\mathbb{P}(E)$ by the embedding given by $\mathcal{O}_{\mathbb{P}(E)}(1)$, see [46].

Two examples of this construction are:

1. $\mathbb{F}(0, \dots, 0) \simeq \mathbb{P}^1 \times \mathbb{P}^{n-1}$;
2. $\mathbb{F}(a, 0)$ is a Hirzebruch surface; see [71].

We have that $\text{Pic}(\mathbb{F}(a_1, \dots, a_n)) \simeq \mathbb{Z}L \oplus \mathbb{Z}M$, where L is the class of a fibre of π and M the class of any monomial $x_1^b x_2^c z_i$, with $b + c = a_i$. If all the $a_i > 0$, then M is the divisor class of the hyperplane section under the embedding $\mathbb{F}(a_1, \dots, a_n) \subset \mathbb{P}^{n+\sum_{i=1}^n a_i - 1}$. Let $\mathcal{O}(d_1, d_2) := \mathcal{O}(d_1 L + d_2 M)$. Thus, a foliation on $\mathbb{F}(a_1, \dots, a_n)$ is a global section of $T\mathbb{F}(a_1, \dots, a_n) \otimes \mathcal{O}(d_1, d_2)$ and has a bidegree (d_1, d_2) . In this case Euler's sequence is given by

$$0 \rightarrow \mathcal{O}^{\oplus 2} \rightarrow \mathcal{O}(1, 0)^{\oplus 2} \oplus \bigoplus_{i=1}^n \mathcal{O}(-a_i, 1) \rightarrow T\mathbb{F} \rightarrow 0,$$

and tensorizing by $\mathcal{O}(d_1, d_2)$ we get the sequence

$$0 \rightarrow \mathcal{O}(d_1, d_2)^{\oplus 2} \rightarrow \mathcal{O}(d_1 + 1, d_2)^{\oplus 2} \oplus \bigoplus_{i=1}^n \mathcal{O}(d_1 - a_i, d_2 + 1) \rightarrow T\mathbb{F} \otimes \mathcal{O}(d_1, d_2) \rightarrow 0.$$

Therefore, a foliation on $\mathbb{F}(a_1, \dots, a_n)$ is given, in homogeneous coordinates, by a vector field

$$X = Q_1 \frac{\partial}{\partial x_1} + Q_2 \frac{\partial}{\partial x_2} + \sum_{i=0}^n P_i \frac{\partial}{\partial z_i},$$

where Q_i is bihomogeneous of bidegree $(d_1 + 1, d_2)$ and P_i bihomogeneous of bidegree $(d_1 - a_i, d_2 + 1)$, modulo $g_1 R_1 + g_2 R_2$, where

$$R_1 = \sum_{i=1}^n z_i \frac{\partial}{\partial z_i}, \quad R_2 = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + \sum_{i=1}^n -a_i z_i \frac{\partial}{\partial z_i}$$

and g_i has bidegree (d_1, d_2) .

Proof of Theorem 2.1.3.

Proof. Let $f_1, f_2, \dots, f_{N+n+r}$ be defining functions for \mathcal{F} -invariant irreducible hypersurfaces, where $N = h^0(\mathbb{P}_\Delta, \mathcal{O}(K_{\mathcal{F}}))$. Let $X = \sum_{i=1}^{n+r} P_i \frac{\partial}{\partial z_i}$ be a polynomial vector field that defines \mathcal{F} in homogeneous coordinates. It follows that

$$\frac{X(f_j)}{f_j} = h_j \in S_{[K_{\mathcal{F}}]}, \quad j = 1, 2, \dots, N + n + r.$$

We get the following relations

$$\lambda_{11} h_1 + \lambda_{12} h_2 + \lambda_{13} h_3 + \dots + \lambda_{1(N+1)} h_{N+1} = 0$$

$$\lambda_{22} h_2 + \lambda_{23} h_3 + \dots + \lambda_{2(N+1)} h_{N+1} + \lambda_{2(N+2)} h_{N+2} = 0$$

$$\lambda_{33} h_3 + \lambda_{34} h_4 + \dots + \lambda_{3(N+2)} h_{N+2} + \lambda_{3(N+3)} h_{N+3} = 0$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$\lambda_{jj} h_j + \lambda_{j(j+1)} h_{j+1} + \dots + \lambda_{j(N+j)} h_{N+j} = 0,$$

where $j = n + r$. We can suppose that $\lambda_{ii} \neq 0$, for all $i = 1, \dots, n$. Define the rational 1-form on \mathbb{C}^{n+r}

$$\eta_k = \sum_{j=k}^{N+k} \lambda_{kj} \frac{df_j}{f_j}, \quad k = 1, \dots, n + r.$$

Observe that by construction $|\eta_i|_\infty \neq |\eta_j|_\infty$ for all $i \neq j$, where $|\cdot|_\infty$ denote the sets of poles. Contracting by X we get

$$i_X \eta_k = \sum_{j=k}^{N+k} \lambda_{kj} \frac{X(f_j)}{f_j} = \sum_{j=k}^{N+k} \lambda_{kj} h_j = 0,$$

for all $k = 1, \dots, n+r$. We claim that $\eta_1, \dots, \eta_{n+r}$ are linearly dependent over the field of rational functions $\tilde{K}(\mathbb{P}_\Delta)$. Otherwise, there exists a rational function $R \neq 0$ such that

$$\eta = \eta_1 \wedge \cdots \wedge \eta_{n+r} = R dz_1 \wedge \cdots \wedge dz_{n+r},$$

Contracting η by $X = \sum_{i=1}^{n+r} P_i \frac{\partial}{\partial z_i}$ we have $R \iota_X (dz_1 \wedge \cdots \wedge dz_{n+r}) = 0$, since $\iota_X \eta_k = 0$, for all $k = 1, \dots, n+r$. But $R \neq 0$, thus

$$0 = \iota_X (dz_1 \wedge \cdots \wedge dz_n) = \sum_{i=1}^{n+r} (-1)^{i+1} P_i dz_1 \wedge \cdots \wedge \widehat{dz_i} \wedge \cdots \wedge dz_{n+r},$$

This implies that $P_1 = \cdots = P_{n+r} = 0$, i.e, $X \equiv 0$, a contradiction. Let V be the $\tilde{K}(\mathbb{P}_\Delta)$ -linear space generated by $\{\eta_1, \dots, \eta_{n+r}\}$, suppose that $\dim_{\tilde{K}(\mathbb{P}_\Delta)} V = k$ and

$$V = \langle \eta_1, \dots, \eta_k \rangle_{\tilde{K}(\mathbb{P}_\Delta)},$$

for some $1 \leq k < n+r$. There exist rational functions $R_1, \dots, R_k, R_{k+1} \in \tilde{K}(\mathbb{P}_\Delta)$, with $R_{k+1} \neq 0$, such that

$$R_1 \eta_1 + \cdots + R_k \eta_k + R_{k+1} \eta_{k+1} = 0,$$

multiplying this equation by $\text{lcm}(R_1, \dots, R_{k+1})$ we obtain

$$Q_1 \eta_1 + \cdots + Q_k \eta_k + Q_{k+1} \eta_{k+1} = 0,$$

where each Q_i is a homogenous polynomial in the Cox ring of \mathbb{P}_Δ . Now, we multiply this equation by $F = \prod_{i=1}^{N+n+r} f_i$

$$Q_1 \tilde{\eta}_1 + \cdots + Q_k \tilde{\eta}_k + Q_{k+1} \widetilde{\eta_{k+1}} = 0, \quad (2.4)$$

where $\tilde{\eta}_i = F \eta_i$. Since $\tilde{\eta}_i$ are all homogeneous of the same degree, we can extract from relation (2.4) a relation

$$Q_{i_1} \widetilde{\eta_{i_1}} + \cdots + Q_{i_\ell} \widetilde{\eta_{i_\ell}} + Q_{k+1} \widetilde{\eta_{k+1}} = 0,$$

where $\deg(Q_{i_j}) = \deg(Q_{k+1})$, $i_j \in \{1, \dots, k\}$ and $j = 1, \dots, \ell \leq k$. Hence, we get

$$F \eta_{k+1} = R_{i_1} F \eta_{i_1} + \cdots + R_{i_\ell} F \eta_{i_\ell}, \quad (2.5)$$

where $R_{i_j} = -\frac{Q_{i_j}}{Q_{k+1}} \in K(\mathbb{P}_\Delta)$. Dividing by F and differentiating

$$0 = dR_{i_1} \wedge \eta_{i_1} + \cdots + dR_{i_\ell} \wedge \eta_{i_\ell},$$

Now, contracting by X results

$$0 = X(R_{i_1}) \eta_{i_1} + \cdots + X(R_{i_\ell}) \eta_{i_\ell}.$$

Since $\ell \leq k$ then $X(R_{i_1}) = \cdots = X(R_{i_\ell}) = 0$. That is, the rational function R_{i_j} , $j = 1, \dots, \ell$, is either a first integral for the foliation \mathcal{F} induced by the vector field X or it is constant. It remains to observe that at least one rational function R_{i_j} is not constant. Indeed, this follows from relation (2.5) and the fact that the set of poles $|\eta_{i_j}|_\infty \neq |\eta_{i_r}|_\infty$, for all $j \neq r$. \square

Example 2.1.5. It follows from [60] that

$$h^0(\mathbb{F}(a_1, \dots, a_n), \mathcal{O}(d_1, d_2)) = \left(\sum_{i=1}^n a_i \right) \binom{d_1 + n - 1}{n} + (d_2 + 1) \binom{d_1 + n - 1}{n - 1}$$

Let \mathcal{F} be a foliation on $\mathbb{F}(a_1, \dots, a_n)$ of bidegree (d_1, d_2) . If \mathcal{F} admits

$$\mathcal{N}(a_1, \dots, a_n, d_1, d_2, n) = \left(\sum_{i=0}^n a_i \right) \binom{d_1 + n - 1}{n} + (d_2 + 1) \binom{d_1 + n - 1}{n - 1} + n + 2$$

invariant irreducible algebraic hypersurfaces, then \mathcal{F} admits a rational first integral.

Example 2.1.6. Let \mathcal{F} be a foliation on $\mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_r}$ of multidegree $(e_1 - 1, \dots, e_r - 1)$. If \mathcal{F} admits

$$\mathcal{N}(n_1, \dots, n_r, e_1, \dots, e_r, r) = \prod_{i=1}^r \binom{e_i + n_i - 1}{n_i} + \sum_{i=1}^r n_i + r$$

invariants irreducible algebraic hypersurfaces, then \mathcal{F} admits a rational first integral.

The extatic hypersurface

The extatic divisor is defined on complex smooth varieties, see chapter 1. The homogeneous coordinates allows us to define the extatic divisor globally for all simplicial toric varieties even for the singular case.

Definition 2.1.3. Let X be a vector field on \mathbb{C}^{n+r} which induces a foliation \mathcal{F} on \mathbb{P}_Δ and consider the linear system $S_\alpha = H^0(\mathbb{P}_\Delta, \mathcal{O}(\alpha))$. The extatic hypersurface of \mathcal{F} associated to the linear system S_α is defined by

$$E(S_\alpha, \mathcal{F}) = \det \begin{pmatrix} s_1 & s_2 & \cdots & s_\ell \\ X(s_1) & X(s_2) & \cdots & X(s_\ell) \\ \vdots & \vdots & \cdots & \vdots \\ X^{\ell-1}(s_1) & X^{\ell-1}(s_2) & \cdots & X^{\ell-1}(s_\ell) \end{pmatrix},$$

where $\dim_{\mathbb{C}} S_\alpha = \ell$ and $\{s_1, \dots, s_\ell\}$ is a base for S_α . The extatic hypersurface is $\mathcal{E}(\mathcal{F}, S_\alpha) = Z(E(\mathcal{F}, S_\alpha))$.

Proposition 2.1.2. *Let \mathcal{F} be a one-dimensional holomorphic foliation on a toric variety \mathbb{P}_Δ and let $|D|$ be a linear system. Then every \mathcal{F} -invariant hypersurface which is contained in the zero locus of some element of $|D|$, must be contained $Z(E(\mathcal{F}, |D|))$.*

Proposition 2.1.3. *Let \mathcal{F} be a one-dimensional holomorphic foliation on a toric variety \mathbb{P}_Δ and let $V \subset |D|$ be a linear system. Then \mathcal{F} admit rational first integral if and only if $\mathcal{E}(V, X) \equiv 0$*

Proof. Let $\{s_1, \dots, s_k\}$ be a \mathbb{C} -base for V . Suppose that $E(V, \mathcal{F})$ vanishes identically. Fix $0 \leq i \leq k-1$, we have that $\deg(X^i(s_1)) = \dots = \deg(X^i(s_k))$. For each $j = 1, \dots, k$ choose a non-zero polynomial f_i such that $\deg(f_i) = \deg(X^i(s_j))$, and consider the matrix

$$E = \begin{pmatrix} \frac{s_1}{f_0} & \frac{s_2}{f_0} & \dots & \frac{s_k}{f_0} \\ \frac{X(s_1)}{f_1} & \frac{X(s_2)}{f_1} & \dots & \frac{X(s_k)}{f_1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{X^{k-1}(s_1)}{f_{k-1}} & \frac{X^{k-1}(s_2)}{f_{k-1}} & \dots & \frac{X^{k-1}(s_k)}{f_{k-1}} \end{pmatrix}.$$

We have that $E \in M_{k \times k}(K(\mathbb{P}_\Delta))$, where $M_{k \times k}(K(\mathbb{P}_\Delta))$ is the $K(\mathbb{P}_\Delta)$ -vector space of matrices with entries in the field $K(\mathbb{P}_\Delta)$. Since $\det(E) = f_0 \cdots f_{k-1} \cdot E(V, \mathcal{F}) \equiv 0$ the columns of the matrix E are dependent over field of rational functions $K(\mathbb{P}_\Delta)$. Hence, there are rational functions $\theta_1, \dots, \theta_k \in K(\mathbb{P}_\Delta)$, such that

$$M_i^\alpha = \sum_{j=1}^k \theta_j X_\alpha^i(s_j) = 0, \quad 0 \leq i \leq k-1. \quad (2.6)$$

The proof follows as in the Theorem 1.2.2. □

Corollary 2.1.1. *Let \mathcal{F} be a foliation of degree d on a weighted projective space of dimension n and \mathcal{V} a hypersurface \mathcal{F} -invariant of degree k . If \mathcal{F} does not admit a rational first integral, then*

$$\mathcal{N}(d, k) \leq h^0(\mathcal{O}(k)) + \frac{(d-1)}{k} \binom{h^0(\mathcal{O}(k))}{2},$$

where $\mathcal{N}(\mathcal{F}, k)$ is the number of \mathcal{F} -invariant hypersurfaces of degree k .

2.2 Multiprojective foliations

Consider the product of complex projective spaces $\mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_r}$ and let $\pi_i : \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_r} \rightarrow \mathbb{P}^{n_i}$ be the natural projections, $i = 1, \dots, r$. Set $\mathbf{P}^{(n_1, \dots, n_r)} = \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_r}$, where $n = n_1 + \dots + n_r$.

We give $\mathbf{P}^{(n_1, \dots, n_r)}$ the natural manifold structure, that is, in \mathbb{C}^{n_j+1} choose coordinates $Z_j = (Z_{0_j}, \dots, Z_{n_j})$, $j = 1, \dots, r$, consider the multihomogeneous system of coordinates $Z = (Z_1, \dots, Z_r) \in \mathbb{C}^{n_1+1} \times \dots \times \mathbb{C}^{n_r+1}$ and cover $\mathbf{P}^{(n_1, \dots, n_r)}$ by the open sets

$$\mathcal{U}_{(i_1, \dots, i_r)} = \{([Z_1], \dots, [Z_r]) \in \mathbf{P}^{(n_1, \dots, n_r)} : Z_{i_1} \neq 0, \dots, Z_{i_r} \neq 0, 0 \leq i_s \leq n_s, 1 \leq s \leq r\}.$$

The changes of coordinates are given by $\varphi_{(i_1, \dots, i_r)} = (\varphi_{i_1}, \dots, \varphi_{i_r})$, with

$$\varphi_{i_s}(Z_{0_s}, \dots, Z_{n_s}) = \left(\frac{Z_{0_s}}{Z_{i_s}}, \dots, \frac{Z_{i_{(s-1)}}}{Z_{i_s}}, \frac{Z_{i_{(s+1)}}}{Z_{i_s}}, \dots, \frac{Z_{n_s}}{Z_{i_s}} \right).$$

The local coordinates are, then

$$z_{k_s} = \frac{Z_{k_s}}{Z_{i_s}}, \quad k_s \neq i_s.$$

Equivalently, we have an action

$$\begin{aligned} (\mathbb{C}^*)^r \times (\mathbb{C}^{n_1+1} \setminus \{0\}) \times \dots \times (\mathbb{C}^{n_r+1} \setminus \{0\}) &\longrightarrow (\mathbb{C}^{n_1+1} \setminus \{0\}) \times \dots \times (\mathbb{C}^{n_r+1} \setminus \{0\}) \\ ((t_1, \dots, t_r), (v_1, \dots, v_r)) &\longmapsto (t_1 v_1, \dots, t_r v_r), \end{aligned}$$

where $v_i = (v_{0_i}, \dots, v_{n_i}) \in \mathbb{C}^{n_i+1} \setminus \{0\}$, and hence a quotient map

$$\pi : (\mathbb{C}^{n_1+1} \setminus \{0\}) \times \dots \times (\mathbb{C}^{n_r+1} \setminus \{0\}) \longrightarrow \mathbf{P}^{(n_1, \dots, n_r)},$$

given by $\pi(v_1, \dots, v_r) = [v_1, \dots, v_r] := ([v_1], \dots, [v_r])$.

Set $\mathcal{O}(0) = \underline{\mathbb{C}}$ and $\mathcal{O}(d_1, \dots, d_r) := \pi_1^* \mathcal{O}(d_1) \otimes \dots \otimes \pi_r^* \mathcal{O}(d_r)$. The Euler sequence over $\mathbb{P}_{\mathbb{C}}^m$

$$0 \longrightarrow \underline{\mathbb{C}} \longrightarrow \mathcal{O}(1)^{\oplus m+1} \longrightarrow T\mathbb{P}_{\mathbb{C}}^m \longrightarrow 0 \quad (2.7)$$

gives, by direct summation, the exact sequence:

$$0 \longrightarrow \underline{\mathbb{C}}^r \longrightarrow \bigoplus_{j=1}^r \mathcal{O}(0, \dots, \underbrace{1}_j, \dots, 0)^{\oplus n_j+1} \longrightarrow T\mathbf{P}^{(n_1, \dots, n_r)} \longrightarrow 0, \quad (2.8)$$

Multiprojective foliations

Definition 2.2.1. *A one-dimensional holomorphic foliation on $\mathbf{P}^{(n_1, \dots, n_r)} = \mathbb{P}_{\mathbb{C}}^{n_1} \times \dots \times \mathbb{P}_{\mathbb{C}}^{n_r}$ of multidegree $d = (d_1, \dots, d_r) \in \mathbb{Z}^r$ is a section of the holomorphic vector bundle $T\mathbf{P}^{(n_1, \dots, n_r)} \otimes \mathcal{O}(d_1 - 1, \dots, d_r - 1)$. We say that d_i is the i -th degree of \mathcal{F} . These can be given by a morphism*

$$\Phi : \mathcal{O}(1 - d_1, \dots, 1 - d_r) \longrightarrow T\mathbf{P}^{(n_1, \dots, n_r)}.$$

We will call these foliations **multiprojective**.

Remark 2.2.1. A priori $d_i \in \mathbb{Z}$, $i = 1, \dots, r$, but we will see below that $d_i \geq 0$.

Example 2.2.1. A foliation on $\mathbb{P} \times \mathbb{P}$ can also be given by a 1-form on homogeneous coordinates. Consider the rational map $\zeta : \mathbb{P} \times \mathbb{P} \dashrightarrow \mathbb{P}^2$, given by $\zeta([x, y], [z, w]) = [xz, yw, xw]$. Let \mathcal{F}_d be a foliation on \mathbb{P}^2 of degree d and $\omega \in \Omega_{\mathbb{P}^2}^1 \otimes \mathcal{O}_{\mathbb{P}^2}(d+2)$ the 1-form that induces \mathcal{F}_d in homogeneous coordinate. Let $\pi : \mathbb{C}^3 \setminus \{0\} \rightarrow \mathbb{P}^2$ and $\varrho : \mathbb{C}^4 \setminus \{0\} \rightarrow \mathbb{P} \times \mathbb{P}$ be the quotient maps. Then we have the following commutative diagram

$$\begin{array}{ccc} \mathbb{C}^4 \setminus \{0\} & \xrightarrow{\tilde{\zeta}} & \mathbb{C}^3 \setminus \{0\} \\ \downarrow & & \downarrow \\ \mathbb{P} \times \mathbb{P} & \xrightarrow{\zeta} & \mathbb{P}^2 \end{array}$$

where $\tilde{\zeta}(x, y, z, w) = (xz, yw, xw)$. We have that $\tilde{\zeta}^*(\omega)$ induces a foliation $\mathcal{F}_{(d,d)} := \zeta^*(\mathcal{F}_d)$ on $\mathbb{P} \times \mathbb{P}$ of bidegree (d, d) . Indeed, since \mathcal{F} has degree d then

$$\omega = a(x, y, z)dx + b(x, y, z)dy + c(x, y, z)dz$$

where a, b and c are homogeneous polynomials of degree $d+1$. Therefore

$$\tilde{\zeta}^*(\omega) = (Az + Cw)dx + Bwdy + Axdz + (By + Cx)dw,$$

where $A = a \circ \tilde{\zeta}$, $B = b \circ \tilde{\zeta}$ and $C = c \circ \tilde{\zeta}$. It is not difficult to see that $(Az + Cw)$ and Bw are bihomogeneous of bidegree $(d, d+1)$, $(By + Cx)$ and Ax are bihomogeneous of bidegree $(d+1, d)$. This shows that $\tilde{\zeta}^*(\omega)$ induces a foliation on $\mathbb{P} \times \mathbb{P}$ of bidegree (d, d) .

Normal form in affine coordinates

Let $P(X_1, \dots, X_r) \in \mathbb{C}[X_1, \dots, X_r]$, where $X_i = (x_{i_1}, \dots, x_{n_i}) \in \mathbb{C}^{n_i}$. Consider P as an element of $(\mathbb{C}[X_1, \dots, \widehat{X}_j, \dots, X_r])[X_j]$, and we will denote the degree of P with respect to the variable X_j by $\deg_{X_j}(P)$.

Proposition 2.2.1. *Let \mathcal{F} be a multiprojective foliation on $\mathbf{P}^{(n_1, \dots, n_r)}$ of multidegree $d = (d_1, \dots, d_r)$. Then \mathcal{F} is given in affine coordinates $(X_1, \dots, X_r) \in \mathcal{U} \simeq \mathbb{C}^{n_1 + \dots + n_r}$, with $X_i = (x_{i_1}, \dots, x_{n_i})$, by a polynomial vector field of the form*

$$\sum_{i=1}^r (P_i(X_1, \dots, X_r) + g_i(X_1, \dots, X_r)R_i),$$

where:

i) $R_i = \sum_{j=1}^{n_i} x_{i_j} \frac{\partial}{\partial x_{i_j}}$ is the radial vector field of \mathbb{C}^{n_i} , $i = 1, \dots, r$.

ii) g_i is a multihomogeneous polynomial of multidegree $(d_1 - 1, \dots, d_i, \dots, d_r - 1)$,

iii) $P_i = \sum_{j=1}^{n_i} P_{ij} \frac{\partial}{\partial x_{ij}}$, satisfying $\deg_{X_i}(P_{ij}) \leq d_i$, for all $j = 1, \dots, n_i$ and $i = 1, \dots, r$.

iv) The hyperplane at infinity $\mathbb{P}_{\mathbb{C}}^{n_1} \times \dots \times H_{\infty}^i \times \dots \times \mathbb{P}_{\mathbb{C}}^{n_r}$ is invariant by \mathcal{F} if, and only if, $g_i \equiv 0$.

Proof. The foliation \mathcal{F} is given by a morphism $\mathcal{X} : \mathcal{O}(k_1, \dots, k_r) \rightarrow T\mathbf{P}^{(n_1, \dots, n_r)}$, with $(k_1, \dots, k_r) \in \mathbb{Z}^r$, and $\mathcal{O}(k_1, \dots, k_r) = K_{\mathcal{F}}^*$. By definition 2.2.1 we have that $k_i = 1 - d_i$. Let (Z_1, \dots, Z_r) be a multihomogeneous coordinate system on $\mathbf{P}^{(n_1, \dots, n_r)}$ and take σ to be the meromorphic section of $\mathcal{O}(k_1, \dots, k_r)$ induced by $z_{0_1}^{k_1} \dots z_{0_r}^{k_r}$. The image of σ by the morphism \mathcal{X} is a meromorphic vector field ζ on $\mathbf{P}^{(n_1, \dots, n_r)}$, that is holomorphic over the open

$$\mathcal{U}_{(0_1, \dots, 0_r)} = \{([Z_1, \dots, Z_r]) \in \mathbf{P}^{(n_1, \dots, n_r)}; z_{0_1} \neq 0, \dots, z_{0_r} \neq 0\}$$

and ζ induces \mathcal{F} in this set. Moreover, each one of the hyperplanes $\{z_{0_i} = 0\}$, $i = 1, \dots, r$, is either a divisor of poles or a divisor of zeros of ζ with multiplicity k_i . Therefore we have

$$\zeta|_{\mathcal{U}_{(0_1, \dots, 0_r)}} = \sum_{i=1}^r \sum_{j=1}^{n_i} P_{ij} \frac{\partial}{\partial x_{ij}},$$

where $P_{ij} \in \mathcal{O}(\mathcal{U}_{(0_1, \dots, 0_r)})$, for all $i = 1, \dots, r$ e $j = 1, \dots, n_i$. We are going to consider the decomposition of these polynomials into multihomogenous parts,

$$P_{ij} = \sum_{s_1, \dots, s_r} P_{ij}^{(s_1, \dots, s_r)},$$

that is, $P_{ij}^{(s_1, \dots, s_r)}$ is multihomogeneous of degree (s_1, \dots, s_r) . We will see what happens when we change to the coordinate system $\mathcal{U}_{(i_1, \dots, i_r)}$, where $i_s \neq 0$ for all $s = 1, \dots, r$. Without loss of generality, it is enough to make the change from the coordinate system $\mathcal{U}_{(0_1, \dots, 0_r)}$ to $\mathcal{U}_{(1_1, \dots, 1_r)} = \{z_{1_1} \neq 0, \dots, z_{1_r} \neq 0\}$. This change is given by

$$\Phi_{0_1}(X_1, \dots, X_r) = (\varphi_{0_1}^1(X_1), \dots, \phi_{0_1}^r(X_r)),$$

where $\varphi_{0_1}^i(X_i) = \left(\frac{1}{x_{1_i}}, \frac{x_{2_i}}{x_{1_i}}, \dots, \frac{x_{n_i}}{x_{1_i}} \right) = (y_{1_i}, \dots, y_{n_i})$, com $i = 1, \dots, r$. The Jacobian matrix of Φ_{0_1} is

$$D\Phi_{0_1} = \begin{pmatrix} D\varphi_{0_1}^1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & D\phi_{0_1}^r \end{pmatrix}$$

where

$$D\varphi_{0_1}^i = \begin{pmatrix} -\frac{1}{x_{1_i}^2} & 0 & \cdots & 0 \\ -\frac{x_{2_i}}{x_{1_i}^2} & \frac{1}{x_{1_i}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{x_{n_i}}{x_{1_i}^2} & 0 & \cdots & \frac{1}{x_{1_i}} \end{pmatrix} = \begin{pmatrix} -y_{1_i}^2 & 0 & \cdots & 0 \\ -y_{2_i}y_{1_i} & y_{1_i} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -y_{n_i}y_{1_i} & 0 & \cdots & y_{1_i} \end{pmatrix}$$

Now, by push-forward we get

$$\begin{aligned} (\Phi_{0_1})_* \zeta|_{\mathcal{U}_{(0_1, \dots, 0_r)}} &= - \sum_{j=1}^r \left[\sum_{s_1, \dots, s_r} y_{1_j}^2 P_{1_j}^{(s_1, \dots, s_r)} \right] \frac{\partial}{\partial y_{1_j}} + \\ &+ \sum_{j=1}^r \sum_{i=2}^{n_i} \left[\sum_{s_1, \dots, s_r} -y_{1_j} y_{i_j} P_{1_j}^{(s_1, \dots, s_r)} + y_{1_j} P_{i_j}^{(s_1, \dots, s_r)} \right] \frac{\partial}{\partial y_{i_j}} \end{aligned}$$

Taking in to account that

$$\begin{aligned} P_{i_j}^{(s_1, \dots, s_r)}(\Phi_{0_1}^{-1}(Y_1, \dots, Y_r)) &= P_{i_j}^{(s_1, \dots, s_r)} \left(\frac{1}{y_{1_1}}, \frac{y_{2_1}}{y_{1_1}}, \dots, \frac{y_{n_1}}{y_{1_1}}, \dots, \frac{1}{y_{1_r}}, \frac{y_{2_r}}{y_{1_r}}, \dots, \frac{y_{n_r}}{y_{1_r}} \right) \\ &= y_{1_1}^{-s_1} \cdots y_{1_r}^{-s_r} P_{i_j}^{(s_1, \dots, s_r)}(1, y_{2_1}, \dots, y_{n_1}, \dots, 1, y_{2_r}, \dots, y_{n_r}) \end{aligned}$$

we get

$$\begin{aligned} (\Phi_{0_1})_* \zeta|_{\mathcal{U}_{(0_1, \dots, 0_r)}} &= - \sum_{j=1}^r \left[\sum_{s_1, \dots, s_r} y_{1_1}^{2-s_1} y_{1_2}^{-s_2} \cdots y_{1_r}^{-s_r} P_{1_j}^{(s_1, \dots, s_r)} \right] \frac{\partial}{\partial y_{1_j}} + \\ &+ \sum_{j=1}^r \sum_{i=2}^{n_i} \left[\sum_{s_1, \dots, s_r} y_{1_1}^{1-s_1} y_{1_2}^{-s_2} \cdots y_{1_r}^{-s_r} (-y_{i_j} P_{1_j}^{(s_1, \dots, s_r)} + P_{i_j}^{(s_1, \dots, s_r)}) \right] \frac{\partial}{\partial y_{i_j}} \end{aligned}$$

Remark that the hyperplane $(z_{0_j} = 0)$ corresponds to the hyperplane $(y_{1_j} = 0)$ on $\mathcal{U}_{(1_1, \dots, 1_r)}$, $j = 1, \dots, r$. If

$$-y_{i_j} P_{1_j}^{(k_1, \dots, k_r)} + P_{i_j}^{(k_1, \dots, k_r)} \equiv 0 \quad (2.9)$$

for all $j = 1, \dots, r$ and $i = 1, \dots, n_r$, $(y_{1_j} = 0)$ is a divisor of poles with order $2 - k_1$ if $j = 1$, and order $-k_i = d_i - 1$ for $i \neq j$ and $k_i \leq 1$. In this case, the equation (3.3.1) gives

$$P_{i_j}^{(k_1, \dots, k_r)} = y_{i_j} P_{1_j}^{(k_1, \dots, k_r)},$$

and changing to the coordinates $y_{i_j} = \frac{x_{i_j}}{x_{1_j}}$ we obtain

$$P_{i_j}^{(k_1, \dots, k_r)} = \frac{x_{i_j}}{x_{1_j}} P_{1_j}^{(k_1, \dots, k_r)}.$$

Defining $P_{1_j}^{(k_1, \dots, k_r)} / x_{1_j} = g_j$, we get that $P_{i_j}^{(k_1, \dots, k_r)} = x_{i_j} g_j$, where g_j is multihomogeneous of multidegree

$$(-k_1, \dots, 1 - k_j, \dots, -k_r) = (d_1 - 1, \dots, d_i, \dots, d_r - 1).$$

Therefore

$$\zeta_{|u_{(0_1, \dots, 0_r)}|} = \sum_{i=1}^r (P_i(X_1, \dots, X_r) + g_i(X_1, \dots, X_r) R_i)$$

We can see that the hyperplane $(z_{0_j} = 0) = \mathbb{P}^{n_1} \times \dots \times (\mathbb{P}^{n_j} - \mathbb{C}^{n_j}) \times \dots \times \mathbb{P}^{n_r}$ is \mathcal{F} -invariant if, and only, if $g_i \equiv 0$. \square

Representation in multihomogeneous coordinates

The Euler sequence over the multiprojective space $\mathbf{P}^{(n_1, \dots, n_r)}$ is given by

$$0 \longrightarrow \mathcal{O}^{\oplus r} \longrightarrow \bigoplus_{j=1}^r \mathcal{O}(0, \dots, \underbrace{1}_j, \dots, 0)^{\oplus n_j + 1} \longrightarrow T\mathbf{P}^{(n_1, \dots, n_r)} \longrightarrow 0, \quad (2.10)$$

Tensorizing this sequence by $\mathcal{O}(d_1 - 1, \dots, d_r - 1)$ we get the exact sequence

$$\begin{aligned} 0 \longrightarrow \mathcal{O}(d_1 - 1, \dots, d_r - 1)^{\oplus r} &\longrightarrow \bigoplus_{i=1}^r \mathcal{O}(d_1 - 1, \dots, d_i, \dots, d_r - 1)^{\oplus n_i + 1} \longrightarrow \\ &\longrightarrow T\mathbf{P}^{(n_1, \dots, n_r)}(d_1 - 1, \dots, d_r - 1) \longrightarrow 0. \end{aligned}$$

We conclude that a foliation on $\mathbf{P}^{(n_1, \dots, n_r)}$ of multidegree (d_1, \dots, d_r) can be represented in multihomogenous coordinates of $\mathbb{C}^{\sum_{i=1}^r (n_i + 1)}$ by a polynomial vector field of the form

$$X = \sum_{i=1}^r X_i$$

with $X_i = \sum_{j=0}^{n_i} P_{i_j} \frac{\partial}{\partial z_{i_j}}$, and P_{i_j} is a multihomogenous polynomial of multidegree $(d_1 - 1, \dots, d_i, \dots, d_r - 1)$ modulo

$$\sum_{i=1}^r g_i R_i,$$

where g_i has multidegree $d = (d_1 - 1, \dots, d_r - 1)$ and $R_i = \sum_{j=0}^{n_i} z_{i_j} \frac{\partial}{\partial z_{i_j}}$.

Geometric interpretation of multidegree

Recall that the Chow group of $\mathbf{P}^{(n_1, \dots, n_r)}$ is given by

$$\mathcal{A}_*(\mathbf{P}^{(n_1, \dots, n_r)}) \simeq \frac{\mathbb{Z}[h_1, \dots, h_r]}{\langle h_1^{n_1}, \dots, h_r^{n_r} \rangle},$$

where $h_i = \pi_i^* H_i$ and H_i is the hyperplane class of \mathbb{P}^{n_i} , $i = 1, \dots, r$.

The following proposition gives a geometric interpretation of the i -th degree d_i of a foliation of multidegree (d_1, \dots, d_r) .

Proposition 2.2.2. *Let \mathcal{F} be a foliation on $\mathbf{P}^{(n_1, \dots, n_r)}$ of multidegree (d_1, \dots, d_r) and h_i a generic hypersurface of multidegree $(0, \dots, \underbrace{1}_i, \dots, 0)$. The i -th degree d_i of \mathcal{F} is given by the intersection number*

$$d_i = \mathcal{T}(\mathcal{F}, h_i) \cdot [h_1^{n_1} \cdots h_i^{n_i-2} \cdots h_r^{n_r}],$$

where $\mathcal{T}(\mathcal{F}, h_i)$ is the cycle of the tangency variety of \mathcal{F} with respect to h_i .

Proof. Since the cycle $\mathcal{T}(\mathcal{F}, h_i)$ is given by the zeros of a section of the line bundle $(K_{\mathcal{F}} \otimes \mathcal{O}(h_i))|_{h_i}$, then

$$\mathcal{T}(\mathcal{F}, h_i) = c_1(K_{\mathcal{F}} \otimes \mathcal{O}(h_i)) \cap h_i \in \mathcal{A}_{n-2}(\mathbf{P}^{(n_1, \dots, n_r)})$$

Since $K_{\mathcal{F}} = \mathcal{O}(d_1 - 1, \dots, d_r - 1)$ and $\mathcal{O}(h_i) = \mathcal{O}(0, \dots, \underbrace{1}_i, \dots, 0)$, we have

$$K_{\mathcal{F}} \otimes \mathcal{O}(h_i) = \mathcal{O}(d_1 - 1, \dots, d_i, \dots, d_r - 1),$$

so $c_1(K_{\mathcal{F}} \otimes \mathcal{O}(h_i)) = (d_1 - 1)h_1 + \cdots + d_i h_i + \cdots + (d_r - 1)h_r$. Therefore

$$\mathcal{T}(\mathcal{F}, h_i) = \left[d_i h_i + \sum_{j \neq i} (d_j - 1) h_j \right] \cap h_i = d_i h_i^2 + \sum_{j \neq i} (d_j - 1) h_j \cap h_i \quad (2.11)$$

Applying the cycle $h_1^{n_1} \cap \cdots \cap h_i^{n_i-2} \cap \cdots \cap h_r^{n_r}$ to equation (2.11) and using that $h_1^{n_1} \cap \cdots \cap h_i^{n_i} \cap \cdots \cap h_r^{n_r} = 1$ and $h_j^{n_j+1} = 0$, for all $j = 1, \dots, r$, we get

$$\mathcal{T}(\mathcal{F}, h_i) \cdot [h_1^{n_1} \cap \cdots \cap h_i^{n_i-2} \cap \cdots \cap h_r^{n_r}] = d_i.$$

□

The degree of a multiprojective foliation via the Segre embedding

The multiprojective space $\mathbf{P}^{(n_1, \dots, n_r)}$ can be embedded into projective space \mathbb{P}^N , where $N = \prod_{i=1}^r (n_i + 1) - 1$, via the Segre embedding $i_{\mathcal{O}(1, \dots, 1)} : \mathbf{P}^{(n_1, \dots, n_r)} \rightarrow \mathbb{P}^N$ through the linear system $|\mathcal{O}(1, \dots, 1)|$. Moreover we have $i_{\mathcal{O}(1, \dots, 1)}^*(\mathcal{O}_{\mathbb{P}^N}(1)) =$

$\mathcal{O}(1, \dots, 1)$. Hence, $c_1[\mathcal{O}(1, \dots, 1)] = c_1[i_{\mathcal{O}(1, \dots, 1)}^*(\mathcal{O}_{\mathbb{P}^N}(1))] = i_{\mathcal{O}(1, \dots, 1)}^*[c_1(\mathcal{O}_{\mathbb{P}^N}(1))]$. If we denote $c_1(\mathcal{O}_{\mathbb{P}^N}(1)) = H$ we conclude that

$$i_{\mathcal{O}(1, \dots, 1)}^*H = c_1[\mathcal{O}(1, \dots, 1)] = h_1 + \dots + h_r.$$

Therefore, if L is a line bundle on $\mathbf{P}^{(n_1, \dots, n_r)}$ then the degree of L with respect to the Segre embedding is given by

$$\deg_{\mathcal{O}(1, \dots, 1)}(L) = \int_{\mathbf{P}^{(n_1, \dots, n_r)}} c_1(L) \cdot (h_1 + \dots + h_r)^{n-1}.$$

Let \mathcal{F} be a foliation on $\mathbf{P}^{(n_1, \dots, n_r)}$ and $i_{\mathcal{O}(1, \dots, 1)}(\mathbf{P}^{(n_1, \dots, n_r)}) = \Sigma_{n_1, \dots, n_r}$. We will determine the degree of \mathcal{F} with respect to the Segre embedding.

Notation: $n = n_1 + \dots + n_r$ and

$$\binom{n}{n_1, \dots, n_r} = \frac{(n_1 + \dots + n_r)!}{n_1! \dots n_r!}.$$

Proposition 2.2.3. *Let \mathcal{F} be a foliation on $\mathbf{P}^{(n_1, \dots, n_r)}$ of degree (d_1, \dots, d_r) . Then*

$$\deg_{\mathcal{O}(1, \dots, 1)}(\mathcal{F}) = \sum_{i=1}^r d_i \binom{n-1}{n_1, \dots, n_i-1, \dots, n_r}.$$

Proof. Since $K_{\mathcal{F}} = \mathcal{O}(d_1 - 1, \dots, d_r - 1)$, we get

$$\deg_{\mathcal{O}(1, \dots, 1)}(\mathcal{F}) = \deg(K_{\mathcal{F}}) + \deg(\Sigma_{n_1, \dots, n_r}) = \deg_{\mathcal{O}(1, \dots, 1)}(\mathcal{O}(d_1 - 1, \dots, d_r - 1)) + \deg(\Sigma_{n_1, \dots, n_r}).$$

We have

$$\begin{aligned} \deg(\mathcal{O}(d_1 - 1, \dots, d_r - 1)) &= \int_{\mathbf{P}^{(n_1, \dots, n_r)}} c_1(\mathcal{O}(d_1 - 1, \dots, d_r - 1)) \cdot (h_1 + \dots + h_r)^{n-1} \\ &= \int_{\mathbf{P}^{(n_1, \dots, n_r)}} \left(\sum_{i=1}^r (d_i - 1) h_i \right) \cdot \sum_{s_1 + \dots + s_r = n-1} \binom{n-1}{s_1, \dots, s_r} h_1^{s_1} \dots h_r^{s_r} \\ &= \sum_{i=1}^r (d_i - 1) \binom{n-1}{n_1, \dots, n_i-1, \dots, n_r} \int_{\mathbf{P}^{(n_1, \dots, n_r)}} h_1^{n_1} \dots h_r^{n_r} \\ &= \sum_{i=1}^r (d_i - 1) \binom{n-1}{n_1, \dots, n_i-1, \dots, n_r}. \end{aligned}$$

On the other hand, $\deg(\Sigma_{n_1, \dots, n_r}) = \frac{(n_1 + \dots + n_r)!}{n_1! \dots n_r!} = \sum_{i=1}^r \binom{n-1}{n_1, \dots, n_i-1, \dots, n_r}$.

Hence

$$\begin{aligned} \deg_{\mathcal{O}(1, \dots, 1)}(\mathcal{F}) &= \sum_{i=1}^r (d_i - 1) \binom{n-1}{n_1, \dots, n_i-1, \dots, n_r} + \sum_{i=1}^r \binom{n-1}{n_1, \dots, n_i-1, \dots, n_r} \\ &= \sum_{i=1}^r d_i \binom{n-1}{n_1, \dots, n_i-1, \dots, n_r}. \end{aligned}$$

□

Example 2.2.2. Let \mathcal{F} be a foliation on $\mathbf{P}^{(n_1, \dots, n_r)}$. If \mathcal{F} has multidegree (d, \dots, d) then

$$\deg_{\mathcal{O}_M(1)}(\mathcal{F}) = d \cdot \deg(\mathbf{P}^{(n_1, \dots, n_r)}),$$

In particular, if $n_i = 1$ for all $i = 1, \dots, r$, we get $\deg_{\mathcal{O}_M(1)}(\mathcal{F}) = d \cdot r!$,

Projections

Let $\mathcal{O}(\mathbf{d} - 1) := \mathcal{O}(d_1 - 1, \dots, d_r - 1)$ and $\mathcal{O}(\mathbf{d}_i) := \mathcal{O}(d_1 - 1, \dots, d_i, \dots, d_r - 1)$. We have the following comutative diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathcal{O}(\mathbf{d} - 1)^{\oplus r} & \rightarrow & \bigoplus_{i=1}^r \mathcal{O}(\mathbf{d}_i)^{\oplus n_i+1} & \rightarrow & T\mathbf{P}^{(n_1, \dots, n_r)} \otimes \mathcal{O}(\mathbf{d} - 1) \rightarrow 0 \\ & & \downarrow \rho_i & & \downarrow \varrho_i & & \downarrow \bar{\varrho}_i \\ 0 & \rightarrow & \mathcal{O}_{\mathbb{P}^{n_i}}(d_i - 1) & \rightarrow & \mathcal{O}_{\mathbb{P}^{n_i}}(d_i)^{\oplus n_i+1} & \rightarrow & T\mathbb{P}^{n_i} \otimes \mathcal{O}_{\mathbb{P}^{n_i}}(d_i - 1) \rightarrow 0 \end{array}$$

where the vertical maps are defined as

$$\rho_i(g_1(Z_1, \dots, Z_r), \dots, g_r(Z_1, \dots, Z_r)) = g_i(1, \dots, 1, Z_i, 1, \dots, 1)$$

and

$$\varrho_i(X_1(Z_1, \dots, Z_r), \dots, X_r(Z_1, \dots, Z_r)) = X_i(1, \dots, 1, Z_i, 1, \dots, 1),$$

with $g_i \in H^0(\mathbf{P}^{(n_1, \dots, n_r)}, \mathcal{O}(\mathbf{d} - 1))$ and $X_i \in H^0(\mathbf{P}^{(n_1, \dots, n_r)}, \mathcal{O}(\mathbf{d}_i)^{\oplus n_i+1})$. Hence, we conclude that there is a rational projection

$$\begin{array}{ccc} \bar{\varrho}_i : \mathbb{P}H^0(\mathbf{P}^{(n_1, \dots, n_r)}, T\mathbf{P}^{(n_1, \dots, n_r)} \otimes \mathcal{O}(\mathbf{d} - 1)) & \dashrightarrow & \mathbb{P}H^0(\mathbb{P}^{n_i}, T\mathbb{P}^{n_i} \otimes \mathcal{O}_{\mathbb{P}^{n_i}}(d_i - 1)) \\ & \mapsto & \mathcal{F}^{d_i} \end{array}$$

that associates to each foliation $\mathcal{F}^{(d_1, \dots, d_r)}$ of multidegree $\mathbf{d} = (d_1, \dots, d_r)$ on $\mathbf{P}^{(n_1, \dots, n_r)}$ a foliation \mathcal{F}^{d_i} on \mathbb{P}^{n_i} of degree d_i , by the construction above. We call \mathcal{F}^{d_i} the *projected foliation* of $\mathcal{F}^{(d_1, \dots, d_r)}$ in \mathbb{P}^{n_i} .

Proposition 2.2.4. Let $H = \{p_1\} \times \cdots \times \mathbb{P}^{n_i} \times \cdots \times \{p_r\} \simeq \mathbb{P}^{n_i}$, where $p_j = (1 : \cdots : 1) \in \mathbb{P}^{n_j}$, $j \neq i$. The i -th degree of a foliation \mathcal{F}^d on $\mathbf{P}^{(n_1, \dots, n_r)}$ is the degree of the foliation on \mathbb{P}^{n_i} given by

$$\mathcal{F}^{d_i} = \mathcal{F}_{|H}^d.$$

Remark 2.2.2. When $d_i = 1$ we have $\bar{\varrho}_i = D\pi_i$.

The number of singularities of a multiprojective foliation

In this section we determine the number of isolated singularities of a multiprojective foliation.

Lemma 2.2.1. Let $\mathbf{P}^{(n_1, \dots, n_r)}$ and $h_i = \pi_i^* H_i$, where H_i is the hyperplane class in \mathbb{P}^{n_i} , with $i = 1, \dots, r$. Then

$$c_k(\mathbf{P}^{(n_1, \dots, n_r)}) = \sum_{i_1 + \dots + i_r = k} \prod_{s=1}^r \binom{n_s + 1}{i_s} h_s^{i_s},$$

for all $1 \leq k \leq n = n_1 + \dots + n_r$.

Proof. From the Euler sequence we have

$$c\left(\bigoplus_{i=1}^r \mathcal{O}(0, \dots, \underbrace{1}_i, \dots, 0)^{\oplus n_i + 1}\right) = c(T\mathbf{P}^{(n_1, \dots, n_r)}) \cdot c(\underline{\mathbb{C}}^r) = c(\mathbf{P}^{(n_1, \dots, n_r)})$$

therefore

$$c(T\mathbf{P}^{(n_1, \dots, n_r)}) = \prod_{i=1}^r (1 + h_i)^{n_i + 1}, \quad (2.12)$$

where $h_i = c_1(\mathcal{O}(0, \dots, \underbrace{1}_i, \dots, 0))$. On the other hand, we have that

$$(1 + h_j)^{n_j + 1} = \sum_{i_j=0}^{n_j} \binom{n_j + 1}{i_j} h_j^{i_j},$$

for all $j = 1, \dots, r$. If we substitute this in equation (2.12) we get

$$c(T\mathbf{P}^{(n_1, \dots, n_r)}) = \sum_{i_1, \dots, i_r} \binom{n_1 + 1}{i_1} \cdots \binom{n_r + 1}{i_r} h_1^{i_1} \cdots h_r^{i_r}.$$

Hence

$$c_k(T\mathbf{P}^{(n_1, \dots, n_r)}) = \sum_{i_1 + \dots + i_r = k} \binom{n_1 + 1}{i_1} \cdots \binom{n_r + 1}{i_r} h_1^{i_1} \cdots h_r^{i_r}.$$

□

Theorem 2.2.1. *Let \mathcal{F} be a multiprojective foliation on $\mathbf{P}^{(n_1, \dots, n_r)}$, of multidegree $d = (d_1, \dots, d_r)$, whose zeros are isolated. Then*

$$\sum_{p \in \text{Sing}(\mathcal{F})} \mu_p(\mathcal{F}) = \sum_{\substack{0 \leq k_1 \leq n_1 \\ \dots \\ 0 \leq k_r \leq n_r}} \binom{k}{k_1, \dots, k_r} \prod_{s=1}^r \binom{n_s + 1}{n_s - k_s} (d_s - 1)^{k_s},$$

where $\mu_p(\mathcal{F})$ is the algebraic multiplicity or Milnor number of \mathcal{F} at $p \in \text{Sing}(\mathcal{F})$.

Proof. We will use the following notation

$$c_n(T\mathbf{P}^{(n_1, \dots, n_r)}(d_1 \dots, d_r)) = c_n(T\mathbf{P}^{(n_1, \dots, n_r)} \otimes \mathcal{O}(d_1 - 1, \dots, d_r - 1)).$$

It follows from Baum-Bott's theorem that

$$\sum_{p \in \text{Sing}(\mathcal{F})} \mu_p(\mathcal{F}) = \int_{\mathbf{P}^{(n_1, \dots, n_r)}} c_n(\mathbf{P}^{(n_1, \dots, n_r)}(d_1 \dots, d_r)).$$

On the other hand,

$$c_n(T\mathbf{P}^{(n_1, \dots, n_r)}(d_1 \dots, d_r)) = \sum_{j=0}^n c_j(T\mathbf{P}^{(n_1, \dots, n_r)}) c_1(\mathcal{O}(d_1 - 1, \dots, d_r - 1))^{n-j}.$$

From lemma 2.2.1 and as $c_1(\mathcal{O}(d_1 - 1, \dots, d_r - 1))^{n-j} = ((d_1 - 1)h_1 + \dots + (d_r - 1)h_r)^{n-j}$ we obtain

$$c_n(T\mathbf{P}^{(n_1, \dots, n_r)}(d_1 \dots, d_r)) = \sum_{j=0}^n \sum_{i_1 + \dots + i_r = j} \prod_{s=1}^r \binom{n_s + 1}{i_s} h_s^{i_s} (d_1 h_1 + \dots + d_r h_r)^{n-j}.$$

Now, $((d_1 - 1)h_1 + \dots + (d_r - 1)h_r)^{n-j} = \sum_{k_1 + \dots + k_r = n-j} (n-j)! \prod_{s=1}^r \frac{(d_s - 1)^{k_s}}{k_s!} h_s^{k_s}$.

Substituting this in the above equation, we get

$$\begin{aligned} c_n(T\mathbf{P}^{(n_1, \dots, n_r)}(d_1 \dots, d_r)) &= \sum_{j=0}^n \sum_{\substack{i_1 + \dots + i_r = j \\ k_1 + \dots + k_r = n-j}} \prod_{s=1}^r \binom{n_s + 1}{i_s} h_s^{i_s} (n-j)! \prod_{s=1}^r \frac{(d_s - 1)^{k_s}}{k_s!} h_s^{k_s} \\ &= \sum_{j=0}^n \sum_{\substack{i_1 + \dots + i_r = j \\ k_1 + \dots + k_r = n-j}} (n-j)! \prod_{s=1}^r \binom{n_s + 1}{i_s} \frac{(d_s - 1)^{k_s}}{k_s!} h_s^{k_s + i_s} \\ &= \sum_{\substack{i_1 + \dots + i_r + \\ + k_1 + \dots + k_r = n}} (n_1 - i_1 + \dots + n_r - i_r)! \prod_{s=1}^r \binom{n_s + 1}{i_s} \frac{(d_s - 1)^{k_s}}{k_s!} h_s^{k_s + i_s} \end{aligned}$$

Integration gives

$$\sum_{\substack{i_1 + \dots + i_r + \\ + k_1 + \dots + k_r = n}} \frac{(n_1 - i_1 + \dots + n_r - i_r)!}{k_1! \dots k_r!} \prod_{s=1}^r \binom{n_s + 1}{i_s} (d_s - 1)^{k_s} \int_{\mathbf{P}^{(n_1, \dots, n_r)}} h_1^{i_1 + k_1} \dots h_r^{i_r + k_r}.$$

It is not difficult to see that the n -form $h_1^{i_1 + k_1} \dots h_r^{i_r + k_r} \neq 0$ if, and only if, $i_s + k_s = n_s$, $s = 1, \dots, r$. That is $n_1 - i_1 + \dots + n_r - i_r = k_1 + \dots + k_r$. Since

$$\int_{\mathbf{P}^{(n_1, \dots, n_r)}} h_1^{n_1} \dots h_r^{n_r} = \int_{\mathbb{P}^{n_1}} H_1^{n_1} \dots \int_{\mathbb{P}^{n_1}} H_1^{n_1} = 1,$$

we get

$$\int_{\mathbf{P}^{(n_1, \dots, n_r)}} c_n(T\mathbf{P}^{(n_1, \dots, n_r)}(d_1, \dots, d_r)) = \sum_{\substack{0 \leq k_1 \leq n_1 \\ \dots \\ 0 \leq k_r \leq n_r}} \binom{k}{k_1, \dots, k_r} \prod_{r=1}^r \binom{n_r + 1}{n_r - k_r} (d_r - 1)^{k_r}.$$

□

Example 2.2.3. Let \mathcal{F} be a multiprojective foliation on $\underbrace{\mathbb{P} \times \dots \times \mathbb{P}}_{n\text{-times}}$, of multidegree $d = (d_1, \dots, d_n)$, with isolated singularities. Then

$$\sum_{p \in \text{Sing}(\mathcal{F})} \mu_p(\mathcal{F}) = \sum_{j=0}^n 2^j (n-j)! \sigma_{n-j}(d_1 - 1, \dots, d_n - 1)$$

where σ_{n-i} is the $(n-i)$ -th elementary symmetric function. In particular, if $d_i = d$, for all $i = 1, \dots, n$, we have that

$$\sum_{p \in \text{Sing}(\mathcal{F})} \mu_p(\mathcal{F}) = n! \sum_{j=0}^n \frac{2^j}{j!} (d-1)^{n-j}.$$

In this case $n_i = 1$, for all $i = 1, \dots, n$ and $\binom{k}{k_1, \dots, k_n} = (k_1 + \dots + k_n)!$. From Theorem 2.2.1 we have that

$$\sum_{p \in \text{Sing}(\mathcal{F})} \mu_p(\mathcal{F}) = \sum_{\substack{1 \leq s \leq n \\ 0 \leq k_r \leq 1}} (k_1 + \dots + k_n)! \prod_{r=1}^n \binom{2}{1 - k_r} (d_r - 1)^{k_r}.$$

Since $\binom{2}{1 - k_r} = 2^{1 - k_r}$ we obtain

$$\begin{aligned} \sum_{p \in \text{Sing}(\mathcal{F})} \mu_p(\mathcal{F}) &= \sum_{j=0}^n (n-j)! 2^j \sum_{k_1 + \dots + k_n = j} (d_1 - 1)^{k_1} \dots (d_n - 1)^{k_n} \\ &= \sum_{j=0}^n (n-j)! 2^j \sigma_{n-j}(d_1 - 1, \dots, d_n - 1). \end{aligned}$$

Corollary 2.2.1. *The number of singularities of a non-degenerated multiprojective foliation on $\mathbf{P}^{(n_1, \dots, n_r)}$, of multidegree $d = (d_1, \dots, d_r)$ is given by*

$$\sum_{\substack{0 \leq k_1 \leq n_1 \\ \dots \\ 0 \leq k_r \leq n_r}} \binom{k}{k_1, \dots, k_r} \prod_{r=1}^r \binom{n_r + 1}{n_r - k_r} (d_r - 1)^{k_r},$$

The number of singularities of a non-degenerated multiprojective foliation on $\mathbf{P}^{(n_1, \dots, n_r)}$, of multidegree $d = (1, \dots, d_i, \dots, 1)$ is

$$\sum_{p \in \text{Sing}(\mathcal{F})} \mu_p(\mathcal{F}) = (n_1 + 1) \cdots (\widehat{n_i + 1}) \cdots (n_r + 1) (d_i^{n_i} + d_i^{n_i-1} + \cdots + d_i + 1)$$

Let $\mathcal{N}(\mathcal{F}^{\mathbf{d}}) = \sum_{p \in \text{Sing}(\mathcal{F}^{\mathbf{d}})} \mu_p(\mathcal{F}^{\mathbf{d}})$, where $\mathcal{F}^{\mathbf{d}}$ is a foliation on $\mathbf{P}^{(n_1, \dots, n_r)}$ of multidegree $\mathbf{d} = (d_1, \dots, d_r)$. Let \mathcal{F}^{d_i} be the projected foliation on \mathbb{P}^{n_i} . A natural question is: what relation there exists between $\mathcal{N}(\mathcal{F}^{\mathbf{d}})$ and $\mathcal{N}(\mathcal{F}^{d_i})$? For instance, if $\mathbf{d} = (1, \dots, d_i, \dots, 1)$ then

$$\mathcal{N}(\mathcal{F}^{\mathbf{d}}) = \prod_{i=1}^r \mathcal{N}(\mathcal{F}^{d_i}).$$

In general we have

Corollary 2.2.2. *Let $\mathcal{F}^{\mathbf{d}}$ be a foliation on $\mathbf{P}^{(n_1, \dots, n_r)}$ of multidegree $\mathbf{d} = (d_1, \dots, d_r)$ and \mathcal{F}^{d_i} the projected foliation on \mathbb{P}^{n_i} . Then*

$$\mathcal{N}(\mathcal{F}^{(d_1, \dots, d_r)}) - \prod_{i=1}^r \mathcal{N}(\mathcal{F}^{d_i}) = \sum_{\substack{0 \leq k_1 \leq n_1 \\ \dots \\ 0 \leq k_r \leq n_r}} \left[\binom{k}{k_1, \dots, k_r} - 1 \right] \cdot \prod_{r=1}^r \binom{n_r + 1}{n_r - k_r} (d_r - 1)^{k_r}.$$

Proof. From theorem 2.2.1 we have that $\mathcal{N}(\mathcal{F}^{d_i}) = \sum_{0 \leq k_i \leq n_i} \binom{n_i + 1}{n_i - k_i} (d_i - 1)^{k_i}$,

therefore

$$\prod_{i=1}^r \mathcal{N}(\mathcal{F}^{d_i}) = \sum_{\substack{0 \leq k_1 \leq n_1 \\ \dots \\ 0 \leq k_r \leq n_r}} \prod_{r=1}^r \binom{n_r + 1}{n_r - k_r} (d_r - 1)^{k_r}.$$

Also, from theorem 2.2.1 we have

$$\mathcal{N}(\mathcal{F}^{(d_1, \dots, d_r)}) = \sum_{\substack{0 \leq k_1 \leq n_1 \\ \dots \\ 0 \leq k_r \leq n_r}} \binom{k}{k_1, \dots, k_r} \prod_{r=1}^r \binom{n_r + 1}{n_r - k_r} (d_r - 1)^{k_r}.$$

The result follows by subtracting $\mathcal{N}(\mathcal{F}^{(d_1, \dots, d_r)}) - \prod_{i=1}^r \mathcal{N}(\mathcal{F}^{d_i})$. \square

2.2.1 Riccati foliations

Definition 2.2.2. [73] Let $\zeta = (E, \pi, B, F)$ be a holomorphic bundle, with total space E , projection π , base B and fibre F . Let \mathcal{F} be a singular holomorphic foliation on E . We say that \mathcal{F} is transversal to almost every fibre of ζ if there is an analytic subset $\Lambda(\mathcal{F}) \subset E$ which is a union of fibers of ζ , such that the restriction of \mathcal{F} to $E_0 = E \setminus \Lambda(\mathcal{F})$ is transversal to the natural subbundle ζ_0 of ζ having E_0 as total space. By a Riccati foliation we mean a foliation \mathcal{F} as above, for which the exceptional set $\Lambda(\mathcal{F})$ is \mathcal{F} -invariant.

F. Santos e B. Scárdua showed the following result about Riccati's foliations.

Theorem 2.2.2. [73] Let \mathcal{F} be a one-dimensional singular holomorphic foliation on $\mathbb{P} \times M$, transversal to almost every fibre of the bundle $\pi : \mathbb{P} \times M \rightarrow \mathbb{P}$, where $\pi(x, y) = x$ and $M = \underbrace{\mathbb{P} \times \cdots \times \mathbb{P}}_{n\text{-times}}$ or $M = \mathbb{P}^2$. Then \mathcal{F} is Riccati. Moreover,

i) If $M = \underbrace{\mathbb{P} \times \cdots \times \mathbb{P}}_{n\text{-times}}$ then \mathcal{F} is given in affine coordinates by a vector field of the form

$$X = p(x) \frac{\partial}{\partial x} + \sum_{i=1}^n (y_i^2 a_{i2}(x) + y_i a_{i1}(x) + a_{i0}(x)) \frac{\partial}{\partial y_i}$$

ii) If $M = \mathbb{P}^2$ then \mathcal{F} is given in affine coordinates by a vector field of the form

$$X = p(x) \frac{\partial}{\partial x} + Q(x, y, z) \frac{\partial}{\partial y} + R(x, y, z) \frac{\partial}{\partial z},$$

where

$$\begin{aligned} Q(x, y, z) &= A(x) + B(x)y + C(x)z + D(x)yz + E(x)y^2 \\ R(x, y, z) &= a(x) + b(x)y + c(x)z + E(x)yz + D(x)y^2 \end{aligned}$$

We will use the multindex notation $J_s = (j_{s_1}, \dots, j_{n_s})$, $|J_s| = j_{s_1} + \cdots + j_{n_s}$ and $Y_s^{J_s} = y_{s_1}^{j_{s_1}} \cdots y_{n_s}^{j_{n_s}}$, with $s = 1, \dots, r$.

Theorem 2.2.3. Let \mathcal{F} be as in theorem 2.2.2 where $M = \mathbf{P}^{(n_1, \dots, n_r)}$. Then \mathcal{F} is Riccati and given in affine coordinates by a vector field of the form

$$X = p(x) \frac{\partial}{\partial x} + \sum_{j=1}^r Z_j,$$

where

$$Z_i = \sum_{k=1}^{n_i} \left(\sum_{\substack{|J_s| \leq 2 \\ 1 \leq s \leq r}} a_{J_1 \dots J_r}^k(x) Y_1^{J_1} \cdots Y_r^{J_r} \right) \frac{\partial}{\partial y_{i_k}}.$$

Moreover, there exists $k \leq \deg(p(x))$ such that

$$\sum_{p \in \text{Sing}(\mathcal{F}) \cap \Lambda(\mathcal{F})} \mu_p(\mathcal{F}) = k(n_1 + 1) \cdots (n_r + 1).$$

Proof. Suppose that \mathcal{F} has multidegree (d, d_1, \dots, d_r) . It follows from Proposition 2.2.1 that \mathcal{F} is given in affine coordinates $(x, Y_1, \dots, Y_r) \in \mathbb{C} \times \mathbb{C}^{n_1} \times \dots \times \mathbb{C}^{n_r}$, with $Y_i = (y_{i1}, \dots, y_{in_i})$, by a polynomial vector field of the form

$$[p(x, Y_1, \dots, Y_r) + g_0(x, Y_1, \dots, Y_r)x] \frac{\partial}{\partial x} + \sum_{i=1}^r Z_i,$$

where $Z_i = \sum_{k=1}^{n_i} (P_i^k(x, Y_1, \dots, Y_r) + g_i(x, Y_1, \dots, Y_r)y_{ik}) \frac{\partial}{\partial y_{ik}}$, satisfying $\deg_{Y_i}(P_i^k) \leq d_i$, for all $j = 1, \dots, n_i$ and $i = 1, \dots, r$. Moreover, g_i is a multihomogeneous polynomial of multidegree $(d-1, d_1-1, \dots, d_i, \dots, d_r-1)$, $i = 0, 1, \dots, r$.

Set $Q(x, Y_1, \dots, Y_r) = p(x, Y_1, \dots, Y_r) + g_0(x, Y_1, \dots, Y_r)x$. Let $\{x_0\} \times \mathbf{P}^{(n_1, \dots, n_r)}$ be a fibre of π which is not invariant by \mathcal{F} . Then, by compactness of fibers of π , \mathcal{F} is transverse to $\{x\} \times \mathbf{P}^{(n_1, \dots, n_r)}$ for all x in a neighborhood U of x_0 . Therefore, the polynomial $Q(x, Y_1, \dots, Y_r) \neq 0$ for all $x \in U$ and all $(Y_1, \dots, Y_r) \in \mathbb{C}^{n_1} \times \dots \times \mathbb{C}^{n_r}$ and thus $Q(x, Y_1, \dots, Y_r) = Q(x)$, and so $g(x, z_1, \dots, z_r) = g(x)$. This implies that $d_i - 1 = \deg_{Y_i}(g) = 0$, i.e. \mathcal{F} has multidegree $(d, 1, \dots, 1)$. Hence $\deg_{Y_i}(P_i^k + g_i y_{ik}) \leq 2$. The fibre $x = c$ is \mathcal{F} -invariant if, and only if, $Q(c) = 0$. Thus, the exceptional set $\Lambda(\mathcal{F}) = \bigcup_{i=1}^k (\{c_i\} \times \mathbf{P}^{(n_1, \dots, n_r)})$, where $k \leq \deg(Q(x))$. Denote by \mathcal{F}_i the one-dimensional foliation on $\{c_i\} \times \mathbf{P}^{(n_1, \dots, n_r)} \simeq \mathbf{P}^{(n_1, \dots, n_r)}$ induced by restriction of \mathcal{F} . Since \mathcal{F}_i is a foliation on $\mathbf{P}^{(n_1, \dots, n_r)}$ of multidegree $(1, \dots, 1)$, it follows from theorem 2.2.1 that

$$\sum_{(c_i, q) \in \text{Sing}(\mathcal{F}) \cap \Lambda(\mathcal{F})} \mu_p(\mathcal{F}) = k \cdot \sum_{q \in \text{Sing}(\mathcal{F}_i)} \mu_q(\mathcal{F}_i) = k(n_1 + 1) \cdots (n_r + 1).$$

□

2.2.2 Totally invariant hypersurfaces

Let \mathcal{V} be a hypersurface on $\mathbf{P}^{(n_1, \dots, n_r)}$ given by zeros of a multihomogeneous polynomial $f \in \mathbb{C}[Z_1, \dots, Z_r]$, where $Z_i = (z_{i0}, \dots, z_{in_i})$, $i = 1, \dots, r$. Consider a foliation \mathcal{F}_X on $\mathbf{P}^{(n_1, \dots, n_r)}$ of multidegree (d_1, \dots, d_r) induced, in multihomogeneous coordinates, by a vector field $X = \sum_{i=1}^r X_i$. We say that \mathcal{V} is *i-invariant* by \mathcal{F} if

$$X_i(f) = h_i f, \tag{2.13}$$

where h_i is a multihomogeneous polynomial of degree $(d_1 - 1, \dots, d_r - 1)$. We say that \mathcal{V} is *totally invariant* if it is *i-invariant* for all $i = 1, \dots, r$. Let \mathcal{CV} be the tangent cone of \mathcal{V} , i.e. $\mathcal{CV} = \pi^{-1}(\mathcal{V})$, where

$$\pi : (\mathbb{C}^{n_1+1} \setminus \{0\}) \times \dots \times (\mathbb{C}^{n_r+1} \setminus \{0\}) \longrightarrow \mathbf{P}^{(n_1, \dots, n_r)}$$

is the quotient projection. The condition (2.13) means that the vector field X_i is tangent to \mathcal{CV} .

Remark 2.2.3. The definition of i -invariant is independent of the vector field X which defines the foliation \mathcal{F} . Indeed, if \mathcal{F} is induced by $Y = \sum_{i=1}^r Y_i$, so there exist polynomials g_i , of degree $(d_1 - 1, \dots, d_r - 1)$, such that

$$Y = \sum_{i=1}^r (X_i + g_i R_i).$$

Therefore, using the Euler formula $R_i(f) = k_i f$, we get

$$Y_i(f) = X_i(f) + g_i R_i(f) = h_i f + k_i g_i f = (h_i + k_i g_i) f.$$

Example 2.2.4. Let \mathcal{F} be a foliation of multidegree (d_1, \dots, d_r) on $\mathbf{P}^{(n_1, \dots, n_r)}$. Then, every hypersurface of multidegree $(0, \dots, d_i, \dots, 0)$ invariant by \mathcal{F} is totally invariant.

2.2.3 The polar divisor

Let \mathcal{F} be a foliation on $\mathbf{P}^{(n_1, \dots, n_r)}$, where $n = n_1 + \dots + n_r$, of multidegree (d_1, \dots, d_r) and with singular set $\text{Sing}(\mathcal{F})$ of codimension at least 2. Consider a pencil of hyperplanes $\mathcal{H}^i = \{H_\lambda^i\}_{\lambda \in \mathbb{P}^1}$, with base locus $\bigcap_{\lambda \in \mathbb{P}^1} H_\lambda^i = \mathbb{L}_i^{n-2}$, where \mathbb{L}_i^{n-2} is a linear subspace of dimension $n - 2$ which is not contained in $\text{Sing}(\mathcal{F})$. The *polar divisor* of \mathcal{F} with respect \mathcal{H}^i is

$$\mathcal{D}_{\mathcal{H}^i} = \bigcup_{\lambda \in \mathbb{P}^1} \mathcal{T}(H_\lambda^i, \mathcal{F}).$$

Lemma 2.2.2. $\mathcal{D}_{\mathcal{H}^i}$ is either $\mathbf{P}^{(n_1, \dots, n_r)}$ or a hypersurface of multidegree

$$(d_1 - 1, \dots, d_i + 1, \dots, d_r - 1).$$

Proof. If all hyperplanes of the pencil \mathcal{H}^i are \mathcal{F} -invariant then $\mathcal{T}(H_\lambda^i, \mathcal{F}) = H_\lambda^i$, for all $\lambda \in \mathbb{P}_\mathbb{C}^1$, then $\mathcal{D}_{\mathcal{H}^i} = \bigcup_{\lambda \in \mathbb{P}_\mathbb{C}^1} \mathcal{T}(H_\lambda^i, \mathcal{F}) = \bigcup_{\lambda \in \mathbb{P}^1} H_\lambda^i = \mathbf{P}^{(n_1, \dots, n_r)}$. On the other hand, if there exists a hyperplane $H_\lambda^i \in \mathcal{H}^i$ that is not \mathcal{F} -invariant, we can set it to be the hyperplane at infinity with respect to the factor $\mathbb{P}_\mathbb{C}^{n_i}$. Thus we choose coordinates in $\mathbf{P}^{(n_1, \dots, n_r)}$ such that \mathcal{H}^i is given, in affine coordinates, by

$$x_{n_i} - \mu = 0,$$

with $\mu \in \mathbb{C}$. It follows from 2.2.1 that in this coordinate system the vector field inducing \mathcal{F} has an expression of the form

$$X_{\mathcal{F}} = \sum_{j=1}^r (P_j(X_1, \dots, X_r) + g_j(X_1, \dots, X_r) R_j),$$

where $X_i = (x_{i_1}, \dots, x_{n_i})$ and $R_j = \sum_{k=1}^{n_j} x_{jk} \frac{\partial}{\partial x_{jk}}$ is the radial vector field on \mathbb{C}^{n_i} , $j = 1, \dots, r$, $g_j(X_1, \dots, X_r)$ is a multihomogeneous polynomial of multidegree $(d_1 - 1, \dots, d_j, \dots, d_r - 1)$ and $P_j(X_1, \dots, X_r) = \sum_{k=1}^{n_j} P_{jk}(X_1, \dots, X_r) \frac{\partial}{\partial x_{jk}}$, satisfying $\deg_{X_j}(P_{jk}(X_1, \dots, X_r)) \leq d_j$, for all $j = 1, \dots, n_i$ and $k = 1, \dots, r$. Moreover $g_i(X_1, \dots, X_r) \neq 0$. Then the polar divisor $\mathcal{D}_{\mathcal{H}^i}$ is given by

$$\mathcal{P}(\mathcal{F}, \mathcal{H}^i) = \overline{\{(x_{n_i} g_i + P_{in_i}) = 0\}}.$$

Note that $\deg(\mathcal{P}(\mathcal{F}, \mathcal{H}^i)) = (d_1 - 1, \dots, d_i + 1, \dots, d_r - 1)$. \square

Let \mathcal{V} be a smooth hypersurface of multidegree (k_1, \dots, k_r) given by the zeros of a multihomogeneous polynomial $f \in \mathbb{C}[Z_1, \dots, Z_r]$. Consider the algebraic subset of \mathcal{V} given by

$$\text{Sing}(\mathcal{V})_i = \left\{ q \in \mathcal{V}; \frac{\partial f}{\partial z_{i_0}}(q) = \dots = \frac{\partial f}{\partial z_{n_i}}(q) = 0 \right\}.$$

If $\text{Sing}(\mathcal{V})_i = \emptyset$, then we define the i -th embedded tangent space of \mathcal{V} at p given, in homogeneous coordinates, by

$$\mathbf{T}_p^i \mathcal{V} = \left\{ [Z_1, \dots, Z_r] \in \mathbf{P}^{(n_1, \dots, n_r)}; \sum_{j=0}^{n_i} z_{i_j} \frac{\partial f}{\partial z_{i_j}}(p) = 0 \right\}.$$

Remark 2.2.4. Observe that \mathcal{V} is i -invariant if, and only if, $p \in \mathcal{T}(\mathbf{T}_p^i \mathcal{V}, \mathcal{F})$ for all $p \in \mathcal{V}$.

We fix a flag, with respect to factor i , of linear subspaces on $\mathbf{P}^{(n_1, \dots, n_r)}$

$$\mathcal{F}\ell(i) : \mathbb{L}_i^{n-k} \subset \mathbb{L}_i^{n-k+1} \subset \dots \subset \mathbb{L}_i^{n-2} \subset \mathbf{P}^{(n_1, \dots, n_r)},$$

where $\text{codim}_{\mathbb{C}}(\mathbb{L}_i^{n-k}) = k$. The k -th polar variety of \mathcal{V} in the factor i , with respect to $\mathcal{F}\ell(i)$ is given by

$$\mathcal{P}_{n-k}^i(\mathcal{V}) = \{p \in \mathcal{V}; \mathbf{T}_p^i \mathcal{V} \supset \mathbb{L}_i^{n-k-1}\}.$$

Observe that

$$\mathcal{P}_{n-1}^i(\mathcal{V}) \subset \mathcal{P}_{n-2}^i(\mathcal{V}) \subset \dots \subset \mathcal{P}_{n-n_i+1}^i(\mathcal{V}) \subset \mathcal{V}.$$

Now, consider the flag $\mathcal{F}\ell(i) : \mathbb{L}_i^{n-k} \subset \mathbb{L}_i^{n-k+1} \subset \dots \subset \mathbb{L}_i^{n-2} \subset \mathbf{P}^{(n_1, \dots, n_r)}$, where $\mathbb{L}_i^{n-k} = \{z_{i_0} = \dots = z_{i_{k-1}} = 0\}$. We conclude that

$$\mathcal{P}_{n-k}^i(\mathcal{V}) = \left\{ p \in \mathcal{V}; \frac{\partial f}{\partial z_{i_k}}(p) = \dots = \frac{\partial f}{\partial z_{n_i}}(p) = 0 \right\}.$$

Thus the class of $\mathcal{P}_{n-k}^i(\mathcal{V})$ is given by

$$[\mathcal{P}_{n-k}^i(\mathcal{V})] = \left[\sum_{j=1}^r k_j h_j \right] \cap \left[(k_i - 1)h_i + \sum_{\substack{j=1 \\ j \neq i}}^r k_j h_j \right]^{\cap(n_i-k+1)}.$$

Lemma 2.2.3. *Let \mathcal{V} be a hypersurface i -invariant by a foliation \mathcal{F} , such that $\text{Sing}(\mathcal{V})_i = \emptyset$. Consider a pencil of hyperplanes $\mathcal{H}_r^i = \{H_\lambda^i\}_{\lambda \in \mathbb{P}^1}$, with base locus $\bigcap_{\lambda \in \mathbb{P}^1} H_\lambda^i = \mathbb{L}_i^{n-2}$. Then*

$$\mathcal{P}_{n-1}^i(\mathcal{V}) \subset \mathcal{D}_{\mathcal{H}^i} \text{ and } \mathcal{V} \not\subset \mathcal{D}_{\mathcal{H}^i}.$$

Proof. If $p \in \mathcal{P}_{n-1}^i(\mathcal{V})$, then $\mathbb{L}_i^{n-2} \subset \mathbf{T}_p^i \mathcal{V}$ and this implies that $\mathbf{T}_p^i \mathcal{V} = H_\lambda^i$ for some $\lambda \in \mathbb{P}^1$. On the other hand, since \mathcal{V} is i -invariant $p \in \mathcal{T}(H_\lambda^i, \mathcal{F}) \subset \mathcal{D}_{\mathcal{H}^i}$, so $\mathcal{P}_{n-1}^i(\mathcal{V}) \subset \mathcal{D}_{\mathcal{H}^i}$. \square

Theorem 2.2.4. *Let \mathcal{V} be a hypersurface of multidegree (k_1, \dots, k_r) , with $k_i > 1$ and $\text{Sing}(\mathcal{V})_i = \emptyset$. If \mathcal{V} is i -invariant by a foliation \mathcal{F} on $\mathbf{P}^{(n_1, \dots, n_r)}$ of multidegree (d_1, \dots, d_r) , then*

$$k_i \leq d_i + 2.$$

Proof. Consider the cycle

$$\mathcal{S}_k^i = h_1^{n_1} \cdots h_{i-1}^{n_{i-1}} h_i^{k-2} h_{i+1}^{n_{i+1}} \cdots h_r^{n_r} \in \mathcal{A}_{n_1 + \cdots + n_{i-1} + k - 2 + n_{i+1} + \cdots + n_r}(\mathbf{P}^{(n_1, \dots, n_r)}).$$

Since $\mathcal{P}_{n-1}^i(\mathcal{V}) \subset \mathcal{D}_{\mathcal{H}^i}$ and $\mathcal{V} \not\subset \mathcal{D}_{\mathcal{H}^i}$, we can conclude that there exists k such that $\mathcal{P}_{n-k}^i(\mathcal{V}) \subset \mathcal{D}_{\mathcal{H}^i}$ and $\mathcal{P}_{n-k-1}^i(\mathcal{V}) \not\subset \mathcal{D}_{\mathcal{H}^i}$, thus $\mathcal{P}_{n-k}^i(\mathcal{V}) \subseteq \mathcal{P}_{n-k-1}^i(\mathcal{V}) \cap \mathcal{D}_{\mathcal{H}^i}$. Then

$$[\mathcal{P}_{n-k}^i(\mathcal{V})] \cap \mathcal{S}_k^i \leq [\mathcal{P}_{n-k-1}^i(\mathcal{V})] \cap [\mathcal{D}_{\mathcal{H}^i}] \cap \mathcal{S}_k^i \in \mathcal{A}_0(\mathbf{P}^{(n_1, \dots, n_r)}) \simeq \mathbb{Z}. \quad (2.14)$$

We have that

$$[\mathcal{P}_{n-k}^i(\mathcal{V})] = \sum_{\ell=1}^r \sum_{s=0}^{n_i-k+1} \binom{n_i-k+1}{s} k_\ell (k_i - 1)^{n_i-k+1-s} h_i^{n_i-k+1-s} h_\ell \left(\sum_{\substack{j=1 \\ j \neq i}}^r k_j h_j \right)^s$$

and

$$\begin{aligned} [\mathcal{P}_{n-k}^i(\mathcal{V})] \cap [\mathcal{D}_{\mathcal{H}^i}] &= \sum_{\ell=1}^r \sum_{s=0}^{n_i-k} \binom{n_i-k}{s} k_\ell (d_i+1) (k_i-1)^{n_i-k-s} h_i^{n_i-k+1-s} h_\ell \left(\sum_{\substack{j=1 \\ j \neq i}}^r k_j h_j \right)^s + \\ &+ \sum_{\substack{t=1 \\ t \neq i}}^r \sum_{\ell=1}^r \sum_{s=0}^{n_i-k} \binom{n_i-k+1}{s} k_\ell (d_t-1) (k_i-1)^{n_i-k-s} h_i^{n_i-k-s} h_t h_\ell \left(\sum_{\substack{j=1 \\ j \neq i}}^r k_j h_j \right)^s. \end{aligned}$$

Therefore

$$[\mathcal{P}_{n-k}^i(\mathcal{V})] \cap \mathcal{S}_k^i = k_i(k_i - 1)^{n_i-k+1} h_1^{n_1} \cdots h_r^{n_r} = k_i(k_i - 1)^{n_i-k+1}$$

and

$$[\mathcal{P}_{n_i-k-1}^i(\mathcal{V})] \cap \mathcal{D}_{\mathcal{H}^i} \cap \mathcal{S}_k^i = k_i(k_i - 1)^{n_i-k}(d_i + 1) h_1^{n_1} \cdots h_r^{n_r} = k_i(k_i - 1)^{n_i-k}(d_i + 1).$$

Now, using the inequality (2.14) we obtain $k_i(k_i - 1) \leq k_i(d_i + 1)$, and this implies that

$$k_i \leq d_i + 2.$$

□

Corollary 2.2.3. *Let \mathcal{V} be a hypersurface with $\text{Sing}(\mathcal{V})_i = \emptyset$ for all $i = 1, \dots, r$. If \mathcal{V} is totally invariant by a foliation \mathcal{F} on $\mathbf{P}^{(n_1, \dots, n_r)}$, then*

$$\deg(\mathcal{V}) \leq \deg(\mathcal{F}) + 2 \deg(\mathbf{P}^{(n_1, \dots, n_r)}),$$

where the degree is with respect to the Segre embedding.

Proof. Multiplying each inequality $k_i \leq d_i + 2$ by $\binom{n-1}{n_1, \dots, n_i-1, \dots, n_r}$ and summing in i we have

$$\deg(\mathcal{V}) = \sum_{i=1}^r k_i \binom{n-1}{n_1, \dots, n_i-1, \dots, n_r} \leq \sum_{i=1}^r (d_i + 2) \binom{n-1}{n_1, \dots, n_i-1, \dots, n_r}.$$

The result follows from proposition 2.2.3 and $\deg(\mathbf{P}^{(n_1, \dots, n_r)}) = \frac{(n_1 + \dots + n_r)!}{n_1! \cdots n_r!} = \sum_{i=1}^r \binom{n-1}{n_1, \dots, n_i-1, \dots, n_r}$. □

2.3 Weighted projective foliations

In this section we study Poincaré's problem for foliations on weighted projective spaces $\mathbb{P}_{\mathbb{C}}(\varpi_0, \dots, \varpi_n)$.

Weighted projective space $\mathbb{P}(\varpi_0, \dots, \varpi_n)$

Let $\varpi_0, \dots, \varpi_n$ be integers ≥ 1 pairwise coprime. Consider the \mathbb{C}^* -action on $\mathbb{C}^{n+1} \setminus \{0\}$ given by

$$\lambda \cdot (z_0, \dots, z_n) = (\lambda^{\varpi_0} z_0, \dots, \lambda^{\varpi_n} z_n),$$

where $\lambda \in \mathbb{C}^*$ and $(z_0, \dots, z_n) \in \mathbb{C}^{n+1} \setminus \{0\}$. The quotient space $\mathbb{P}(\varpi_0, \dots, \varpi_n) = (\mathbb{C}^{n+1} \setminus \{0\} / \sim)$, induced by the action above is the *weighted projective space of type* $(\varpi_0, \dots, \varpi_n)$. We will abbreviate $\mathbb{P}(\varpi_0, \dots, \varpi_n) := \mathbb{P}(\varpi)$.

Consider the open $\mathcal{U}_i = \{[z_0 : \cdots : z_n] \in \mathbb{P}(\varpi_0, \dots, \varpi_n); z_i \neq 0\} \subset \mathbb{P}(\varpi_0, \dots, \varpi_n)$, with $i = 0, 1, \dots, n$. Let $\mu_{\varpi_i} \subset \mathbb{C}^*$ be the subgroup of ϖ_i -th roots of unity. We can define the homeomorphisms $\phi_i : \mathcal{U}_i \longrightarrow \mathbb{C}^n / \mu_{\varpi_i}$, by

$$\phi_i([z_0 : \cdots : z_n]) = \left(\frac{z_0}{z_i^{\varpi_0/\varpi_i}}, \dots, \frac{\widehat{z}_i}{z_i}, \dots, \frac{z_n}{z_i^{\varpi_n/\varpi_i}} \right)_{\varpi_i}$$

where the symbol " $\widehat{}$ " means omission and $(\cdot)_{\varpi_i}$ is a ϖ_i -conjugacy class in $\mathbb{C}^n / \mu_{\varpi_i}$ with μ_{ϖ_i} acting on \mathbb{C}^n by

$$\lambda \cdot (z_0, \dots, z_n) = (\lambda^{\varpi_0} z_0, \dots, \widehat{z}_i, \dots, \lambda^{\varpi_n} z_n), \lambda \in \mu_{\varpi_i}.$$

On $\phi_i(\mathcal{U}_i \cap \mathcal{U}_j) \subset \mathbb{C}^n / \mu_{\varpi_i}$ we have the transitions maps

$$\phi_i \circ \phi_j^{-1}((z_1, \dots, z_n)_{\varpi_i}) = \left(\frac{z_0}{z_j^{\varpi_0/\varpi_j}}, \dots, \frac{\widehat{z}_j}{z_j}, \dots, \frac{1}{z_j^{\varpi_i/\varpi_j}}, \dots, \frac{z_n}{z_j^{\varpi_n/\varpi_j}} \right)_{\varpi_j}$$

We conclude that $\{\phi_i, \mathcal{U}_i\}_{i=0}^n$ is a holomorphic orbifold atlas for $\mathbb{P}(\varpi_0, \dots, \varpi_n)$. Also, $\{\mathbb{C}^n, \mu_{\varpi_i}, \pi \circ \phi_i\}_{i=0}^n$ is an n -dimensional uniformizing system for $\mathbb{P}(\varpi_0, \dots, \varpi_n)$.

Since $\varpi_0, \dots, \varpi_n$ are pairwise coprime, the singular set of $\mathbb{P}(\varpi_0, \dots, \varpi_n)$ is the set of $n + 1$ points

$$[1, 0, \dots, 0], [0, 1, \dots, 0], \dots, [0, \dots, 0, 1].$$

There exists another orbifold structure for $\mathbb{P}(\varpi_0, \dots, \varpi_n)$. This is induced by the action of the group $(\mu_{\varpi_0} \times \cdots \times \mu_{\varpi_n})$ on \mathbb{P}^n given by

$$\begin{aligned} (\mu_{\varpi_0} \times \cdots \times \mu_{\varpi_n}) \times \mathbb{P}^n &\longrightarrow \mathbb{P}^n \\ ((\lambda_0, \dots, \lambda_n), [z_0, \dots, z_n]) &\longmapsto [\lambda_0 z_0, \dots, \lambda_n z_n]. \end{aligned}$$

Now consider the map $\varphi : \mathbb{P}^n \longrightarrow \mathbb{P}(\varpi_0, \dots, \varpi_n)$ defined by $\varphi([z_0, \dots, z_n]) = [z_0^{\varpi_0}, \dots, z_n^{\varpi_n}]$. φ induces a homeomorphism

$$\widetilde{\varphi} : \mathbb{P}^n / (\mu_{\varpi_0} \times \cdots \times \mu_{\varpi_n}) \longrightarrow \mathbb{P}(\varpi_0, \dots, \varpi_n)$$

and $\mathbb{P}(\varpi_0, \dots, \varpi_n) \simeq \mathbb{P}^n / (\mu_{\varpi_0} \times \cdots \times \mu_{\varpi_n})$ is an orbifold structure given as a global quotient.

Remark 2.3.1. The degree of the map $\varphi : \mathbb{P}^n \longrightarrow \mathbb{P}(\varpi)$ is equal the order of the group $(\mu_{\varpi_0} \times \cdots \times \mu_{\varpi_n})$, i.e, $\deg(\varphi) = \varpi_0 \cdots \varpi_n$. For details see [1].

\mathbb{Q} -line bundles on $\mathbb{P}(\varpi_0, \dots, \varpi_n)$

Let $\frac{d}{r} \in \mathbb{Q}$, with $\gcd(r, d) = 1$ and $r > 0$. Consider the \mathbb{C}^* -action $\zeta_{(\frac{d}{r})}$ on $\mathbb{C}^{n+1} \setminus \{0\} \times \mathbb{C}$ given by

$$\begin{aligned} \zeta_{(\frac{d}{r})} : \mathbb{C}^* \times \mathbb{C}^{n+1} \setminus \{0\} \times \mathbb{C} &\longrightarrow \mathbb{C}^{n+1} \setminus \{0\} \times \mathbb{C} \\ (\lambda, (z_0, \dots, z_n), t) &\longmapsto ((\lambda^{r\varpi_0} z_0, \dots, \lambda^{r\varpi_n} z_n), \lambda^{-d} t). \end{aligned}$$

We denote the quotient space induced by the action $\zeta_{(\frac{d}{r})}$ by

$$\mathcal{O}_{\mathbb{P}(\varpi)}(d/r) := (\mathbb{C}^{n+1} \setminus \{0\} \times \mathbb{C}) / \sim \zeta_{(\frac{d}{r})}.$$

The space $\mathcal{O}_{\mathbb{P}(\varpi)}(d/r)$ is a line orbifold on $\mathbb{P}(\varpi)$. We shall describe the global holomorphic sections of $\mathcal{O}_{\mathbb{P}(\varpi)}(d/r)$, for $d > 0$.

Proposition 2.3.1. *Let $\mathbb{P}(\varpi) := \mathbb{P}(\varpi_0, \dots, \varpi_n)$, then*

$$H^0(\mathbb{P}(\varpi), \mathcal{O}_{\mathbb{P}(\varpi)}(d/r)) = \bigoplus_{\varpi_0 k_0 + \dots + \varpi_n k_n = \frac{d}{r}} \mathbb{C} \cdot (z_0^{k_1} \dots z_n^{k_n}).$$

Proof. A global section of this line orbifold is a linear combination of the monomials $z^k = z_0^{k_1} \dots z_n^{k_n}$, invariant by the action $\zeta_{(\frac{d}{r})}$, that is, $\zeta_{(\frac{d}{r})}([z, z^k]) = [z, z^k]$. Using this action we obtain

$$\begin{aligned} [(z_0 \dots, z_n), (z_0^{k_1} \dots z_n^{k_n})] &= [(\lambda^{r\varpi_0} z_0, \dots, \lambda^{r\varpi_n} z_n), \lambda^{\sum_{i=0}^n r\varpi_i k_i} (z_0^{k_1} \dots z_n^{k_n})] \\ &= [(z_0 \dots, z_n), \lambda^{-d + \sum_{i=0}^n r\varpi_i k_i} (z_0^{k_1} \dots z_n^{k_n})]. \end{aligned}$$

Therefore $\sum_{i=0}^n r\varpi_i k_i = d$, hence the proposition follows. \square

The orbifolds $\mathcal{O}_{\mathbb{P}(\varpi)}(d/r)$ can therefore be considered as elements of the rational Picard group of $\mathbb{P}(\varpi)$, that is, as \mathbb{Q} -line bundles. It is possible to show that the \mathbb{Q} -Picard group is generated by $\mathcal{O}_{\mathbb{P}(\varpi)}(1)$, that is

$$\text{Pic}(\mathbb{P}(\varpi)) \otimes \mathbb{Q} := \text{Pic}(\mathbb{P}(\varpi))_{\mathbb{Q}} = \mathbb{Q} \cdot \mathcal{O}_{\mathbb{P}(\varpi)}(1).$$

Remark 2.3.2. It is possible to show that $\mathcal{O}_{\mathbb{P}(\varpi)}(1) = \varphi^*(\mathcal{O}_{\mathbb{P}^n}(1))$, where $\mathcal{O}_{\mathbb{P}^n}(1)$ is the hyperplane bundle on \mathbb{P}^n , see [58].

The Euler sequence on $\mathbb{P}(\varpi)$ reads

$$0 \longrightarrow \underline{\mathbb{C}} \xrightarrow{\varsigma} \bigoplus_{i=0}^n \mathcal{O}_{\mathbb{P}(\varpi)}(\varpi_i) \longrightarrow T\mathbb{P}(\varpi) \longrightarrow 0$$

where $\underline{\mathbb{C}}$ is the trivial line orbifold on $\mathbb{P}(\varpi)$. The map ς is given by $\varsigma(1) = (\varpi_0 z_0, \dots, \varpi_n z_n)$, see [58].

Definition 2.3.1. Let X be a n -dimensional compact complex orbifold with uniformizing system $\{\mathcal{U}_i, G_i, \pi_i\}_{i \in \Lambda}$ and $\omega \in \Omega_X^n$ a n -form. The orbifold integral of ω over X is defined by

$$\int_X^{orb} \omega = \sum_{i \in \Lambda} \frac{1}{|G_i|} \int_{\mathcal{U}_i} \pi_i^* \omega,$$

where $|G_i|$ is the order of the group G_i , see [2].

Remark 2.3.3. Let $Ker(X) = \{g \in \prod_{i \in \Lambda} G_i; g(x) = x, \forall x \in X\}$ and $X_{reg} = X \setminus \text{Sing}(X)$. Then

$$\int_X^{orb} \omega = \frac{1}{\#Ker(X)} \int_{X_{reg}} \omega,$$

see [58].

Proposition 2.3.2. [58] Let $\mathcal{O}_{\mathbb{P}(\varpi)}(1)$ be the hyperplane bundle on $\mathbb{P}(\varpi)$. Then

$$\int_{\mathbb{P}(\varpi)}^{orb} c_1(\mathcal{O}_{\mathbb{P}(\varpi)}(1))^n = \frac{1}{\varpi_0 \dots \varpi_n},$$

Proof. From the definition of orbifold integral we have

$$\int_{\mathbb{P}(\varpi)}^{orb} c_1(\mathcal{O}_{\mathbb{P}(\varpi)}(1))^n = \frac{1}{\#Ker(\mathbb{P}(\varpi))} \int_{\mathbb{P}(\varpi)_{reg}} c_1(\mathcal{O}_{\mathbb{P}(\varpi)}(1))^n.$$

Since $\mathbb{P}(\varpi_0, \dots, \varpi_n) \simeq \mathbb{P}^n / (\mu_{\varpi_0} \times \dots \times \mu_{\varpi_n})$ we conclude that

$$Ker(\mathbb{P}(\varpi)) = \bigcap_{i=0}^n \mu_{\varpi_i} = \{1\},$$

hence $\#Ker(\mathbb{P}(\varpi)) = 1$. On the other hand, since $\varphi^*(\mathcal{O}_{\mathbb{P}(\varpi)}(1)) = \mathcal{O}_{\mathbb{P}^n}(1)$ we get

$$\int_{\mathbb{P}(\varpi)}^{orb} c_1(\mathcal{O}_{\mathbb{P}(\varpi)}(1))^n = \int_{\mathbb{P}(\varpi)_{reg}} c_1(\mathcal{O}_{\mathbb{P}(\varpi)}(1))^n = \frac{1}{\deg(\varphi)} \int_{\mathbb{P}^n} c_1(\mathcal{O}_{\mathbb{P}^n}(1))^n = \frac{1}{\varpi_0 \dots \varpi_n}.$$

□

ℚ-line bundles on simplicial toric varieties and intersection numbers

Let X be a normal toric variety. Since a Weil divisor is a cycle in X of real dimension $2n - 2$, we have a homomorphism

$$\vartheta : \mathcal{W}(X) \longrightarrow H_{2n-2}(X, \mathbb{Z})$$

which associates to each Weil divisor its homology class. On the other hand, there exists (see [47]) an isomorphism

$$\alpha : \mathcal{C}(X) \xrightarrow{\cong} \text{Pic}(X)$$

between the group of classes of Cartier divisors and the Picard group. This latter is the group of isomorphism classes of line bundles (or isomorphism classes of invertible sheaves) on X . By composition of α with the morphism $c_1 : \text{Pic}(X) \rightarrow H^2(X, \mathbb{Z})$, we obtain a morphism

$$c_1 : \mathcal{C}(X) \longrightarrow H^2(X, \mathbb{Z}).$$

When X is smooth we have that $c_1(\mathcal{O}(D))$ is the Poincaré dual of the cycle represented by $D \in \mathcal{C}(X)$. In the general case, we cannot guarantee this, but we will see that it is true if D is an invariant divisor by a torus action.

Let \mathbb{T} be the torus which acts in X . Denote by $\mathcal{C}^{\mathbb{T}}(X)$ and $\mathcal{W}^{\mathbb{T}}(X)$, respectively, the groups of \mathbb{T} -invariant divisors of Cartier and Weil, modulo equivalence of principal \mathbb{T} -invariant divisors .

Theorem 2.3.1. [7] *Let X be a compact toric variety. There exists a commutative diagram*

$$\begin{array}{ccc} \mathcal{C}^{\mathbb{T}}(X) & \hookrightarrow & \mathcal{W}^{\mathbb{T}}(X) \\ \downarrow \simeq & & \downarrow \simeq \\ H^2(X, \mathbb{Z}) & \xrightarrow{\simeq^{[X]}} & H_{2n-2}(X, \mathbb{Z}) \end{array}$$

where the vertical isomorphisms correspond to the morphisms c_1 and ϑ .

When X is simplicial we have $\text{Pic}(X) \otimes \mathbb{Q} \simeq \mathcal{C}^{\mathbb{T}}(X) \otimes \mathbb{Q} = \mathcal{W}^{\mathbb{T}}(X) \otimes \mathbb{Q}$. Using these identifications and tensorizing the diagram of the theorem 2.3.1 by \mathbb{Q} we have

$$\begin{array}{ccc} \text{Pic}(X) \otimes \mathbb{Q} & \xrightarrow{\simeq} & \mathcal{C}^{\mathbb{T}}(X) \otimes \mathbb{Q} \\ \downarrow \simeq & & \downarrow \simeq \\ H^2(X, \mathbb{Q}) & \xrightarrow{\simeq^{[X]}} & H_{2n-2}(X, \mathbb{Q}) \end{array}$$

Let X be a two dimensional complete simplicial toric variety and let D be a \mathbb{Q} -Cartier divisor on X . Then from theorem 2.3.1 we conclude that $c_1(\mathcal{O}(D))$ is the Poincaré dual of the cycle represented by D . Therefore, we have the intersection numbers with rational coefficients. For instance, let $D_1, D_2 \in \mathcal{W}^{\mathbb{T}}(X)$ the intersection number is the rational number

$$D_1 \cdot D_2 = \langle c_1(\mathcal{O}(D_1)) \cap c_1(\mathcal{O}(D_2)), [X] \rangle \in \mathbb{Q},$$

as in the case with integer coefficients.

We will use the Poincaré-Satake duality to express the number of intersection in terms of the orbifold integral.

Proposition 2.3.3. *Let X be a simplicial compact toric variety and $L_1, L_2 \in \text{Pic}(X) \otimes \mathbb{Q}$. Then*

$$L_1 \cdot L_2 = \int_X^{orb} c_1(L_1) \wedge c_1(L_2).$$

Proof. Since X be a simplicial compact toric variety, it follows from theorem 2.1.2 that X is a compact complex orbifold. Let $H^i(X)$ be the cohomology group of i -forms on X (in orbifold's sense). We have the following Poincaré duality for orbifolds showed by Satake in [74]:

$$\begin{aligned} H^i(X) \otimes H^{n-i}(X) &\longrightarrow \mathbb{Q} \\ \alpha \wedge \eta &\longmapsto \int_X^{orb} \alpha \wedge \eta. \end{aligned}$$

From this we get

$$L_1 \cdot L_2 = \langle c_1(L_1) \cap c_1(L_2), [X] \rangle = \int_X^{orb} c_1(L_1) \wedge c_1(L_2).$$

□

Therefore, if $D_1, D_2 \in \mathcal{W}^{\mathbb{T}}(X)$ we have

$$D_1 \cdot D_2 = \int_X^{orb} c_1(\mathcal{O}(D_1)) \wedge c_1(\mathcal{O}(D_2)).$$

Example 2.3.1. Let $D_1 \in H^0(\mathbb{P}(\varpi_0, \varpi_1, \varpi_2), \mathcal{O}(d_1))$ and $D_2 \in H^0(\mathbb{P}(\varpi_0, \varpi_1, \varpi_2), \mathcal{O}(d_2))$. It follows from propositions 2.3.2 and 2.3.3 that

$$D_1 \cdot D_2 = \int_X^{orb} c_1(\mathcal{O}(d_1)) \wedge c_1(\mathcal{O}(d_2)) = \int_X^{orb} (d_1 d_2) \cdot c_1(\mathcal{O}(1))^2 = \frac{d_1 d_2}{\varpi_0 \varpi_1 \varpi_2}.$$

2.3.1 Poincaré's problem for quasi-homogeneous foliations

In this section we consider the question of bounding the degree of curves which are invariant by a holomorphic foliation of a given degree on a well-formed weighted projective plane.

Foliations on weighted projective planes

Let us to denote $\mathbb{P}(\varpi) := \mathbb{P}(w_0, w_1, w_2)$. The Chern-Weil theory of Chern classes holds as well in $\mathbb{P}(\varpi)$ as in projective spaces, see [58]. Denoting by $\zeta = c_1(\mathcal{O}_{\mathbb{P}(\varpi)}(1))$ we have, from Euler sequence,

$$c(T\mathbb{P}(\varpi)) = (1 + w_0 \zeta)(1 + w_1 \zeta)(1 + w_2 \zeta) \tag{2.15}$$

and hence

$$c_i(T\mathbb{P}(\varpi)) = \sigma_i(w_0, w_1, w_2) \quad (2.16)$$

where σ_i is the i -th elementary symmetric function.

Now, let X be a quasi-homogeneous vector field of type (w_0, w_1, w_2) and degree d in \mathbb{C}^3 , that is, writing $X = \sum_{i=0}^2 P_i(z) \frac{\partial}{\partial z_i}$ we have that $P_i(\lambda^{w_0} z_0, \lambda^{w_1} z_1, \lambda^{w_2} z_2) = \lambda^{d+w_i-1} P_i(z_0, z_1, z_2)$. These descend well to $\mathbb{P}(\varpi)$. In fact, we may tensorize the Euler sequence by $\mathcal{O}_{\mathbb{P}(\varpi)}(d-1)$ to get

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}(\varpi)}(d-1) \longrightarrow \bigoplus_{i=0}^2 \mathcal{O}_{\mathbb{P}(\varpi)}(d+w_i-1) \longrightarrow T\mathbb{P}(\varpi) \otimes \mathcal{O}_{\mathbb{P}(\varpi)}(d-1) \longrightarrow 0. \quad (2.17)$$

It follows that a quasi-homogeneous vector field X induces a foliation \mathcal{F} of $\mathbb{P}(\varpi)$ and that $gR_w + X$ defines the same foliation as X , where R_w is the adapted radial vector field $R_w = w_0 z_0 \frac{\partial}{\partial z_0} + w_1 z_1 \frac{\partial}{\partial z_1} + w_2 z_2 \frac{\partial}{\partial z_2}$, with g a quasi-homogeneous polynomial of type (w_0, w_1, w_2) and degree $d-1$.

Dually, noting that $|w| = w_0 + w_1 + w_2$, we have the exact sequence

$$0 \rightarrow \Omega_{\mathbb{P}(\varpi)}^1 \otimes \mathcal{O}_{\mathbb{P}(\varpi)}(d+|w|-1) \rightarrow \bigoplus_{i=0}^2 \mathcal{O}_{\mathbb{P}(\varpi)}(d+|w|-w_i-1) \rightarrow \mathcal{O}_{\mathbb{P}(\varpi)}(d+|w|-1) \rightarrow 0. \quad (2.18)$$

Hence, a foliation \mathcal{F} of $\mathbb{P}(\varpi)$ is also induced by a 1-form $\eta = A_0 dz_0 + A_1 dz_1 + A_2 dz_2$, with A_i a quasi-homogeneous polynomial of type (w_0, w_1, w_2) , degree $d+|w|-w_i-1$ and $\iota_{R_w} \eta = w_0 z_0 A_0 + w_1 z_1 A_1 + w_2 z_2 A_2 \equiv 0$.

Example 2.3.2. (logarithmic foliations)

Let f_1, \dots, f_k quasi-homogeneous polynomial of type $(\varpi_0, \varpi_1, \varpi_2)$ and degrees d_1, \dots, d_k , respectively, with $k \geq 3$. Let $\lambda_1, \dots, \lambda_k \in \mathbb{C}^*$ be such that $\sum_{i=1}^k \lambda_i d_i = 0$. Define the 1-form

$$\eta = (f_1 \cdots f_k) \cdot \sum_{i=1}^k \lambda_i \frac{df_i}{f_i}.$$

By Euler's formula, $\iota_{\mathcal{R}_\varpi}(\eta) = (f_1 \cdots f_k) \cdot \left(\sum_{i=1}^k \lambda_i d_i \right) = 0$. Therefore, η define a foliation on $\mathbb{P}(\varpi)$ of degree $\sum_{i=1}^k d_i - |\varpi| + 1$.

From now on we shall assume that

$$\text{Sing}(\mathcal{F}) \cap \text{Sing}(\mathbb{P}(\varpi)) = \emptyset. \quad (2.19)$$

This assumption is fairly generic in that it requires X , or η , not to have zeros along the coordinate axes of \mathbb{C}^3 and it assures us that the leaves of \mathcal{F} are orbifolds.

We proceed now to define the ‘‘degree’’ of such a foliation. Recall that, in the usual projective situation, $\text{deg } \mathcal{F}$ is the degree of the variety of tangencies of \mathcal{F} with a generic hyperplane.

An analogous geometric interpretation holds in the weighted situation and we similarly have the corresponding canonical \mathbb{Q} -bundles $K_{\mathbb{P}(\varpi)}$, $K_{\mathcal{F}}$ and the \mathbb{Q} -bundles $T_{\mathcal{F}}$, $N_{\mathcal{F}}$, $N_{\mathcal{F}}^*$, all lying in $\text{Pic}(\mathbb{P}(\varpi)) \otimes \mathbb{Q}$. The adjunction formula

$$K_{\mathbb{P}(\varpi)} = K_{\mathcal{F}} \otimes N_{\mathcal{F}}^* \quad (2.20)$$

still holds and we point out that $K_{\mathbb{P}(\varpi)} = \mathcal{O}_{\mathbb{P}(\varpi)}(-|w|)$, $K_{\mathcal{F}} = \mathcal{O}_{\mathbb{P}(\varpi)}(d-1)$ and $N_{\mathcal{F}} = \mathcal{O}_{\mathbb{P}(\varpi)}(d+|w|-1)$.

Let C be a compact connected curve (possibly singular), whose irreducible components are not \mathcal{F} -invariant. Then, for $p \in C$, the index $\text{tang}(\mathcal{F}, C, p)$ is defined as in [12] and, writing $\text{tang}(\mathcal{F}, C) = \sum_{p \in C} \text{tang}(\mathcal{F}, C, p)$, we have that

$$\text{tang}(\mathcal{F}, C) = K_{\mathcal{F}} \cdot C + C \cdot C \geq 0. \quad (2.21)$$

We define the degree of \mathcal{F} just as in the usual projective situation, that is,

$$\text{deg}(\mathcal{F}) := \text{tang}(\mathcal{F}, H) \quad (2.22)$$

where H is a generic element of the linear system $H^0(\mathbb{P}(\varpi), \mathcal{O}_{\mathbb{P}(\varpi)}(1))$.

Poincaré's duality holds, as shown by I. Satake (see [74] and [58]). Hence, (2.22) reads

$$\begin{aligned} \text{deg}(\mathcal{F}) &= K_{\mathcal{F}} \cdot H + H \cdot H = \int_H^{orb} c_1(\mathcal{O}_{\mathbb{P}(\varpi)}(d-1)) + \int_H^{orb} c_1(\mathcal{O}_{\mathbb{P}(\varpi)}(1)) \\ &= \int_{\mathbb{P}(\varpi)}^{orb} c_1(\mathcal{O}_{\mathbb{P}(\varpi)}(d-1)) \wedge c_1(\mathcal{O}_{\mathbb{P}(\varpi)}(1)) + \int_{\mathbb{P}(\varpi)}^{orb} c_1(\mathcal{O}_{\mathbb{P}(\varpi)}(1)) \wedge c_1(\mathcal{O}_{\mathbb{P}(\varpi)}(1)) \\ &= \frac{d-1}{w_0 w_1 w_2} + \frac{1}{w_0 w_1 w_2} = \frac{d}{w_0 w_1 w_2}. \end{aligned} \quad (2.23)$$

Now, suppose that

(i) C is a quasi-smooth curve in \mathbb{P}_w , that is, is defined by a quasi-homogeneous polynomial $P(z_0, z_1, z_2)$, of degree d^o , whose only singularity is at $0 \in \mathbb{C}^3$.

(ii) C contains no codimension 2 singular stratum of $\mathbb{P}(\varpi)$.

Then the usual adjunction formula holds (see [9]):

$$K_C = K_{\mathbb{P}(\varpi)|_C} \otimes N_C. \quad (2.24)$$

With this at hand we have, using Poincaré's duality,

$$\text{deg}(C) = \int_C^{orb} c_1(\mathcal{O}_{\mathbb{P}(\varpi)}(1)) = \int_{\mathbb{P}(\varpi)}^{orb} c_1(\mathcal{O}_{\mathbb{P}(\varpi)}(d^o)) \wedge c_1(\mathcal{O}_{\mathbb{P}(\varpi)}(1)) = \frac{d^o}{w_0 w_1 w_2}. \quad (2.25)$$

We show the following result:

Theorem 2.3.2. *Let \mathcal{F} be a singular holomorphic foliation on $\mathbb{P}(\varpi) = \mathbb{P}(w_0, w_1, w_2)$, of degree $\deg(\mathcal{F})$, C a quasi-smooth curve of degree $\deg(C)$, which avoids the singular locus of $\mathbb{P}(\varpi)$ and is invariant by \mathcal{F} . Then,*

$$\deg(C) \leq \deg(\mathcal{F}) + \frac{w_0 + w_1 + w_2 - 2}{w_0 w_1 w_2}.$$

Remark 2.3.4. This bound cannot be improved. Let $f(x, y, z) = x^{mk} + y^{mk} - z^k$ and $g(x, y, z) = ax^m + by^m + cz$, $m, k \in \mathbb{N}$. These are quasi-homogeneous polynomials of type $(1, 1, m)$ and degrees km and m , both defining quasi-smooth curves of degrees k and 1 , respectively, which avoid the singularity of $\mathbb{P}(1, 1, m)$. The 1-form $\omega = k f dg - g df$ defines a foliation \mathcal{F} on $\mathbb{P}(1, 1, m)$ of degree $\deg(\mathcal{F}) = k - 1/m$. The orbifold $C = (f = 0)$ is \mathcal{F} -invariant and

$$\deg(C) = k \leq k - \frac{1}{m} + 1 = \deg(\mathcal{F}) + 1.$$

Proof of theorem 2.3.2.

Suppose C is quasi-smooth, avoids the singularities of $\mathbb{P}(\varpi)$ and is \mathcal{F} -invariant.

The sum of the Camacho-Sad indices, $CS(\mathcal{F}, C)$, over $C \cap \text{Sing}(\mathcal{F})$ satisfies (see [11])

$$CS(\mathcal{F}, C) = \sum_{p \in C \cap \text{Sing}(\mathcal{F})} CS(\mathcal{F}, C, p) = C \cdot C \quad (2.26)$$

and, since the adjunction formula (2.24) holds, we have

$$C \cdot C = \frac{\deg(C)^2}{w_0 w_1 w_2} > 0 \quad (2.27)$$

so that $C \cap \text{Sing}(\mathcal{F}) \neq \emptyset$. On the other hand, by (2.15) and (2.24),

$$\begin{aligned} & \int_C^{\text{orb}} c_1(TC \otimes \mathcal{O}_{\mathbb{P}_w}(d-1)) \\ &= \frac{\deg(C)(w_0 + w_1 + w_2 - \deg(C))}{w_0 w_1 w_2} + \frac{(d-1)\deg(C)}{w_0 w_1 w_2} \quad (2.28) \\ &= \deg(C) \frac{w_0 + w_1 + w_2 - \deg(C) - 1 + d}{w_0 w_1 w_2}. \end{aligned}$$

Now, $\mathcal{F}|_C$ induces a non-zero holomorphic section of $TC \otimes \mathcal{O}_{\mathbb{P}_w}(d-1)$ and the number in (2.28) is the degree of this line \mathbb{Q} -bundle. Since $C \cap \text{Sing}(\mathcal{F})$ is non-empty and finite, this degree is positive and it follows that $\deg(C) \leq \deg(\mathcal{F}) + \frac{|w| - 2}{w_0 w_1 w_2}$.

Chapter 3

Bound for the sectional genus of a variety invariant by Pfaff fields

The problem of bounding the genus of an invariant curve in terms of the degree of a one-dimensional foliation on $\mathbb{P}_{\mathbb{C}}^n$ has been considered, for instance, by A. Campillo, M. Carnicer and J. García de la Fuente which, in [16], showed that if C is a reduced curve which is invariant by a one-dimensional foliation \mathcal{F} on $\mathbb{P}_{\mathbb{C}}^n$ then

$$\frac{2p_a(C) - 2}{\deg(C)} \leq \deg(\mathcal{F}) - 1 + a, \quad (3.1)$$

where $p_a(C)$ is the arithmetic genus of C and a is an integer obtained from the concrete problem of imposing singularities to projective hypersurfaces. For instance, if C has only nodal singularities then $a = 0$, and thus formula (3.1) follows from [38]. This bound has been improved by E. Esteves and S. L. Kleiman in [35].

In [34], Esteves and Kleiman extended Jouanolou's work on algebraic Pfaff equations to smooth schemes V . An algebraic Pfaff equation of rank s on a smooth scheme X of pure dimension n is, according to Jouanolou [49, pp. 136-38], a nonzero map $u : E \rightarrow \Omega_X^1$ where E is a locally free sheaf of constant rank s with $1 \leq s \leq n - 1$. Esteves and Kleiman introduced the notion of a *Pffaf field* in V , which is a nontrivial sheaf map $\eta : \Omega_V^s \rightarrow L$, where L is an invertible sheaf on V , and the integer $1 \leq s \leq n - 1$ is called the rank of η . A subvariety $X \subset V$ is said to be invariant under η if the map η factors through the natural map $\Omega_V^s|_X \rightarrow \Omega_X^s$. A Pfaff system on V induces, via exterior powers and the perfect pairing of differential forms, a Pffaf field on V . However, the converse is not true; see [34, Section 3] for more details.

In this chapter we establish upper bounds for the sectional genus of Gorenstein varieties which are invariant under Pfaff fields on \mathbb{P}_k^n , where k is an algebraically closed field of characteristic zero.

3.1 Sectional genus of a polarized variety

We work over a fixed an algebraically closed field. Let (V, L) be a Gorenstein projective variety V of dimension n equipped with a very ample line bundle L ; recall that, since V is Gorenstein, the canonical divisor K_V is a Cartier divisor.

Definition 3.1.1. *The sectional genus of V with respect to L , denoted $g(V, L)$, is defined by the formula:*

$$2g(V, L) - 2 = (K_V + (\dim(V) - 1)L) \cdot L^{\dim(V)-1}.$$

This quantity has the following geometric interpretation. Suppose that V is nonsingular, and let H_1, \dots, H_{n-1} be generic elements in the linear system $|L|$, such that $\text{Bs}|L| = \emptyset$, where $\text{Bs}|L| = \emptyset$ denote the base locus of $|L|$. By Bertini's theorem, one can assume that the curve $V_{n-1} = H_1 \cap \dots \cap H_{n-1}$ is nonsingular. Then $g(V, L)$ coincides with the geometric genus of V_{n-1} , see [39, Remark 2.5].

3.2 Pfaff fields

Definition 3.2.1. *A holomorphic Pfaff field \mathcal{F} of rank k on V is a global holomorphic section of $\bigwedge^k \Theta_V \otimes N$, where Θ_V is the tangent sheaf and N is a line bundle.*

A Pfaff field of rank k on \mathbb{P}^n is a section of $\bigwedge^k \Theta_{\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^n}(s)$, and $\deg_{\mathcal{O}_{\mathbb{P}^n}(1)}(\mathcal{F}) = s + k$ is by definition the degree of the Pfaff field \mathcal{F} . It follows from Bott's formula that $\deg(\mathcal{F}) \geq 0$. For Bott's formula see the reference [63, Chapter I, section 1.1]).

Since $\bigwedge^k \Theta_V \otimes N \simeq \mathcal{H}om(N^*, \bigwedge^k \Theta_V) \simeq \mathcal{H}om(\Omega_V^k, N)$, a Pfaff field can also be regarded as a sheaf map $\xi_{\mathcal{F}} : N^* \rightarrow \bigwedge^k \Theta_V$. The *singular set* of \mathcal{F} is given by

$$\text{Sing}(\mathcal{F}) = \{x \in V; \xi_{\mathcal{F}}(x) \text{ is not injective}\} = \{x \in V; \xi_{\mathcal{F}}^{\vee}(x) \text{ is not surjective}\}.$$

Alternatively, a holomorphic Pfaff field can also be defined as a global holomorphic section of $\Omega_V^{n-k} \otimes N'$, where $N' = N \otimes K_V^{-1}$. If V is nonsingular, this definition is equivalent to the one above.

Let $X \subset V$ be a closed subscheme of dimension larger than or equal to the rank of a holomorphic Pfaff field \mathcal{F} . Following [25, Subsection 2.2], we introduce the following definition.

Definition 3.2.2. *We say X is invariant under \mathcal{F} if there exists a morphism of sheaves $\phi : \Omega_X^1 \rightarrow N|_X$ such that the following diagram*

$$\begin{array}{ccc} \Omega_V^k|_X & \xrightarrow{\xi_{\mathcal{F}}^{\vee}|_X} & N|_X \\ \downarrow & \nearrow \phi & \\ \Omega_X^k & & \end{array}$$

commute.

Applying the functor $\mathcal{H}om(\cdot, \mathcal{O}_X)$ to the above diagram, we get the following commutative diagram:

$$\begin{array}{ccc} N^*|_X & & \\ \downarrow & \searrow \xi_{\mathcal{F}|_X} & \\ (\Omega_X^k)^\vee & \longrightarrow & \bigwedge^k \Theta_V|_X \end{array}$$

Therefore, X is invariant under \mathcal{F} if it induces a global section of $(\Omega_X^k)^\vee \otimes N|_V$.

Our first main result is the following.

Theorem 3.2.1. *Let $X \subset \mathbb{P}^n$ be a Gorenstein projective variety which is invariant under a holomorphic Pfaff field \mathcal{F} on \mathbb{P}^n whose rank is equal to the dimension of X , and such that $\text{cod}(\text{Sing}(X), X) \geq 2$. Then*

$$\frac{2g(X, \mathcal{O}_X(1)) - 2}{\text{deg}(X)} \leq \text{deg}(\mathcal{F}) - 1, \quad (3.2)$$

where $g(X, \mathcal{O}_X(1))$ is the sectional genus of X with respect to the line bundle $\mathcal{O}_X(1)$ associated to the hyperplane section.

Proof. Let $X \subset \mathbb{P}^n$ be a Gorenstein variety such that $\text{cod}(\text{Sing}(X), X) \geq 2$; let $X_0 := X - \text{Sing}(X)$. Then there exists a canonical map $\gamma_X : \Omega_X^k \rightarrow \omega_X$, where ω_X is the dualizing sheaf of X , see [26, p. 7]. Clearly, γ_X is an isomorphism away from the singular set of X , thus so is also the map

$$\tilde{\gamma}_X = \gamma_X^\vee \otimes \mathbf{1}_{\mathcal{O}_X(d-k)} : \omega_X^\vee \otimes \mathcal{O}_X(d-k) \rightarrow (\Omega_X^k)^\vee \otimes \mathcal{O}_X(d-k).$$

Since X is Gorenstein, ω_X^\vee is locally-free, hence, in particular, reflexive. From [52, Proposition 5.21], we also conclude that ω_X^\vee is normal, since $\text{cod}(\text{Sing}(X)) \geq 2$.

If X is invariant under a holomorphic Pfaff field \mathcal{F} on \mathbb{P}^n of rank k and degree d , then we have a global section $\zeta_{\mathcal{F}}$ of $(\Omega_X^k)^\vee \otimes \mathcal{O}_X(d-k)$; consider its restriction $\zeta_{\mathcal{F},0} = \zeta_{\mathcal{F}}|_{X_0}$ to X_0 . Composing it with the the inverse of $\tilde{\gamma}_X|_{X_0}$, the restriction of the map $\tilde{\gamma}_X$ to X_0 , we obtain a section

$$\tilde{\gamma}_X|_{X_0}(\zeta_{\mathcal{F},0}) \in H^0(X_0, \omega_X^\vee \otimes \mathcal{O}_X(d-k)|_{X_0}).$$

However, $\omega_X^\vee \otimes \mathcal{O}_X(d-k)|_{X_0}$ is a normal sheaf, so the above section extends to a global section of $\omega_X^\vee \otimes \mathcal{O}_X(d-k)$. In particular, $H^0(X_0, \omega_X^\vee \otimes \mathcal{O}_X(d-k)) \neq 0$, therefore

$$\text{deg}(\omega_X^\vee \otimes \mathcal{O}_X(d-k)) \geq 0. \quad (3.3)$$

Let $i : X \rightarrow \mathbb{P}^n$ be an embedding, and set, as usual, $\mathcal{O}_X(1) = i^*\mathcal{O}_{\mathbb{P}^n}(1)$. Let K_X be a Cartier divisor such that $\mathcal{O}_X(K_X) = \omega_X$.

Now, consider the following difference

$$(2g(X) - 2) - [\mathcal{O}_X(d-k) + (k-1)\mathcal{O}_X(1)] \cdot \mathcal{O}_X(1)^{k-1} =$$

$$= - (K_X^{-1} + \mathcal{O}_X(d - k)) \cdot \mathcal{O}_X(1)^{k-1} = - \deg(\omega_X^\vee \otimes \mathcal{O}_X(d - k)) \leq 0$$

It follows from (3.3) that the first expression must be less than or equal to zero, hence

$$2g(X) - 2 \leq [\mathcal{O}_X(d - k) + (k - 1)\mathcal{O}_X(1)] \cdot \mathcal{O}_X(1)^{k-1} = \deg(X)(d - 1),$$

as desired. □

Corollary 3.2.1. *Let X be a smooth Fano variety of Picard number one that is invariant under a Pfaff field \mathcal{F} of rank $k = \dim(X)$. Then*

$$\deg_{K_X^{-1}}(X) \leq k^k (\deg(\mathcal{F}) + 2)^k,$$

where $\deg_{K_X^{-1}}(X)$ is the degree of X with respect to anticanonical polarization.

Proof. Indeed, in this case we have

$$2g(X, K_X^{-1}) - 2 = (k - 2) \deg_{K_X^{-1}}(X).$$

Thus, it follows from Theorem 3.2.1 that $k \leq \deg(\mathcal{F}) + 1$. On the other hand, it follows from [59] that $d(X) \leq k + 1$ and $\deg(X) \leq (d(X)k)^k$, where $d(X)$ is the least positive number integer d for which X can be covered by rational curves of (anticanonical) degree at most d , see [59, Subsection 1.3]. □

Finally, we also consider the case when the invariant variety is Calabi-Yau, i.e. $\deg(K_X) = 0$.

Corollary 3.2.2. *If X is Calabi-Yau and invariant by \mathcal{F} then $\dim(X) \leq \deg(\mathcal{F})$.*

In other words, holomorphic Pfaff fields of small degree do not admit invariant Calabi-Yau varieties.

Complete intersections invariant by Pfaff field

Let us now consider the application of Theorem 3.2.1 to the case when the invariant variety X is a complete intersection, one obtains the following statement.

Corollary 3.2.3. *Let X be a k -dimensional complete intersection variety of multi-degree (d_1, \dots, d_{n-k}) , and such that $\text{cod}(\text{Sing}(X), X) \geq 2$. If X is invariant under a holomorphic Pfaff field \mathcal{F} of rank k on \mathbb{P}^n , then*

$$d_1 + \dots + d_{n-k} \leq \deg(\mathcal{F}) + n - k + 1.$$

Proof. Notice that

$$2g(X) - 2 = \deg(X) (d_1 + \cdots + d_{n-k} - n + k - 2).$$

By Theorem 3.2.1 this is less than or equal to $(\deg(\mathcal{F}) - 1) \deg(X)$, and the desired inequality follows easily. \square

Remark 3.2.1. It follows from [26, Corollary 4.5] that if X and \mathcal{F} are as above and

$$\dim(\text{Sing}(\mathcal{F}) \cap X) < k,$$

then

$$d_1 + \cdots + d_{n-r} \leq \begin{cases} \deg(\mathcal{F}) + n - k, & \text{if } \rho \leq 0, \\ \deg(\mathcal{F}) + n - k + \rho, & \text{if } \rho > 0 \end{cases}$$

where $\rho := \sigma + n - r + 1 - d_1 - \cdots - d_{n-r}$, with σ denoting the Castelnuovo-Mumford regularity of the singular locus of X . Therefore, Corollary 3.2.3 allow us to conclude that if $\text{cod}(\text{Sing}(X), X) \geq 2$, then one can take $\rho = 1$, regardless of $\dim(\text{Sing}(\mathcal{F}) \cap X)$.

Let V be an algebraic manifold with $\text{Pic}(V) \simeq \mathbb{Z}$. If D is a divisor on V then $\mathcal{O}_V(D) = \mathcal{O}_V(d_D)$, for some $d_D \in \mathbb{Z}$. In this case, we denote $\kappa(V) = d_{K_V}$. A Pfaff field of rank k on V is a section of $\bigwedge^k \Theta_V \otimes \mathcal{O}_V(s)$, for some $s \in \mathbb{Z}$. Thus, we define $d_{\mathcal{F}} := s + k$, naturally. In this case, we get the following.

Proposition 1. Let V be a n -dimensional algebraic manifold with $\text{Pic}(V) \simeq \mathbb{Z}$. Let X be a k -dimensional smooth complete intersection of nonsingular hypersurfaces D_1, \dots, D_{n-k} on V . If X is invariant under a holomorphic Pfaff field \mathcal{F} of rank k on V , then

$$d_{D_1} + \cdots + d_{D_{n-k}} \leq d_{\mathcal{F}} - k - \kappa(V).$$

Proof. Since X is invariant by \mathcal{F} we have that $H^0(X, \bigwedge^k \Theta_X \otimes \mathcal{O}_V(d_{\mathcal{F}} - k)|_X) \neq \{0\}$, then $\deg(\bigwedge^k \Theta_X \otimes \mathcal{O}_V(d_{\mathcal{F}} - k)|_X) \geq 0$. Let $\mathcal{O}_V(D_i)$ be the line bundle associated to hypersurface D_i , $i = 1, \dots, n - k$. We have the following adjunction formula

$$\bigwedge^k \Theta_X = \bigwedge^k \Theta_V|_X \otimes \mathcal{O}_V(-D_1)|_X \otimes \cdots \otimes \mathcal{O}_V(-D_{n-k})|_X.$$

Therefore $\bigwedge^k \Theta_X = \mathcal{O}_V(-\kappa(V) - d_{D_1} - \cdots - d_{D_{n-k}})|_X$, thus

$$\deg(\mathcal{O}_V(d_{\mathcal{F}} - k - \kappa(V) - d_{D_1} - \cdots - d_{D_{n-k}})|_X) = \deg(\bigwedge^k \Theta_X \otimes \mathcal{O}_V(d_{\mathcal{F}} - k)|_X) \geq 0.$$

\square

Note that this last inequality coincides with the given in Corollary 3.2.3 when $V = \mathbb{P}^n$ and X is a non-singular complete intersection.

3.3 Bound for invariant varieties with stable tangent bundle

Our second main result uses the hypothesis of stability (in the sense of Mumford-Takemoto) of the tangent bundle of X to establish another upper bound for the sectional genus in terms of the degree and rank of a holomorphic Pfaff field. It generalizes the previous result for nonsingular invariant varieties by allowing its dimension to be larger than the rank of the Pfaff field. To the best of our knowledge, this is the first time that the stability of the tangent bundle is used to obtain such bounds. Notice that if Θ_X is stable, then each $\bigwedge^k \Theta_X$ is semistable, see [3]. Examples of projective varieties with stable tangent bundle are Calabi-Yau [78], Fano [36, 48, 68, 76] and complete intersection [68, 77] varieties.

Definition 3.3.1. *Let E be a torsion-free sheaf on V . The ratio $\mu_L(E) = \deg_L(E)/\text{rk}(E)$ is called the slope of E , where $\deg_L(E) = \deg_L(\det(E))$. Recall that a E is semistable (in the sense of Mumford-Takemoto) if every torsion-free subsheaf E' of E satisfies $\mu_L(E') \leq \mu_L(E)$. Furthermore, E is stable if the strict inequality is satisfied.*

Theorem 3.3.1. *Let X be a nonsingular projective variety of dimension m which is invariant under a holomorphic Pfaff field \mathcal{F} of rank k on \mathbb{P}^n ; assume that $m \leq k$. If the tangent bundle Θ_X is stable, then*

$$\frac{2g(X, \mathcal{O}_X(1)) - 2}{\deg(X)} \leq \frac{\deg(\mathcal{F}) - k}{\binom{m-1}{k-1}} + m - 1. \quad (3.4)$$

Proof. The proof follows the same argument of the proof of Theorem 3.2.1. Since X is invariant under \mathcal{F} , we can conclude that $H^0(X, \bigwedge^k \Theta_X \otimes \mathcal{O}_X(d-k)) \neq \{0\}$. It then follows from the semistability of $\bigwedge^k \Theta_X$ that $\bigwedge^k \Theta_X \otimes \mathcal{O}_X(d-k)$ is also semistable, thus $\deg(\bigwedge^k \Theta_X \otimes \mathcal{O}_X(d-k)) \geq 0$. On the other hand, note that

$$\deg\left(\bigwedge^k TX\right) = -\binom{\dim(X) - 1}{k - 1} \deg(K_X).$$

Now, it is enough to consider the difference

$$(2g(X) - 2) - \left[\frac{\mathcal{O}_X(d-k)}{\binom{m-1}{k-1}} + (m-1)\mathcal{O}_X(1) \right] \cdot \mathcal{O}_X(1)^{m-1}.$$

A straight forward calculation leads to the inequality in Theorem 3.3.1. □

However, the inequality of Theorem 3.3.1 is not sharp in general. To see this, let X be a complete intersection variety of dimension m and multidegree

(d_1, \dots, d_{n-m}) , which is invariant under a k -dimensional Pfaff field \mathcal{F} on \mathbb{P}^n ; assume that $m \geq k$. Then

$$d_1 + \dots + d_{n-m} \leq \frac{\deg(\mathcal{F}) - k}{\binom{m-1}{k-1}} + n + 1.$$

Setting $m = k = 1$, the inequality reduced to $d_1 \leq \deg(\mathcal{F}) + n$. However, Marcio Soares has shown, under the same circumstances, that $d_1 \leq \deg(\mathcal{F}) + 1$ [75, Theorem B].

Bibliography

- [1] Abd'Allah Al Amrami, *Cohomological study of weighted projective space*. In Algebraic Geometry, Lecture Notes in pure and applied mathematics, **193**, Edited by Sinan Sertöz.
- [2] A. Adem, J. Leida and Y. Ruan, *Orbifolds and stringy topology*, *Cambridge Tracts in Mathematics*, **171**, Cambridge University Press, Cambridge, 2007.
- [3] V. Ancona, G. Ottaviani, *Stability of special instanton bundles on \mathbb{P}^{2n+1}* . *Trans. Am. Math. Soc.* **341**, 677-693 (1994).
- [4] M. Audin, *The Topology of Torus Actions on Symplectic Manifolds*. Progress in Math. **93**, Birkhäuser, Basel Boston Berlin, 1991.
- [5] L. Autonne, *Sur la théorie des équations différentielles du premier ordre et du premier degré*. *J. École Polytech.* **61** (1891) 35-122, **62** (1892) 47-180.
- [6] V. Batyrev and D. Cox, *On the Hodge structure of projective hypersurfaces in toric varieties*, *Duke Math. J.* **75** (1994), 293-338.
- [7] G. Barthel, J-P. Brasselet, K-H. Fieseler et L. Kaup; *Diviseurs invariants et homomorphisme de Poincaré des variétés toriques complexes*. *Tôhoku Math. J.* **48** (1996), 363-390.
- [8] M. Beltrametti, A. Howard, M. Schneider, A. Sommese. *Projections from subvarieties*. arXiv:math/9804048, 1998.
- [9] C. Boye, K. Galicki, M. Nakamaye, *Sasakian geometry, homotopy spheres and positive Ricci curvature*. *Topology* **42** (2003), 981-1002.
- [10] M. Brunella. *Some remarks on indices of holomorphic vector fields*. *Publications Mathématiques* 41: 527-544, 1997.
- [11] M. Brunella, *Foliations on complex projective surfaces*. arXiv:math/0212082v1 [math.CV] 5 Dec 2002.
- [12] M. Brunella, *Birational Geometry of Foliations*. First Latin American Congress of Mathematicians, IMPA, ISBN 85-244-0161-3 (2000).

- [13] M. Brunella, *Inexistence of invariant measures for generic rational differential equations in the complex domain*. Bol. Soc. Mat. Mexicana (3) **12** (2006), no. 1, 43-49.
- [14] M. Brunella and L. G. Mendes, *Bounding the degree of solutions to Pfaff equations*, Publ. Mat. **44** (2000), 593-604
- [15] M. Brunella, M. Nicolau, *Sur les hypersurfaces solutions des équations de Pfaff*. C. R. Acad. Sci. Paris S´er. I Math., 329 (1999), no 9, 793-795.
- [16] A. Campillo, M. M. Carnicer, and J. García de la Fuente: *Invariant Curves by Vector Fields on Algebraic Varieties* . J. London Math. Soc. 2000, **62**, 56-70.
- [17] S. Cantat, *Invariant hipersurfaces in holomorphic dynamics*. 2010, Pre-print.
- [18] M.M. Carnicer. *The Poincaré problem in the non-dicritical case*. Ann. Math. 140: 289-294, 1994.
- [19] V. Cavalier and D. Lehmann, *On the Poincaré inequality for one-dimensional foliations*. Com. Compositio Math. **142** (2006) 529-540.
- [20] D. Cerveau, J.F. Mattei, *Formes intégrables holomorphes singulières*. Asterisque, 97 (1983).
- [21] D. Cerveau and A. Lins Neto, *Holomorphic foliations in $\mathbb{P}_{\mathbb{C}}^2$ having an invariant algebraic curve*. Ann. Inst. Fourier **41** (1991), 883-903.
- [22] S. S. Chern, *Meromorphic vector fields and characteristic numbers*. Selected Papers, Springer-Verlag New York (1978), 435-443.
- [23] C. Christopher, J. Llibre, *A family of quadratic polynomial differential systems with invariant algebraic curves of arbitrarily high degree without rational first integrals*. Proc. Amer. Math. Soc., 2002, **130**, 2025-2030.
- [24] C. Christopher, J. Llibre, J.V. Pereira, *Multiplicity of invariant algebraic curves in polynomial vector fields*. Pacific J. Math **229**, no. 1, (2007) 63-117.
- [25] S. C. Coutinho, J. V. Pereira, *On the density of algebraic foliations without algebraic invariant sets*. Journal für die reine und angewandte Mathematik, Alemanha, v. 594, p. 117-136, 2006.
- [26] J. D. A. S. Cruz, E. Esteves, *Regularity of subschemes invariant under Pfaff fields on projective spaces*. To appear in *Comment. Math. Helv.* (2010).
- [27] D. Cox, *The homogeneous coordinate ring of a toric variety*. J. Algebraic Geom. **4** (1995), 17-50.

- [28] D. Cox, J. Little and H. Schenck, *Toric Varieties*. <http://www.cs.amherst.edu/dac/toric.html>, In preparation.
- [29] D. A. Cox and S. Katz, *Mirror Symmetry and Algebraic Geometry*. Math. Surveys Monogr. **68**, Amer. Math. Soc., Providence, 1999.
- [30] V. I. Danilov, *The Geometry of Toric Varieties*. Russian Math Surveys 33:2 (1978) 97.
- [31] G. Darboux, *Memoire sur les équations différentielles algébriques du premier ordre et du premier degré*. (Mélanges), Bull. Sci. Math. **2** (1878) 60-96, 123-144, 151-200.
- [32] P. Deligne, *Theorie de Hodge, II*. Inst. Hautes Etudes Sci. Publ. Math., 40 (1971), 5-57.
- [33] Dobrovolskii V A, Lokot' N V and Strelcyn J M, *Mikhail Nikolaevich Lagutinskii (1871-1915): an unrecognized mathematician*. Hist. Math. **25** 245-64, 1998.
- [34] E. Esteves and S. Kleiman, *Bounding solutions of Pfaff equations*. Comm. Algebra **31** (2003), 3771-3793.
- [35] E. Esteves and S. Kleiman, *Bounds on leaves of one-dimensional foliations*. Bull. Braz. Mat. Soc. (NS) **34** (2003), 145-169.
- [36] R. Fahlaoui, *Stabilité du fibre tangent des surfaces de del Pezzo*. Math. Ann. 283 (1989).
- [37] R. Friedman, *Algebraic Surfaces and Holomorphic Vector Bundles*. Columbia University, New York, 1998.
- [38] J. García de la Fuente, *Multiplicity of a foliation on projective spaces along an integral curve*. Re. Mat. Uni. Complut. Madrid 6 (1993) 207-217.
- [39] Y. Fukuma, *On the sectional geometric genus of quasi-polarized varieties*. I, Comm. Algebra **32** 2004 1069-1100.
- [40] W. Fulton, *Intersection Theory*. second edition, Springer, 1998.
- [41] W. Fulton, *Introduction to Toric Varieties*. Princeton University Press, 1993.
- [42] É. Ghys, *À propos d'un théorème de J.-P. Jouanolou concernant les feuilles fermées des feuilletages holomorphes*. Rend. Circ. Mat. Palermo (2) 49 (2000), no. 1, 175-180.
- [43] X. Gómez-Mont and G. Kempf, *Stability of meromorphic vector fields in projective spaces*. Comm. Math. Helv. **64** (1989), pp 462-473.

- [44] X. Gómez-Mont, L. Ortiz-Bobadilla. *Sistemas Dinámicos Holomorfos en Superficies*. Sociedad Matematica Mexicana, 1989.
- [45] P. Griffiths, J. Harris. *Principles of algebraic geometry*. John Wiley & Sons, New York, 1978.
- [46] J. Harris, *A bound on the geometric genus of projective varieties*. Ann. Scuola Norm. Sup. Pisa Cl. Sci. **8** 4 (1981) pp 35-68.
- [47] R. Hartshorne, *Algebraic Geometry*. Springer-Verlag Graduate Texts in Mathematics **52**, 1977.
- [48] J.M. Hwang, *Stability of tangent bundles of low dimensional Fano manifolds with Picard number 1*. Math. Ann. **312** (1998), 599-606.
- [49] J-P. Joaunolou, *Équations de Pfaff algébriques*. Lecture Notes in Math. **708**, Springer, 1979.
- [50] J-P. Jouanolou *Hypersurfaces solutions d'une équation de Pfaff analytique*. Math. Ann. 232 (1978) no 3, 239-245.
- [51] E. Kunz, *Holomorphe Differentialformen auf algebraischen Varietäten mit Singularitäten*. Manuscripta math. **25** (1975), 91-108.
- [52] S. Kobayashi, *Differential Geometry of complex Vector Bundle*. Publication of the Mathematical Society of Japan, Princeton University Press, 1987.
- [53] A. Lins Neto, *Some examples for Poincaré and Painleve problem*. Ann. Scient. Ec. Norm. Sup., 4e série, **35**, 2002, p. 231 a 266.
- [54] A. Lins Neto, M.G. Soares, *Algebraic solutions of one-dimensional foliations*. J. Differential Geom. **43** (1996), no. 3, 652–673.
- [55] J. Lipman, *Dualizing sheaves, differentials and residues on algebraic varieties*. Astérisque, vol. **117**, Soc. Math. de France, 1984.
- [56] J. Llibre, J.C. Medrado, *On the invariant hyperplanes for d-dimensional polynomial vector fields*. J. Phys. A: Math. Theor. **40** (2007) 8385-8391.
- [57] J. Llibre, X. Zhang, *Rational first integral in Darboux theory of integrability in \mathbb{C}^n* . Bull. Sci. Math. (2007), in press doi:10.1016/j.bulsci.2007.12.001
- [58] E. Mann, *Cohomologie quantique orbifold des espaces projectifs à poids*. J. Algebraic Geom. **17** (2008), 137-166.
- [59] A. M. Nadel, *The Boundedness of Degree of Fano Varieties with Picard Number One*. Journal of the American Mathematical Society, Vol. **4**, No. 4 (Oct., 1991), pp. 681-692

- [60] U. Nagel, *Arithmetically Buchsbaum divisors on varieties of minimal degree*. Trans. Amer. Math. Soc., (1999) **351**, N. 11, 4381-4409
- [61] J. Noguchi, *A short analytic proof of closedness of logarithmic forms*. Kodai Math. J. 18.(2). (1995), 295-299.
- [62] T. Oda, *Convex Bodies and Algebraic Geometry*, Springer-Verlag, 1985.
- [63] O. Okonek, M. Schneider, H. Spindler, *Vector bundles on complex projective spaces*. Boston: Birkhauser (1980).
- [64] J. M. Ollagnier, *About a conjecture on quadratic vector fields*. Journal of Pure and Applied Algebra **165** (2001) 227-234.
- [65] P. Painlevé, *Sur les intégrales algébrique des équations différentielles du premier ordre* and *Mémoire sur les équations différentielles du premier ordre*, Oeuvres de Paul Painlevé; Tome II, Éditions du Centre National de la Recherche Scientifique, **15**, quai Anatole-France, 75700, Paris, 1974.
- [66] J. V. Pereira, *Vector Fields, Invariant Varieties and Linear Systems*. Annales de L'Institut Fourier **51**, no.5 (2001), 1385-1405.
- [67] J. V. Pereira, *On the Poincaré problem for foliations of general type*. *Math. Ann.* **323** (2002), 217-226.
- [68] T. Peternell, A. Wisniewski, *On stability of tangent bundles of Fano manifolds with $b_2 = 1$* . J. Alg. Geom. 4 (1995) 363-384.
- [69] H. Poincaré, *Sur l'intégration algébrique des équations différentielles du premier ordre et du premier degré*. Rend. Circ Mat Palermo, **5** (1891), 161-191.
- [70] H. Poincaré, *Sur l'intégration algébrique des équations différentielles du premier ordre et du premier degré I*. Rend. Circ Mat Palermo, **11** (1897), 169-193-239.
- [71] M. Reid, *Chapters on algebraic surfaces*. AMS(1998).
- [72] S-S. Roan: *Picard groups of hypersurfaces in toric varieties*. Publ. Res. Inst. Math. Sci. **32** (1996), 797-834.
- [73] F. Santos and B. Scárdua. *Vector fields and foliations associated to groups of projective automorphisms*. To appear in Conformal Geometry and Dynamics, AMS, 2010.
- [74] I. Satake. *On a generalization of the notion of manifold*. Proc. of the Nat. Acad. of Sc. U.S.A. 42 (1956), pp. 359-363.

- [75] M. G. Soares, *The Poincaré problem for hypersurfaces invariant by one-dimensional foliations*. Inventiones Mathematicae, Alemanha, v. **128**, p. 495-500, 1997.
- [76] A. Steffens, *On the stability of the tangent bundle of Fano manifolds*. Math. Ann. **304** (1996) 635-643
- [77] S. Subramanian, *Stability of the tangent bundle and existence of a Kähler-Einstein metric*. Math. Ann. **291** (1991), 573-577
- [78] H. Tsuji, *Stability of tangent bundles of minimal algebraic varieties*. Topology **27**, 4 429-442 (1988)
- [79] F. L. Zak, *Tangents and secants of algebraic varieties*. Translations of Mathematical Monographs, vol. **127**, Amer. Math. Soc. 1993.
- [80] F.L. Zak, *Projections of algebraic varieties*. (English transl.), Math USSR-Sb. 116(158) (1981), 593-608.
- [81] A. G. Zamora, *Foliations in Algebraic Surfaces having a rational first integral*. Publicacions Matemàtiques **41** (1997), 357-373.

Livros Grátis

(<http://www.livrosgratis.com.br>)

Milhares de Livros para Download:

[Baixar livros de Administração](#)

[Baixar livros de Agronomia](#)

[Baixar livros de Arquitetura](#)

[Baixar livros de Artes](#)

[Baixar livros de Astronomia](#)

[Baixar livros de Biologia Geral](#)

[Baixar livros de Ciência da Computação](#)

[Baixar livros de Ciência da Informação](#)

[Baixar livros de Ciência Política](#)

[Baixar livros de Ciências da Saúde](#)

[Baixar livros de Comunicação](#)

[Baixar livros do Conselho Nacional de Educação - CNE](#)

[Baixar livros de Defesa civil](#)

[Baixar livros de Direito](#)

[Baixar livros de Direitos humanos](#)

[Baixar livros de Economia](#)

[Baixar livros de Economia Doméstica](#)

[Baixar livros de Educação](#)

[Baixar livros de Educação - Trânsito](#)

[Baixar livros de Educação Física](#)

[Baixar livros de Engenharia Aeroespacial](#)

[Baixar livros de Farmácia](#)

[Baixar livros de Filosofia](#)

[Baixar livros de Física](#)

[Baixar livros de Geociências](#)

[Baixar livros de Geografia](#)

[Baixar livros de História](#)

[Baixar livros de Línguas](#)

[Baixar livros de Literatura](#)
[Baixar livros de Literatura de Cordel](#)
[Baixar livros de Literatura Infantil](#)
[Baixar livros de Matemática](#)
[Baixar livros de Medicina](#)
[Baixar livros de Medicina Veterinária](#)
[Baixar livros de Meio Ambiente](#)
[Baixar livros de Meteorologia](#)
[Baixar Monografias e TCC](#)
[Baixar livros Multidisciplinar](#)
[Baixar livros de Música](#)
[Baixar livros de Psicologia](#)
[Baixar livros de Química](#)
[Baixar livros de Saúde Coletiva](#)
[Baixar livros de Serviço Social](#)
[Baixar livros de Sociologia](#)
[Baixar livros de Teologia](#)
[Baixar livros de Trabalho](#)
[Baixar livros de Turismo](#)