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Instituto de Matemática Pura e Aplicada

Mathematical Methods in Finance: Real Options on Mean-Reverting Cash-Flow Processes

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Abstract

Corporations are often face with the possibility of investing in specific projects that might impact on their cash flow. The Real Option Analysis approach became, in the last few years, an important instrument for managers to take investment decision.

The Real Option approach consists in the analysis of contracts, using the same tools developed for financial derivative markets. In the case of stocks, the Black and Scholes model assumes that the underlying asset of the contract follows a Geometric Brownian Motion. Nevertheless, if we are interested in modeling the cash flow that arises from a project that might be negative, then a GBM cannot possibly be a good model. In this case we return to the Bachelier model (which pre-dated B&S by almost 100 years), and show that, together with some extensions, it yields consistent results. The approach was implemented computationally and numerical results are presented.

To my mother, with love...

Acknowledgements

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Introduction

Large corporations often face the possibility to invest in projects to impact their cash flow. Many of these opportunities can determine if the company will be a major player in the market or will sunk down. For this reason, these projects must be submitted to a careful analysis by the managers of the company. As stated by [Mun02], the first approach to deal with this challenge is to recur to the Net Present Value. Nevertheless, this methodology can underestimate the value of the project once it does not take in account the value of managerial flexibilities, like postponement, abandonment or creation of new opportunities. To give a more realistic framework, Real Options Analysis has become, in the last years, an important instrument that quantifies the value of these optionalities, helping the managers to decide if a project must be implemented.

As stated by [BL10], an acceptable model should have requirements like capturing the of reality of real worlds situations, absence of arbitrage opportunities, use of market data, proper incorporation of risk, being empirically testable, being mathematically transparent and being computationally efficient.

In the existing literature, there are many studies in which the underlying of the project cannot assume a negative value. Models consider that the underlying asset follows a Geometric Brownian Motion, an Exponential Mean Reverting or a Jump Process as described in [DP96]. However, sometimes it is more realistic to allow the asset to assume negative values. For example, suppose that our asset is the cash flow of a project. It is natural that this underlying becomes negative in some states due to debts, judicial issues, poor sales performance and others debts. To better reflect these factors, it would be reasonable to adopt a dynamics for the asset that can assume negative values in some states of nature.

In Chapter 2 of this work, the usual approach adopted by managers in process of valuation is described. Concepts like Net Present Value, CAPM and Systematic Risk are explained in the same context of [Mun02] and [RWJ01]. The strength and the weakness of these methods are pointed out and exam-

ples are presented to clarify the ideas.

In Chapter 3 we explain another method of valuation called the Real Option Analysis. This methodology uses the tools developed to derivatives in financial markets presented in [KK01]. We present the most common assumptions and show how to proceed to evaluate the prices of some contracts, making hypothesis over the dynamics of the underlying asset.

In Chapter 4, we analyze several optionalities that can be priced using Real Options Analysis. We explain several contracts, as described in [Hul05], and provide examples to clarify the procedure.

In Chapter 5, we propose a new dynamics to the underlying asset and develop a methodology to price derivatives based on this mathematical model. A particular case of the model is the Bachelier one, analyzed in [MR05]. Properties of the model, like the so-called greeks are explained.

In Chapter 6, numerical results from the model developed in Chapter 5 are presented and discussed. The numerical implementation of the model and the algorithm are shown in the Appendix. [Bra06] and [Hul05] are used as references for the implementation of the algorithms.

In Chapter 7, we present the conclusion of the work and suggest avenues for future research.

Chapter 1

Classical Approach

1.1 Net Present Value

An investment is only justified if it gives return to the shareholders. Managers need to identify projects that worth more than their cost of acquisition. There are several ways to measure how an investment contributes to generate value to the company.

The Net Present Value is the main tool used by companies to value projects [Hul05]. It consists in the comparison of the discounted future cash flow(DCF) with the investment(I) needed to implement the project. Only if the difference between DCF and I is positive, the project is initiated. This rule is extremely simple and it has some other advantages like

- Comparability between projects

- It is clear and consistent

- Understandable for the majority of managers

- It considers the value of the money through time

Figure 1.1 illustrates a possible scenario over which the NPV is applied. The NPV can be expressed as, in the moment of the analysis,

$$NPV = \sum_{i=1}^{+\infty} \frac{CF_i}{(1 + R_{ad})^i} - I \qquad (1.1)$$

where CF_i is the cash flow at time i. It remains to determine the parameter R_{ad}. Here lies one of the major problems of this analysis. It assumes that the rate R_{ad} is well known. Such rate needs to reflect the risk of the project from the point of view of the shareholders, and consequently it is the rate that
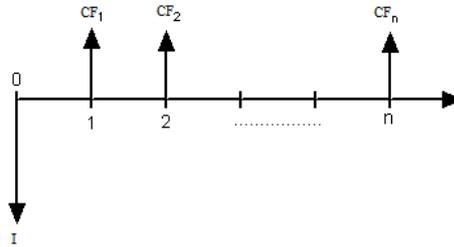
Figure 1.1: Discounted Cash Flow

they want to be rewarded to assume that risk. One way to solve this problem is adopt r as the unorder Weighted Avarage Cost of Capital(WACC). The latter is the return that a company must earn given its current assets or the return required by shareholders on new investments that share approximately the same risk as the current operations. For details, see [RWJ01].

Despite the simple calculation and interpretation, some criticism is made on NPV analysis due to its strong assumptions. The assumption that future cash flows are known is, in most practical cases, unrealistic. Uncertainties related to political elections, central banks meetings, emergence of a market competitor, abrupt changes in weather jeopardizing the harvests are some of the factors that can substantially influence the performance of a company.

Another key point is the assumption of constant discounted adjustment rate. The cash flow analysis of a company, is made through a considerable period of time. For this reason, some authors have been considering it is not reasonable to assume a constant interest free rate, which implies a variable risk adjust discount rate.

A more realistic model can be built introducing stochastic behavior for some variables. However, the great number of parameters to estimate demands more data, which, in most of cases, are unavailable.

Back to the simple NPV, the central point is the good estimation of the rate r. Another approach to calculate this rate is to extract it from the market. For this, a general equilibrium model is suggested to allow us to extract the desired rate to reward the agents for assuming such risk.

1.2   CAPM

CAPM is an equilibrium market model. From the behavior of the market agents, we are able to price the assets in which the market will be in equilibrium. The model is constructed over the following assumptions:

1. Absence of transaction cost

2. Infinitely divisible assets

3. Absence of taxes

4. Perfect Competition

5. Agents make decisions following a Markowitz Analysis

6. Short selling is allowed

7. Unlimited borrowing and lending at a risk-free rate

8. Homogeneous expectations

9. All assets are negotiable

Despite of being a simple model, CAPM provides the necessary parameters not only for the NPV calculation but also, in future sections, for the calculation of more complex models with stochasticity. It also allows us to extract the market price of risk. Once we have this quantity, we can change to the risk neutral world and price derivatives on the approach of Real Options Analysis for several proposal dynamics for the underlying of the option. CAPM can be develop into a Financial framework as in [Pli97]

Consider the Mean-Variance Portfolio problem. The interest rate r is deterministic, there are no arbitrage opportunities, and there exists some portfolio with $E[R] = \rho \neq r$. Our aim is

$$
\begin{aligned}
& \text{minimize var}(R) \\
& \text{subject } to \ E(R) = \rho \\
& \text{R is a portfolio return}
\end{aligned}
\tag{1.2}
$$

where ρ is a specified scalar. If $\rho \geq r$, the feasible region in (1.2) is non-empty and the solution of the problem is well defined. Now, let's take a look at the following problem

$$
\begin{aligned}
& \text{minimize var}(V_1) \\
& \text{subject } to \ E(V_1) = v(1 + \rho) \\
& V_0 = v
\end{aligned}
\tag{1.3}
$$

To realize that these problems are equivalent, notice that if $\hat{V}_1$ is a solution of (1.3), then $\hat{R} \equiv (\hat{V}_1 - v)/v$ satisfies the constraints in (1.2). If R is any other return that is feasible to (1.2), then $V_1 \equiv v(1 + R)$ satisfies $E(V_1) = v(1 + r)$,

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which means that V_1 is feasible to (1.3) and $\text{var}(\hat{R}) \leq \text{var}(\hat{R})$, therefore $\hat{R}$ is a solution of (1.2).

At this point, we use Lagrange Multipliers to solve (1.3). Introducing the scalar β, we are interested in solving

$$\text{maximize } E\left[-\frac{1}{2}V_1^2 + \beta V_1\right]$$
$$\text{subject to } V_0 = v \tag{1.4}$$

where the solution is given by

$$\hat{V} = \beta\frac{E_Q[L] - L}{E_Q[L]} + v(1+r)\frac{L}{E_Q[L]} \tag{1.5}$$

and $L = Q/P$ is the state price density. Taking the expectation,

$$E[\hat{V}] = \beta(E_Q[L] - 1)/E_Q[L] + v(1+r)L/E_Q[L] \tag{1.6}$$

It follows that

$$E[\hat{V}] = v(1+\rho) \leftrightarrow \beta = \frac{v[(1+\rho)E_Q[L] - (1+r)]}{E_Q[L] - 1} \tag{1.7}$$

Substituting this in (1.5), we see that $\hat{V}$ is not only feasible to (1.2), but is also optimal.

Conversely, suppose $\hat{V}$ is a solution of (1.2) with β as in (1.7), it is also the optimal solution for (1.4). If we assume that the investors utility function is taken to be the quadratic function $u(w) = -w^2 + \beta w$ and due to the one to one correspondence between (1.2) and (1.4), the solution of (1.2) must be an affine function of the state price density, leading to the following expression to the return $\hat{R}$ corresponding to $\hat{V}$:

$$\hat{R} = \frac{\rho E_Q[L] - r}{E_Q[L] - 1} - \frac{\rho - r}{E_Q[L] - 1}L \tag{1.8}$$

This shows that the optimal solution R of the mean-variance portfolio problem (1.2) is an affine function of the state price density.

The CAPM can be expressed as,

If R' is a solution of the mean-variance portfolio problem (1.2) for $\rho \geq r$ and R is the return of an arbitrary portfolio, then

$$E[R] - r = \frac{\sigma_{R'R}}{\sigma_{R'}^2}(E[R'] - r) \tag{1.9}$$

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Another way to achieve at the same result is following Markowitz Theory. From it, we now the possibility that each agent faces to borrow and lending at the risk-free rate fixes its portfolio, independent of it risk aversion, being the same portfolio of risky assets for all market participants. This imply that all agents will have the same portfolio of risky assets, and at the equilibrium, this will be the market portfolio. What it tells us is that each agent will have a linear combination of the market portfolio and applications in the interest rate.

The curve that represents the linear combination of the risky portfolio and the riskless portfolio is called Capital Market Line(CML). It can be represented by the equation

$$E(R_i) = R_f + \frac{E(R_M) - R_f}{\sigma_M^2}\sigma_i \qquad (1.10)$$

where $E(R_i)$ is the expected return of the risky portfolio, $E(R_M)$ is the expected return of the market, σ_i is the volatility of the risky portfolio and σ_M is the volatility of the market portfolio.

The term $E(R_M) - R_f/\sigma_M^2$ is the market price of risk. In financial terms, it can be thought as the additional return that can be gained adding one unity of risk. As stated by [EGG06], the financial interpretation of (1.10) is

Expected Return = Price of Time + (Price of Risk x Amount of Risk)

At this point it is important to highlight that the equation 1.10 only gives the expected return of efficient portfolios in the sense of Markowitz. Thus, to develop a relation between inefficient portfolios and market price of risk we need to investigate the properties of the systemic risk of a portfolio.

1.2.1 Systematic Risk

The Systematic Risk is the risk that affects a large number of assets. Uncertainties related with the inflation rate and Gross Domestic Product(GDP) affect, in different levels, most of the assets. These forces of economy are the core of Systematic Risk. As explained in [RWJ01], the diversification of the portfolio cannot eliminate the Systematic Risk. For this reason, the agents on the economy are only rewarded by assuming this type of risk. Thus, the expected return of an asset depends exclusively on it Systematic Risk. The main point is how such risk is quantified. For this purpose, the measure used is the β coefficient. Given an base asset, usually a market index, the β of a specific asset represents how much Systematic Risk this asset is compared to

the base asset, which has by definition $\beta_M = 1$. It is shown in [EGG06] that the β of the asset can be expressed as

$$\beta_i = \frac{\sigma_{iM}}{\sigma_M^2},$$

where σ_{iM} is the covariance between the i-th asset and the market portfolio, and σ_M^2 is the market portfolio variance.

Using no arbitrage arguments, it can be shown that all assets and portfolios lie in a curve called the Security Market Line(SML) in the space expected return versus beta. To identify this line we can take into account the market portfolio with $\beta_M = 1$ and expected return $E(R_M)$. As we know, that interception of the line is the risk-free rate. Thus, we are able to determine its equation as

$$E(R_i) = R_f + \beta_i \left(E(R_M) - R_f \right) = R_f + \frac{\sigma_{iM}}{\sigma_M^2} \left(E(R_M) - R_f \right) \qquad (1.11)$$

As is shown in the previous section, this is the famous CAPM equation. It describes the expected return of all assets and portfolios on the economy. This model allows us to establish the adjusted discount rate to calculate the value of a project as proposed in Section (1.1). To calculate it, we will follow the steps described by [Hul05], and it consists of

- Sample a good number of companies whose main business is in line with the contemplated project

- Average their betas to have a proxy for the beta of the project

- Use the equation of CAPM to extract the adjusted discount rate

Now, all the required parameters to evaluate the NPV analysis is known and the manager can evaluate the project. However, this analysis will not take into account the value of some embedded optionalities like expansion, abandonment or waiting to invest. The Real Options Approach can handle some of these optionalities. One of its advantages is it does not need to compute the adjusted discount rate. The reason for this is once we are working in a risk neutral world, the rate of discount is the risk-free rate. But to pass from the physical measure to the risk neutral one we need to know the market price of risk and use Girsanov Theorem [1].

The natural step is to find a formulation that gives us the market price of risk. Remember the market price of risk is defined by

$$\frac{E(R_i) - R_f}{\sigma_i} = \lambda_i \qquad (1.12)$$

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[1]for more explanations about Girsanov Theorem, see [KK01]

Substituting equation (1.12) into (1.11) we arrive at,

$$\lambda_i = \frac{\beta_i}{\sigma_i} \left(E(R_i) - R_f \right) \tag{1.13}$$

Equation (1.13) will be extensively used to price derivatives in the risk neutral measure. Before going further in Real Options analysis, we provide an example to illustrate the procedure developed above.

1.3 A Clarifying Example

Suppose that an investor wants to open a full equity fast food company and the initial investment required is U\$\$ 20.000.000,00. He expects to have an annual cash flow of U\$\$ 1.000.000,00 with the new business.The risk-free rate is 4% per year.

Our first step is to find the beta for the investment. We will look at the betas from fast food companies and take the average of risk adjusted discount rate. In 01/09/2010, the historical betas, extracted from Yahoo.com for companies in this line of business are presented in the Table 1.1

Table 1.1: Historical Beta from Fast Food Companies

Company	Beta
McDonalds	0.55
Burger King	0.34
Yum!Brands	1.04
Wendy's	0.86

The average β is approximately $\beta_A = 0.7$. We use the average of the returns of the NYSE US 100 from 2004 to 2008 as a proxy for the expected return of the market. The calculations give $E(R_M) = 6.2\%$. Using Equation (1.11) to extract the risk adjusted rate, we have

$$E(R_p) = 0.04 + 0.7 \left(0.062 - 0.04 \right) = 0.0554$$

At this rate, we calculate the NPV from (1.1).

$$NPV = 10^6 \left(\frac{1}{1 + 0.0554} + \frac{1}{(1 + 0.0554)^2} + \frac{1}{(1 + 0.0554)^3} + ... \right) \tag{1.14}$$
$$- 2 \; 10^7$$

As the $NPV = -1.949.458,48$ is a negative value, the manager rejects its implementation. Note that if the cost of implementation was two million dollars cheaper, the project should be implemented.

All the weakness of the NPV are well exposed in the example. The next Chapter shows how to use the Real Options Approach to give more flexibility for variables and to get a more realistic model.

Chapter 2

The Real Options Approach

The main idea of the Real Options approach is to use the quantitative finance modeling in the study of real assets to provide a better analysis on flexibilities that are not taken into account in NPV. In the stock market framework, it is supposed that the assets have a known dynamics, the Geometric Brownian Motion(GMB). One of the attempt to use Real Options to value a project was proposed by [MS86], where the dynamics of the project follows a GBM. Nevertheless, even for stock assets this assumptions can be strong as stated by [Man63] and [Fam65]. For this reason, it is important to have in mind that other dynamics should be considered when one is trying to model the dynamics of a project. Therefore, throughout this chapter, we will digress on the most common hypothesis on project's dynamics.

2.1 Project's Dynamic as a Geometric Brownian Motion

In this section we will work in the framework proposed by [MS86] and follow the notation of [DP96]. Our aim is to price the value of waiting to invest, an optionality not quantified by NPV.

Suppose that the project value, V, follows a GBM such as

$$dV(t) = \nu V(t)dt + \sigma V(t)dW(t), \tag{2.1}$$

where ν is its expected growth and σ is its volatility.

As time evolves, the value of the projects are known but its future values are always unknown. Based on its value at time t, it is possible to establish a clear rule of investment. This bring us naturally the same problem of the American option valuation. Here the payoff at time t is defined as $F(V(t)) =$

$\max(V(t) - I, 0)$, where I is the sunk cost of implementing the project $V(t)$. Hence, the problem can be solved with dynamic programming techniques[1]. However, there is a slight difference between this problem and the traditional one. In the simplest version, we consider the firm's investment opportunity, $F(V(t))$, as a perpetual call option, and naturally it cannot depends on time, because the investment opportunity yields no cash flows up to the time that the investment is undertaken, the only return from holding it is its capital appreciation.

Once we are the holder of the option, we want a rule that maximize its expected present value:

$$F(V(t)) = \max \mathbb{E}[e^{-\rho(T-t)}(V(T) - I)^+], T \geq t, \tag{2.2}$$

where T is the unknown time where the decision is made, ρ is the discount rate and the maximization is subject to Equation (2.2) for V.[2]

Instead of solving this problem by dynamic programming, we will use contingent claim techniques. Because similar arguments will be applied in some models from the next sections. It also has the advantage of not requiring assumptions on risk preferences and discount rates. To simplify the problem, we will assume that V is tradable.Consider a self-financing trading strategy (x_0, x_1) in a project and a bond, with price $S_0(t)$, such that the wealth process includes exactly one sold contract. Therefore, the portfolio $\Pi(t)$ can be represented by

$$\Pi(t) = x_0(t)S_0(t) + x_1(t)V(t) - F(t, V(t)) \tag{2.3}$$

Consider F is smooth enough to apply Ito's Formula and remember that it is a perpetuity $(\partial_t F = 0)$.Thus,

$$dF = \partial_v F dV(t) + \frac{1}{2}\sigma^2 V(t)^2 \partial_{vv} F d[V(t), V(t)]. \tag{2.4}$$

Equation (2.3) becomes, in differential form,

$$d\Pi(t) = [x_0(t)S_0(t)r + x_1(t)V(t)\nu] - (\partial_v F\nu + \frac{\sigma^2}{2}V^2(t)\partial_{vv}F)dt$$
$$+ (x_1(t)V(t)\sigma - \partial_v F V(t)\sigma)\,dW(t). \tag{2.5}$$

To vanish the diffusion coefficient, we must have

$$x_1(t) = \partial_v F(t, V(t)) \tag{2.6}$$

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[1]for more details, see [Bra06]

[2]for integrability issues, we need to assume that $\nu < \rho$

For arbitrage reasons, the wealth process must be a multiple of the bond price minus the consumption of dividend,

$$dΠ(t) = rΠ(t)dt - δx_1(t)V(t)dt. \tag{2.7}$$

Hence,

$$
\begin{aligned}
rΠ(t) - ∂_v F δV(t) &= x_0(t)S_0(t)r + x_1(t)V(t)ν - (∂_v Fν + \frac{1}{2}σ^2 S_1^2(t)) \\
&= r\left[x_0(t)S_0(t) + x_1(t)V(t) - F\right] + Fr - x_1(t)V(t)(ν - r) \\
&\quad - (ν∂_v F + \frac{σ^2}{2}V^2(t)∂_{vv}F) \\
&= rΠ(t) + rF - (rV(t)∂_v F + \frac{σ^2}{2}V^2(t)∂_{vv}F) \tag{2.8}
\end{aligned}
$$

It follows that

$$\frac{1}{2}σ^2 v^2 ∂_{vv}F + (r - δ)v∂_v F - rF = 0. \tag{2.9}$$

Since we are dealing with a free boundary problem, we need three conditions to determine uniquely the optimal point to invest, thus, they are

1. $F(0) = 0$

2. $F(v^*) = V^* - I$

3. $F'(v^*) = 1$

where v^* is this optimal price for investment. The two first conditions are natural. If the project is worthless, then a contract on it must be worthless. Second, at the point on the boundary, the contract must have the same value of its payoff. The third is the "smooth pasting" condition[3].

The general solution of (2.9) is

$$F(v) = A_1 v^{γ_1} + A_2 v^{γ_2}, \tag{2.10}$$

where A_1, A_2 are constants and $γ_1$, $γ_2$ are the roots of the equation

$$\frac{1}{2}σ^2 γ(γ - 1) + (r - δ)γ - r = 0. \tag{2.11}$$

Solving (2.11), we have

$$γ = \frac{1}{2} - \frac{r - δ}{σ^2} ± \sqrt{\left[\frac{r - δ}{σ^2} - \frac{1}{2}\right]^2 + \frac{2r}{σ^2}}, \tag{2.12}$$

---

[3]For details, see [WHD05]

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with $\gamma_1 > 1$ and $\gamma_2 < 0$. The first boundary condition gives us that $A_2 = 0$, leading to the solution of

$$F(v) = A_1 v^{\gamma_1}. \tag{2.13}$$

Substituting (2.13) in the second boundary condition, we obtain

$$v^* = \frac{\gamma_1}{\gamma_1 - 1} I. \tag{2.14}$$

Finally, we differentiate (2.13) with respect to v, take $v = v^*$ and use the third boundary condition, to determine the coefficient

$$A_1 = \frac{1}{\gamma_1 (v^*)^{\gamma_1}}. \tag{2.15}$$

Substituting (2.14) in (2.15), we have

$$A_1 = \frac{(\gamma_1 - 1)^{\gamma_1 - 1}}{\gamma_1^{\gamma_1} I^{\gamma_1 - 1}}. \tag{2.16}$$

Once we determined the free boundary condition, we are able to establish the investment rule. It can be summarize as follow

- Hold the option if $V(t) < v^*$ (consequently, you must have $V(t) - I < F(V(t))$)

- Exercise the option if $V(t) \geq v^*$ (consequently, you must have $V(t) - I = F(V(t))$)

In other words, invest only if the value of the project is the value of the option to wait plus the sunk cost.

Table 2.1: The sensitivities of v^* with respect of the parameters of the model.

Derivatives	Interpretation
$\frac{\partial v^*}{\partial I} > 0$	Lower sunk cost encourages investments
$\frac{\partial v^*}{\partial \sigma} > 0$	Lower uncertainty encourages investments
$\frac{\partial v^*}{\partial \delta} < 0$	Higher cash flow encourages investments
$\frac{\partial v^*}{\partial r} > 0$	Lower interest rate encourages investments

Note that if the project adds a huge value to the company and generates huge cash flow($\delta \to \infty$), this means that you must invest immediately and

the optionality to wait is worthless(F(V)=0). This would make $\delta \to \infty$ in (2.12), and we have $\gamma \to \infty$. By (2.14), $v^* \to I$ which means that, by the second boundary condition, the rule is exactly the NPV.

This is the main result of this section. The NPV rule, largely used in the industry, is an asymptotic case of this Real Options Analysis in this framework, which means that in every other situations, the value of waiting is worthy and must be considered.

Conversely, if we have a project that generates poor cash flow $(\delta \to 0)$, this means that you should not invest and the optionality to wait has a lot of value. Making $\delta \to 0$ in (2.12),we have $\gamma \to 1$ and, by (2.14), $v^* \to \infty$ revealing that the investments should not be made.

2.1.1 An Example

Suppose that an oil company is interested in building a refinery. The sunk cost for implementing it is estimated in $I = 10^8$. The cash flow generated by this project is due to sales of the oil projects that is reflected on the prices of the company shares. Thus, assume the dynamics of the company's shares are perfectly correlated with the dynamics of the project's value. After a calibration procedure[4], suppose that the following equation models the project in the risk neutral measure,

$$dV(t) = 0.1V(t)dt + 0.3V(t)dW(t). \tag{2.17}$$

Due to large investments, the company will try to reinvest all the profits and pay 5% of dividends. According to the theory developed in this section, we need first to calculate the parameter γ, that is given by

$$\gamma = \frac{1}{2} - \frac{0.1 - 0.05}{0.3^2} \pm \sqrt{\left[\frac{0.1 - 0.05}{0.3^2} - \frac{1}{2}\right]^2 + 2\frac{0.1}{0.3^2}}$$
$$\gamma_1 = 1.43 \text{ and } \gamma_2 = -1.54.$$

The next step is essential to obtain the critical value for which the manager would invest, and according to (2.14)

$$v^* = \frac{1.43}{1.43 - 1}10^8$$
$$v^* = 332.558.140$$

Substituting v^* in (2.15) we have $A_1 = 4.554 \ 10^{-13}$.

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[4]see [Tsa02] for calibration procedure to GBM

With all these quantities calculated above, the value of the option can be expressed as

$$F(v) = \begin{cases} 4.554 \ 10^{-13} v^{1.43} & \text{if } v \ \leq \ 332.558.140 \\ v - 100.000.000 & \text{if } v \ > \ 332.558.140 \end{cases}$$

At this stage, it is interesting to compare this method with the NPV. Note that with this analysis, the value of the asset must be more than three times the sunken cost for the project to be set up. With the NPV, it is only necessary that the static discounted cash flow to be as high as sunken cost to the manager invest.

The figures below illustrate the sensitivities of Table 2.1.

Figure 2.1: Sensitivities
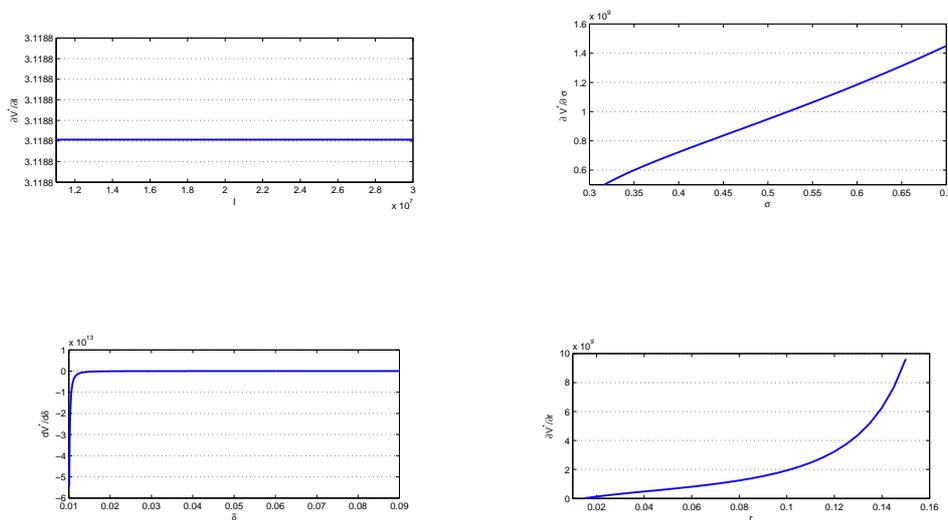
2.2 Project's Dynamic as an Exponential Mean Reverting Process

Exponential Mean Reverting Process are largely used to model contracts on interest rate. It was first proposed by [Vas77] and was largely accepted because it has the interesting characteristics of, in a long run, reverting to a mean. This is observed in interest rate and commodities price. The extension to business valuation was made by [MS00] to value the shares of a company.

Suppose now that the project's value $V(t)$ follows

$$dV(t) = \eta(\bar{V} - V(t))V dt + V(t)\sigma dW(t). \tag{2.18}$$

Now, we will suppose too that the dividend rate is a function that depends only on $V(t) = v$. Following the same arguments developed in Section (2.1), we arrive at

$$\frac{1}{2}\sigma^2 v^2 \partial_{vv} F + (r - \delta(v)) v \partial_v F - rF = 0. \tag{2.19}$$

At a first glance, this equation has the advantage of being independent of bias from discount rates and risk preferences. For this section we will follow the approach proposed by [DP96] to range more possibilities of valuation.

Let μ be the risk-adjusted discount rate, i.e. the rate that the agent is rewarded by assuming the systematic risk of the project. The gains from the project comes from the capital gain and from the dividend payment. Note that for the Exponential Mean Reverting Process the rate of growth is $\eta(\bar{V} - V)$. If we express the dividends as a function of v, we have

$$\delta(v) = \mu - \eta(\bar{V} - v) \tag{2.20}$$

Substituting (2.20) in (2.21)

$$\frac{1}{2}\sigma^2 v^2 \partial_{vv} F + (r - \mu + \eta(\bar{V} - v))v\partial_v F - rF = 0. \tag{2.21}$$

The boundary conditions remain the same due to the same arguments. The next step is to find the solution to the problem. Suppose that the solution has the form of

$$F(v) = Av^\theta h(v), \tag{2.22}$$

where A and θ are constant coefficients and $h(V)$ is a function satisfying a differential equation with known solution. Substituting (2.22) in (2.21) we have the following equation

$$0 = v^\theta h(v) \left[\frac{1}{2}\sigma^\theta(\theta - 1) + (r - \mu - \eta\bar{V})\theta - r\right]$$

$$+ v^{\theta+1} \left[\frac{1}{2}\sigma^2 v h''(v) + (\sigma^2\theta + r - \mu + \eta\bar{V} - \eta v)h'(v) - \eta\theta h(v) - \eta\theta h(v)\right]$$

$$. \tag{2.23}$$

Since (2.23) must hold for every v, the terms in brackets must be zero. Equating the first term to zero we obtain

$$\frac{1}{2}\sigma^\theta(\theta - 1) + (r - \mu - \eta\bar{V})\theta - r = 0, \tag{2.24}$$

where the positive solution is given by

$$\theta = \frac{1}{2} + \frac{\mu - r - \eta\bar{V}}{\sigma^2} + \sqrt{\left[\frac{\mu - r - \eta\bar{V}}{\sigma^2} - \frac{1}{2}\right]^2 + \frac{2r}{\sigma^2}}. \tag{2.25}$$

Equating the second term to zero,

$$\frac{1}{2}\sigma^2 vh''(v) + (\sigma^2\theta + r - \mu + \eta\bar{V} - \eta v)h'(v) - \eta\theta h(v) - \eta\theta h(v) = 0. \tag{2.26}$$

Making $x = 2\eta v/\sigma^2$, $g(x) = h(v)$ and substituting in (2.26) we arrive at Kummer's Equation

$$xg''(x) + (b - x)g'(x) - \theta g(x) = 0, \tag{2.27}$$

where

$$b = 2\theta + \frac{2(r - \mu + \eta\bar{V})}{\sigma^2}, \tag{2.28}$$

with known solution given by the confluent hypergeometric series

$$H(x; \theta, b). \tag{2.29}$$

This verify that the solution of (2.21) is

$$F(V) = Av^\theta H(\frac{2\eta}{\sigma^2}v; \theta, b). \tag{2.30}$$

We need now to determine A and v^*. The third boundary condition gives us that

$$F'(v^*) = 1$$

$$F'(v^*) = A\theta(v^*)^{\theta-1}H + A(v^*)^\theta\frac{\partial H}{\partial v},$$

which implies that

$$1 = A\theta(v^*)^{\theta-1}H + A(v^*)^\theta\frac{\partial H}{\partial v}$$

$$A = \left(\theta(v^*)^{\theta-1}H + (v^*)^\theta\frac{\partial H}{\partial v}\right)^{-1}. \tag{2.31}$$

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Substituting (2.31) in (2.30) and using the second boundary condition

$$v^* H / \left[\theta H + v^* \frac{\partial H}{\partial v} \right] = v^* - I$$

$$(v^*)^2 \frac{\partial H}{\partial v} + v^* H (\theta - 1) - I v^* \frac{\partial H}{\partial v} - I H \theta = 0. \qquad (2.32)$$

This nonlinear equation can be solved numerically. After calculations, we substitute the value of v^* in (2.30) and use the secondary boundary condition to determine the value of A.

2.2.1   An Example

Suppose again that an oil company is interested in building a refinery. The sunken cost to implement this is estimated in $I = 1$ per volume traded in the market. The cash flow generated by this building is due to the sales of the oil products that is reflected on the prices of the company shares. But now, we will assume that the dynamics of the oil products prices(which follows an exponential mean reverting process) is perfectly correlated with the project. After a calibration procedure, the suggested model, in the physical measure, is given by

$$dV = 2(3 - V)V \, dt + 2V \, dW(t). \qquad (2.33)$$

Substituting the values of (2.33) in (2.25), we have

$$\theta = \frac{1}{2} + \frac{0.07 - 0.05 - 3^3}{2^2} + \sqrt{\left[\frac{0.07 - 0.05 - 3^2}{2^2} - \frac{1}{2} \right]^2 + 2\frac{0.05}{2^2}}$$

$$\theta = 1.0063$$

Calculating b from (2.28)

$$b = 2 \; 1.0063 + \frac{2(0.05 - 0.07 + 2 \; 3)}{2^2}$$

$$b = 5.0026$$

Once we have the values from b and θ, we can determine the value of v^* from (2.32). Solving the nonlinear equation, we find that $v^* = 2.5147$. Consequently, $A = 0.3324$.

The solution is given by

$$F(v) = \begin{cases} 0.3324 v^{0.0063} H(v, 1.0063, 5.0026) & \text{if } v \leq 2.5147 \\ v - 1 & \text{if } v > 2.5147 \end{cases}$$

Again, note that the price of the commodity must be as high as twice and a half the sunk cost to be interesting for the manager to implement the project.

To analyze the sensitivity of v^* with respect to the parameters of the stochastic process, we vary them separately and see what happen with the solution v^*. Then, we plot the numerical derivatives.

Figure (2.2) illustrate the sensitivities of v^* with respect to its parameters

Figure 2.2: Sensitivities

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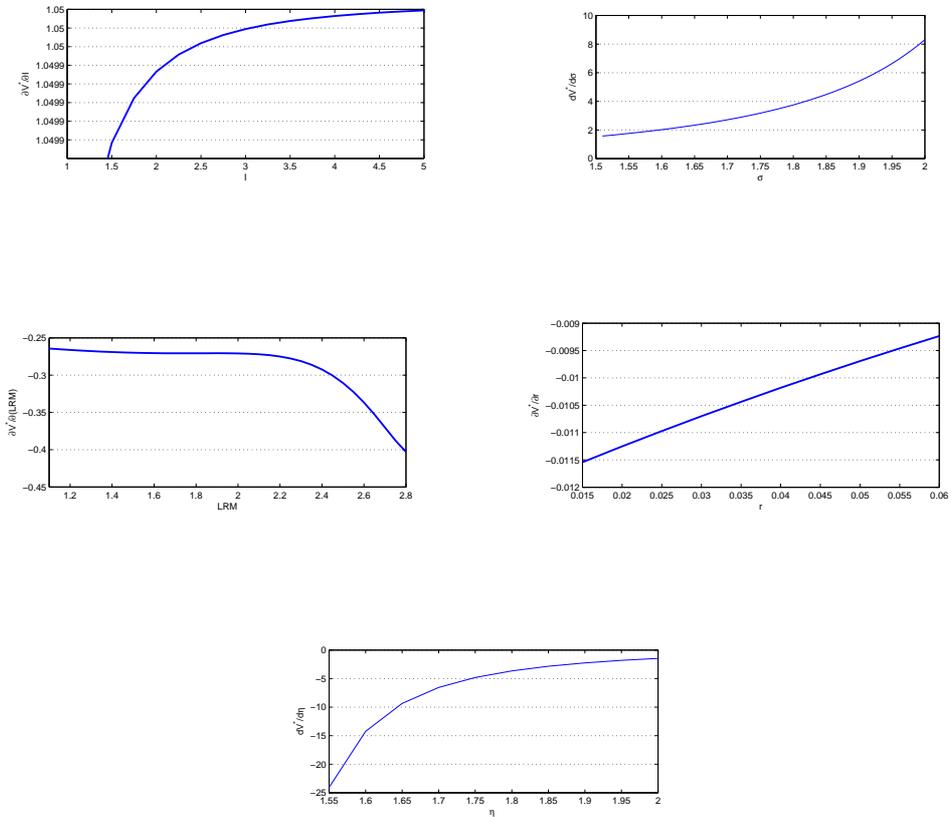
Chapter 3

Types of Contracts

The goal of the previous chapter was to provide an analysis of the project's dynamics. This chapter intends to answer questions related with the optionalities embedded on projects. For didactic reasons, we calculated, in the previous chapter, the cost of waiting to enter in a project. This is one type of optionality that can be valued using Real Options techniques. Now, we will discuss other types of contracts that can have their values evaluated.

3.1 The Abandonment Option

According to [KP06], abandonment options are presented virtually in every project. They are important to avoid large losses coming from projets that may affect company's finances. This optionality allows its owner to abandon a project(V_t) for a specific value (I), before incurring in major losses. For this characteristic, these options are largely used in research projects. Once the owner of the contract pays a value to avoid a major loss, the option to abandon can be viewed as an American put option, where its payoff, at time t, is $\max(I - V_t, 0)$. All numerical techniques to solve problems of pricing puts can be used to price the Abandonment option. The approach recommended by [KP06] and [Mun02] is the binomial tree due to its easy implementation and easy understanding. However, both of them do not explicitly say that they are assuming that the project follows a GBM. [Hul05] implements a trinomial tree for an EMR process of oil prices to calculate the value of a project with the embedded option to abandon it.

These examples illustrate how important it is to know the stochastic model suggested to a process. The calibration or the estimation procedure plays a central role in the results of the analysis. Once one has a good model to describe the dynamics of the project, it can be an important tool for

pricing financial derivatives.

3.1.1 Examples

Geometric Brownian Motion Context

Suppose that a commodity company is developing a new oil fuel for spaceships. Due to market potential, it is estimated that the sales of the project value today are $US\$500.000.000$. The company is faced, for the next 10 years, with the possibility of continuing the research or selling its intellectual property. The possibility of selling its rights gives reward it in 200.000.000. The estimated volatility is 50% for the log returns of the cash flow and the interest rate for the next 10 years is 6%.

To calculate the value to abandon the project, we calculate the price of an American put option with the parameters

- $S_0 = 500.000.000$

- $K = 200.000.000$

- $\sigma = 0.5$

- $r = 0.06$

- $T = 10$

To solve the problem, we can use any numerical method for calculating the price of an American put option. Thus, by using a binomial model, we obtain the price numerically and we get

Price	35.450.722

Since we are dealing with large money values, it is recommended that we use a fine grid in the numerical method adopted. Other methods could be used like Crank-Nicolson and Longstaff-Schwartz, described in [Bra06], to obtain the price. These methods can be interesting if we had a more exotic path dependent derivative. It is important to point out that a very small grid implies in large computational efforts that sometimes results in long calculations.

Table 3.1: Future Prices

Year	Future Price
2011	54
2012	63
2013	60
2014	68

Exponential Mean Reverting Context

Suppose that we are in the same context of the previous problem, but now the underlying asset follows an exponential mean reverting process as

$$d\mathrm{log}V = \left[\hat{V} - \mathrm{log}V\right]dt + dW(t). \tag{3.1}$$

The spot price of a similar product, that is supposed to replicate our project, is US$ 54,00. Its future prices are in Table 3.1.

Suppose that the interest rate is the same as in the previous example, therefore the estimated recovery for each product is US$ 40,00 and the time of development is the next four years. To evaluate the price of the contract, we will use the trinomial tree method described in Appendix A.

After calculations, the price of each contract is

Price	5.72

3.2   Option to Expand

Suppose that a company is performing well on market and it intends to expand its production to keep growing. In this process of expansion many possibilities must be analyzed. They include the building of a new factory, acquisition of competitor or simply the expansion of the industrial park. Note that all of them offer the opportunity of receiving a revenue(V_t) by the price of the sunk cost of implementation(I), discounted to the exercise time. Usually, as described in [Hul05], the strike price depends on the initial investment. Drawing the parallel with the financial market, the manager is facing with a problem of pricing an American call option, where the payoff is $\max(V_t - I, 0)$, on the value of additional capacity installed. Again, it can be solved with traditional techniques.

3.2.1 Examples

Geometric Brownian Motion Context

Suppose that a commodity company is performing very well due to an economic boom in a developing country. It faces with the opportunity to expand its current operations at a total cost of US\$ 50 millions, during the next 4 years. The present value of the expansion is US\$ 20 millions and the volatility is estimated in 50%. The interest rate is 6% a year.

To calculate the value of expanding the operation, we compute the price of an American call option with the parameters

- $S_0 = 20.000.000$

- $K = 50.000.000$

- $\sigma = 0.5$

- $r = 0.06$

- $T = 4$

To solve the problem, we can use any numerical method for calculating the price of an American put option. The following table presents the price calculated using a binomial tree technique

Price	610.019,12

Exponential Mean Reverting Context

Suppose that we are faced with the possibility of expansion, but the dynamics follow the same process of Section 3.1.1. Our problem can be solved by the same method with a different final condition.

Suppose that all the previous parameters are the same and that the additional cost to the expansion is US\$ 40. The interest rate is the same as in the previous example. The estimated recovery for each product is US\$ 150,00 and the time of development is the next four years. To evaluate the price of the contract, we will use the trinomial tree method described in Appendix A.

After the calculation, the price of each contract is

Price	27.73

3.3 Abandonment and Expansion Simultaneously

Another case of interest is when a project has both options. In most cases, this optionalities are dependent, meaning that the value of having both optionalities is not the sum of the value that each of them has when analyzed separately. Indeed, the manager needs to be careful in considering the options once some exercise order doesn't make sense (for example, if the project was already abandoned it cannot be expanded). This lead us to four possibilities at any arbitrary time:

1. Not abandon and not expand

2. Expanded and not abandoned

3. Abandoned and consequently not expanded

4. Expanded and after this abandoned

To calculate the price of an option with these two optionalities requires a more sophisticated technique. Since the problem is a path-dependent option, we can use the trinomial tree and compare the four possibilities in each node, taking the one that increase the payoff or use Longstaff-Schwartz method.

To illustrate the procedure, suppose a contract with these two optionalities and at the same conditions as stated before. If we use the trinomial tree to find the price of this contract, we have

Price	29.96

One important remark needs to be made. Note that in this section we simply transfer the techniques from financial market and apply them to real assets. But all these techniques were developed for financial instruments where the time plays a crucial role. One of the previous assumptions was that the price of the contract was time independent, since we were dealing with a perpetuity.

The modeling adopted in this chapter is not better or worst than that developed in Chapter 2. It is simply different. The manager can justified that is only reasonable to assume the process of the project in a finite horizon of time. After this, it can consider there are so many uncertainties that the cash flow cannot be estimated and it would be a huge error if the analysis tried to model such cash flow.

To maintain the coherence of the work, we will analyze the same kind of contract assuming a perpetuity of the cash flow of the project, and consequently the time independence of the contract value.

3.4 Abandonment and Expansion Simultaneously in a Perpetuity

For this section we will follow the approach suggested by [DP96]. Suppose the company has the opportunity of investing in a project that demands an initial capital investment I to be activated. Remember that the option's price is a function of the stochastic process V_t. We will consider now the cash flow of the project is due to product's sales which prices follow

$$dP = \alpha P dt + \sigma P dW(t) \tag{3.2}$$

This price will determine if the company will invest and, consequently exercise the option to expand, but also will ascertain if the operations must be suspended, meaning that the abandonment option could be exercised.

Let's call P_H the value that makes the investment attractive, P_L the price for which machines should be turned off, by a fixed cost E and $F_0(P)$ the price of the option of investing. The prices determine three regions within all price possibilities. The main idea will identify the ODE that must be satisfied by the options in each region. After this, we will be able to determine the boundaries prices and the value of the optionalities.

Suppose the price of the product that a company sells is in the interval $P_L < P < P_H$. Thus, the option of expanding, F_0, is alive and using the same argument in Section 2.1 we have that F_0 must satisfy

$$\frac{1}{2}\sigma^2 v^2 F_0'' + (r - \delta) v F_0' - r F_0 = 0. \tag{3.3}$$

If we impose $F_0(0) = 0$ must be satisfied, which have the same interpretation as before, we should have a solution that is given by

$$F_0(P) = A_1 P^{\gamma_1}, \tag{3.4}$$

where A_1 is a constant that will be determined by the other boundary conditions and γ_1 can be determined as the positive roots of the quadratic equation obtained when one substitutes (3.4) in (3.3).

Suppose the sales are doing well, the price P_H is reached and the expansion is taken. After the expansion, the option to abandon comes alive and, if there is a big drop in the prices, it can be exercised.

To arrive at the ODE that is satisfied by the option to abandon, we repeat the same arguments used in (2.1) with a slight difference. Once the project was implemented, the instantaneous profit that flows from the project is $\max(P - C, 0)$, where C is the cost flow of the project.

Thus, proceeding exactly as before, we arrive at the following ODE

$$\frac{1}{2}\sigma^2 v^2 F_1'' + (r-\delta)v F_1' - rF_1 + P - C = 0. \qquad (3.5)$$

The general solution is given by the sum of the solutions for the homogeneous parts and a particular solution. Given that $F_1(P) = \frac{P}{\delta} - \frac{C}{r}$ is a particular solution, we have that

$$F_1(P) = B_1 P^{\gamma_1} + B_2 P^{\gamma_2} + \frac{P}{\delta} - \frac{C}{r}. \qquad (3.6)$$

The parameters γ are determined as before. But now, if the prices become too high, the option of abandonment must have a zero value. This implies that $B_1 = 0$ to the solution converges to zero as $P \to \infty$, which reduces the solution to

$$F_1(P) = B_2 P^{\gamma_2} + \frac{P}{\delta} - \frac{C}{r}. \qquad (3.7)$$

For no arbitrage arguments, at the critical prices P_L e P_H the value of the options must be equal to their payoff. This condition with the smooth pasting condition lead us to the following boundary condition

$$\begin{aligned}
F_1(P_H) - I &= F_0(P_H) \\
F_0'(P_H) &= F_1'(P_H) \\
F_1(P_L) &= F_0(P_L) - E \\
F_1'(P_H) &= F_1'(P_H).
\end{aligned} \qquad (3.8)$$

Note that the system depends on the derivatives of both functions. Differentiating (3.4) and (3.7), we have

$$F_0'(P) = A_1 \gamma_1 P^{(\gamma_1 - 1)} \qquad (3.9)$$

$$F_1'(P) = B_2 \gamma_2 P^{(\gamma_2 - 1)} + \frac{1}{\delta} \qquad (3.10)$$

Substituting (3.4), (3.7), (3.9) and (3.10) in (3.8),

$$A_1 P_H^{\gamma_1} = B_2 P_H^{\gamma_2} + \frac{P_H}{\delta} - \frac{C}{r} - I$$

$$A_1 \gamma_1 P_H^{(\gamma_1 - 1)} = B_2 \gamma_2 P_H^{(\gamma_2 - 1)} + \frac{1}{\delta}$$

$$A_1 \gamma_1 P_L^{(\gamma_1 - 1)} - E = B_2 P_L^{(\gamma_2)} + \frac{P_L}{\delta} - \frac{C}{r}$$

$$A_1 \gamma_1 P_L^{(\gamma_1 - 1)} = B_2 \gamma_2 P_L^{(\gamma_2 - 1)} + \frac{1}{\delta}.$$

Finally, all the unknown parameters (A_1, B_2, P_L, P_H) can be obtained numerically solving the system above.

3.4.1 An Example

Suppose after a calibration procedure, the estimated process, in the risk neutral measure, for a project is

$$dP = 0.1Pdt + 0.5PdW(t) \tag{3.11}$$

The company pays $\delta = 5\%$ in dividends and the operating cost is $C = 100$. The fixed cost to abandon is $E = 30$ and the investment to enter in the project is $I = 70$. Considering $r = 10\%$ and using (2.12), we have $\gamma_1 = 1.24$ and $\gamma_2 = -0.64$. To solve the system, we used the function 'fsolve' from MATLAB. After calculations,

$$A_1 = 4.4$$
$$B_2 = 7637.52$$
$$P_L = 69.12$$
$$P_H = 160.49$$

$$\tag{3.12}$$

Chapter 4

A New Perspective

After the understanding of the classical models, this chapter deals with the problem of pricing real options when the dynamic of the project is a pure mean reverting process. This model can sometimes be found in projects of commodities companies. As described in [MR05], it was proposed by Bachelier to model derivatives securities.

Suppose the cash flow of a commodity company follows a pure mean reverting process like

$$dS(t) = \kappa(\mu - S(t))dt + \sigma dW(t). \tag{4.1}$$

Let us assume that $V(t)$ is the PV associated, at time t, to the cash flow $S(t)$. Thus, we can define the PV as

$$V(t) := \mathbb{E}\Big[\int_t^\infty e^{-\rho(s-t)} S(s) ds | S(t) \Big]. \tag{4.2}$$

Note the difference between this approach and the previous one. In the previous chapter, the cash flow was considered the dividends payed by the project. Now, we are suggesting to first model the cash flow with a stochastic behavior and then define the value of the project as a function of the cash flow.

Since $S(t)$ follows Ornstein-Uhlenbeck process, we know that $S(t)$ is normal with conditional mean

$$\mathbb{E}[S(s)|S(t)] = S(t)e^{-\kappa(s-t)} + \mu(1 - e^{-\kappa(s-t)}), \ \ t < s, \tag{4.3}$$

and conditional variance given by

$$\text{var}[S(s)|S(t)] = \sigma^2 \frac{1 - \exp(-2\kappa(s-t))}{2\kappa}, \ \ t < s. \tag{4.4}$$

With the central absolute moment for normal random variables one can verify that Fubini's Theorem hypothesis are satisfied. Therefore, we can reverse the order of integration and by the property (4.4), we can evaluate the expression (4.2) as

$$
\begin{aligned}
V(t) &= \mathbb{E}\Big[\int_t^\infty e^{-r(s-t)} S(s) ds | S(t) \Big] \\
&= \int_t^\infty e^{-r(s-t)} \mathbb{E}[S(s)|S(t)] ds \\
&= \int_t^\infty e^{-r(s-t)} \Big(S(t) e^{-\kappa(s-t)} + \mu(1 - e^{-\kappa(s-t)}) \Big) ds \\
&= (S(t) - \mu) \int_t^\infty e^{-(r+\kappa)(s-t)} ds + \mu \int_t^\infty e^{-r(s-t)} ds \\
&= \frac{S(t) - \mu}{r + \kappa} + \frac{\mu}{r} \\
&= \frac{S(t) + \mu\kappa/r}{r + \kappa}.
\end{aligned}
$$

Expressing $S(t)$ in terms of $V(t)$, we have the following relation

$$
S(t) = (r + \kappa)\Big(V(t) - \frac{\mu}{r}\Big) + \mu. \tag{4.5}
$$

In the differential form, the expression becomes

$$
dS(t) = (r + \kappa)dV(t). \tag{4.6}
$$

By the previous relations, one can determine the dynamics of $V(t)$ substituting (4.5) and (4.6) in (4.1)

$$
\begin{aligned}
dS(t) &= \kappa(\mu - S(t))dt + \sigma dW(t) \\
(r+\kappa)dV(t) &= \kappa\Big(\mu - (r+\kappa)\big(V(t) - \frac{\mu}{r}\big) - \mu\Big)dt + \sigma dW(t) \\
(r+\kappa)dV(t) &= \kappa\Big(\mu + (r+\kappa)(\frac{\mu}{r} - V(t)) - \mu\Big)dt + \sigma dW(t) \\
(r+\kappa)dV(t) &= \kappa(r+\kappa)\Big(\frac{\mu}{r} - V(t)\Big)dt + \sigma dW(t) \\
dV(t) &= \kappa\Big(\frac{\mu}{r} - V(t)\Big)dt + \frac{\sigma}{r+\kappa}dW(t).
\end{aligned}
$$

Note that the dynamics of $V(t)$ also follows an Ornstein-Uhlenbeck process in the physical measure. As we are interested in pricing derivatives on the project, we need to find both the dynamics in the risk neutral measure to be able to price using Contingent Claim analysis.

Since we are dealing with a commodity company, it is reasonable assume the project could be replicated by the existing contracts on the commodity market. This is reasonable since the inflow of the project comes mainly from the production of the commodity. At this stage, we will consider we are in a complete market framework. We will assume that is possible for the investor to replicate the payoff of one option on $V(t)$, just trading the bond and a contract $V(t)$. We will assume the dynamics of $V(t)$ follows the equation

$$dV(t) = \kappa(\mu_2 - V(t))dt + \sigma_2 dW(t),$$

where $\mu_2 = \frac{\mu}{r}$ and $\sigma_2 = \frac{\sigma}{r+\kappa}$.

By Girsanov, we are able to change the measure to price the contingent claim in the risk neutral world. We introduce a market price of risk to eliminate the parameter μ_2 of the dynamics of $V(t)$. The expected growth must be the risk free rate. The following equation must be satisfied by the market price of risk θ for the required expected growth

$$\kappa(\mu_2 - V(t)) - \sigma_2\theta_t = rV(t). \tag{4.7}$$

Rearranging the terms, we have the following expression for θ_t

$$\theta_t = \frac{\kappa(\mu_2 - V(t)) - rV(t)}{\sigma_2}, \tag{4.8}$$

which gives the risk neutral dynamic below for $V(t)$

$$dV(t) = rV_t dt + \sigma_2 dW^{\mathbb{Q}}(t). \tag{4.9}$$

To obtain the solution for this equation, we take $f(t, V(t)) = V(t)e^{-rt}$ and apply Ito's formula to the expression, obtaining

$$df = -rV(t)e^{-rs}dt + e^{-rs}dV(s)$$
$$d(V(s)e^{-rs}) = -rV(s)e^{-rs}dt + rV(s)e^{-rs} + e^{-rs}\sigma dW^{\mathbb{Q}}(s)$$
$$d(V(s)e^{-rs}) = e^{-rs}\sigma dW^{\mathbb{Q}}(s). \tag{4.10}$$

Integrating from 0 to t,

$$V(t)e^{-rt} - V(0) = \int_0^t e^{-rs}\sigma dW^{\mathbb{Q}}(s)$$

$$V(t)e^{-rt} = V(0) + \int_0^t e^{-rs}\sigma dW^{\mathbb{Q}}(s)$$

$$V(t) = e^{rt}\left(V(0) + \int_0^t e^{-rs}\sigma dW^{\mathbb{Q}}(s)\right). \tag{4.11}$$

The expression (4.11) shows that $V(t)e^{-rt}$ is a gaussian variable with zero mean and variance given by

$$\operatorname{var}^2(0,t) = \int_0^t \left(e^{-rs}\sigma\right)^2 ds$$

$$= \sigma^2 \int_0^t e^{-2rs} ds$$

$$= \frac{\sigma^2}{2r}\left(1 - e^{-2rt}\right). \tag{4.12}$$

To calculate the price of an European call option, we evaluate the expression

$$P(0, V(0)) = \mathbb{E}^{\mathbb{Q}}\left[e^{-rT}(V(t) - K)^+ \big| \mathcal{F}_0\right]$$

$$P(0, V(0)) = e^{-rT} \int_{-d_1}^{\infty} \left(V(0)e^{rT} + e^{rT}\operatorname{var}(0,T)x - K\right) e^{-\frac{x^2}{2}} \frac{1}{\sqrt{2\pi}} dx$$

$$P(0, V(0)) = V(0)\left(1 - \Phi(-d_1)\right) + \int_{-d_1}^{\infty} \operatorname{var}(0,T)x e^{-\frac{x^2}{2}} \frac{1}{\sqrt{2\pi}} dx - Ke^{-rT}\left(1 - \Phi(-d_1)\right)$$

$$P(0, V(0)) = \left(V(0) - Ke^{-rT}\right) \Phi(d_1) + \operatorname{var}(0,T)\varphi(-d1),$$

where d_1 is defined by

$$d_1 := \frac{V(0) - Ke^{-rT}}{\operatorname{var}(0,T)},$$

and φ is the normal density. Since it is symmetric in y axis, we have $\varphi(-d_1) = \varphi(d_1)$. Thus

$$P(0, V(0)) = \left(V(0) - Ke^{-rT}\right)\Phi(d_1) + \operatorname{var}(0,T)\varphi(d_1).$$

The price for an arbitrary time t can be found proceeding exactly as before, considering that $V(t) \in \mathcal{F}_t$ and $W^{\mathbb{Q}}(T) - W^{\mathbb{Q}}(t)$ is independent of $\mathcal{F}_t$ under $\mathbb{Q}$. The price at time t is given by

$$P(t, V(t)) = \left(V(t) - Ke^{-r(T-t)}\right)\Phi(d_1) + \operatorname{var}(t,T)\varphi(d_1), \tag{4.13}$$

where d_1 is now defined by

$$d_1 := \frac{V(t) - Ke^{-r(T-t)}}{\operatorname{var}(t,T)}.$$

On the other hand, we can use the approach introduced by Black and Scholes to derive the PDE associated with this stochastic process. Let $P_t = P(t, V(t))$ be a contingent claim on $V(t)$. We will follow the approach

described in [KK01], that consists in following a self-financing trading strategy (x_0, x_1) in project and the bond, with price $S_0(t)$, such that the wealth process including exactly one sold contract

$$\Pi(t) = x_0(t)S_0(t) + x_1(t)V(t) - P(t, V(t)) \qquad (4.14)$$

In differential form,

$$d\Pi(t) = x_0(t)dS_0(t) + x_1(t)dV(t) - dP(t, V(t)). \qquad (4.15)$$

We will suppose that the function $P(t, V(t))$ is smooth enough to apply Ito Formula and obtain

$$dP = \partial_t P dt + \partial_v P dV(t) + \frac{1}{2}\partial_{vv}P d[V(t), V(t)], \qquad (4.16)$$

that leads us to

$$dΠ(t) = \left[x_0(t)S_0(t)r + x_1(t)\kappa(\mu_2 - V(t)) - \left(\partial_t P dt + \partial_v P\kappa(\mu_2 - V(t)) + \frac{\sigma_2^2}{2}\partial_{vv}P\right)\right]dt$$
$$+ (x_1(t)V(t)\sigma - \partial_v P V(t)\sigma)\, dW(t). \qquad (4.17)$$

For $P(t)$ to be a wealth process corresponding to a riskless portfolio, the last term of (4.17) must vanish. Hence,

$$x_1(t) = \partial_v P(t, V(t)) \qquad (4.18)$$

For arbitrage reasons, all the variations of the wealth process $\Pi(t)$ must be due to inflow of the interest rate less the consumption of dividends, implying that

$$d\Pi(t) = r\Pi(t)dt - x_1(t)D(v)dt \qquad (4.19)$$

Substituting (4.19) and (4.18) in (4.17),

$$r\Pi(t) - x_1(t)D(v) = x_0(t)S_0(t)r + x_1(t)P\kappa(\mu_2 - V(t))$$
$$- (\partial_t P dt + \partial_v P\kappa(\mu_2 - V(t)) + \frac{1}{2}\sigma^2 P)$$
$$= r[x_0(t)S_0(t) + x_1(t)V(t) - P] + Pr - x_1(t)V(t)r$$
$$- \left(rV(t)\partial_v P - \partial_t P + \partial_v P\kappa(\mu_2 - V(t)) + \frac{\sigma^2}{2}\partial_{vv}P\right)$$
$$+ x_1(t)P\kappa(\mu_2 - V(t))$$
$$= r\Pi(t) - \left[\partial_t P + rV(t)\partial_v P + \frac{\sigma_2^2}{2}\partial_{vv}P - rP\right]. \qquad (4.20)$$

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Hence, for $V(t) = v$ and $t \geq 0$

$$\partial_t P + (rv - D(v))\partial_v P + \frac{\sigma_2^2}{2}\partial_{vv}P - rP = 0. \quad (4.21)$$

Note that this equation is similar to Black and Scholes PDE except by the term

$$\frac{\sigma_2^2}{2}\partial_{vv}P,$$

that does not have the dependency on the underlying squared. To better understand the behavior of this PDE, we will simplify it taking $D(v) = 0$ and analyzing this simple case.

From an economical perspective, at a first glance, one can think that this is the case when a company does not pay dividends. In fact, this is true, but this is not as immediate as one may realize. The reason for this is explained in the next section, when more properties of the dynamics of the process would be enlightened. Now, assume this is true, then the equation becomes

$$\partial_t P + rv\partial_v P + \frac{\sigma^2}{2}\partial_{vv}P - rP = 0. \quad (4.22)$$

Conversely, we know by Feymann-Kac's Formula, the associated stochastic differential equation is

$$dV(t) = rV(t)dt + \sigma_2 dW^{\mathbb{Q}}(t). \quad (4.23)$$

Now, we will obtain the sensibility of the price of the option with respect to the parameters, we derive (4.13) with respect to the desired parameter.

We first compute the sensitivity with respect to the underlying project. This is

$$\Delta = \frac{\partial P}{\partial V(t)}$$

$$\Delta = \Phi(d_1) + \left(V(t) - Ke^{-r(T-t)}\right)\varphi(d_1)\frac{\partial d_1}{\partial V(t)} + \mathrm{var}(t,T)\varphi'(d_1)\frac{\partial d_1}{\partial V(t)}. \quad (4.24)$$

Using the fact that

$$\varphi'(d_1) = -d_1\varphi(d_1) = -\frac{V(t) - Ke^{-r(T-t)}}{\mathrm{var}(t,T)}\varphi(d_1), \quad (4.25)$$

it follows that

$$\left(V(t) - Ke^{r(T-t)}\right)\varphi(d_1) + \mathrm{var}(t,T)\varphi'(d_1) = 0, \quad (4.26)$$

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and (4.24) reduces to

$$\Delta = \Phi(d_1). \tag{4.27}$$

Note that, as in the Black and Scholes model, the delta has the same expression.

To calculate the Gamma, we differentiate (4.27 with respect to $V(t)$, obtaining

$$\Gamma = \frac{\partial^2 P}{\partial V(t)^2}$$

$$\Gamma = \varphi(d_1)\frac{\partial d_1}{\partial V(t)}$$

$$\Gamma = \frac{\varphi(d_1)}{\text{var}(t,T)}. \tag{4.28}$$

To obtain the Theta, we differentiate (4.11) with respect to t, obtaining

$$\Theta = \frac{\partial P}{\partial t}$$

$$\Theta = -rKe^{-r(T-t)}\Phi(d_1) + \left[V(t) - Ke^{-r(T-t)}\right]\varphi(d_1)\frac{\partial d_1}{\partial t}$$

$$+ \frac{\partial \text{var}(t,T)}{\partial t}\varphi(d_1) + var(t,T)\varphi'(d_1)\frac{\partial d_1}{\partial t}$$

$$\Theta = -rKe^{-r(T-t)}\Phi(d_1) + \frac{\partial \text{var}(t,T)}{\partial t}\varphi(d_1). \tag{4.29}$$

Evaluating $\frac{\partial \text{var}(t,T)}{\partial t}$

$$\frac{\partial \text{var}(t,T)}{\partial t} = -\sqrt{\frac{\sigma^2}{2r}\frac{1}{2}}\left(e^{-2rt} - e^{-2rT}\right)^{1/2} re^{-2rt}$$

$$\frac{\partial \text{var}(t,T)}{\partial t} = -\frac{\sigma^2 e^{-2rt}}{2\text{var}(t,T)}, \tag{4.30}$$

and substituting in (4.31),

$$\Theta = -rKe^{-r(T-t)}\Phi(d_1) - \frac{\sigma^2 e^{-2rt}}{2\text{var}(t,T)}\varphi(d_1). \tag{4.31}$$

To calculate the Λ, we will use again the relation (4.26) to simplify the

calculation, obtaining

$$\Lambda = \frac{\partial P}{\partial K}$$

$$\Lambda = -e^{r(T-t)}\Phi(d_1) - \left[(V(t) - Ke^{-r(T-t)})\varphi(d_1) + \text{var}(t,T)\varphi'(d_1)\right]\frac{\partial d_1}{\partial K}$$

$$\Lambda = -e^{r(T-t)}\Phi(d_1). \tag{4.32}$$

Finally, to calculate the $Vega$, we use again (4.26) and obtain

$$Vega = \frac{\partial P}{\partial \sigma}$$

$$Vega = \left(\left[V(t) - Ke^{-r(T-t)}\right]\varphi(d_1) + \text{var}(t,T)\varphi'(d_1)\right)\frac{\partial d_1}{\partial \sigma} + \frac{\partial \text{var}(t,T)}{\partial \sigma}\varphi(d_1)$$

$$Vega = \varphi(d_1)\frac{\partial d_1}{\partial \sigma}. \tag{4.33}$$

As $\frac{\partial d_1}{\partial \sigma} = -\frac{d_1}{\sigma}$, we have

$$Vega = -\frac{d_1\varphi(d_1)}{\sigma}. \tag{4.34}$$

Table (4.1) summarizes the behavior of the contract with respect to these parameters.

Table 4.1: Sensitivities

Derivatives
$\frac{\partial P}{\partial V(t)} > 0$
$\frac{\partial P}{\partial V(t)^2} < 0$
$\frac{\partial P}{\partial t} < 0$
$\frac{\partial P}{\partial \sigma} > 0$
$\frac{\partial P}{\partial K} < 0$

4.1　An Example

As an example, suppose P_t is a call option over an investment amount of 10MM. Suppose also the time to maturity is 2 years and σ of the cash flow of the company is 50% in a year. The interest rate for this model is assumed to

be 0% in a year. We vary the NPV to show the behavior of the price of the option. We use (4.13) to calculate the option price, varying the investment and maintaining all other parameters fixed.

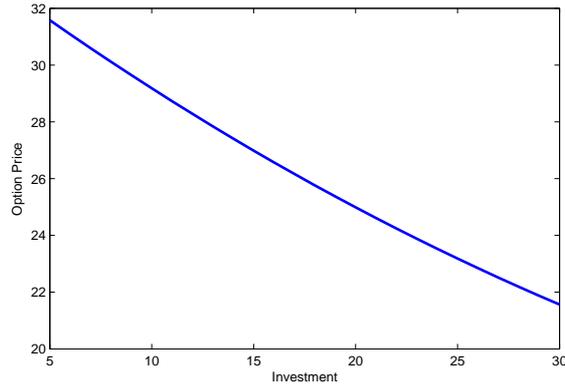Figure (4.1) shows how the value of the call option varies with changes in the strike.

Figure 4.1: Investment vs Option Price

This model is good to get a feeling of how worth is the option to postpone a project. But many of assumption are not reasonable. Both hypothesis are very strong and we want to develop a model that can consider interest and dividends. In the next chapter we try to relax these hypothesis.

4.2 Fixed Dividend Policies

Most of company has well established dividend policy. They pay a fixed amount of the yearly cash flow. In this section, we will assume that the company pays a fraction of the cash flow as dividends. It is important to note the difference between this model and others at this point. In the Black and Scholes framework, the dividend is a fraction of the underlying of the option. In our case, our underlying is the process $V(t)$ and not $S(t)$, over which the dividend is paid. This leads us to derive the relation between $V(t)$ and $S(t)$ in the risk neutral measure, once we have this only in the physical measure.

Again by Girsanov's Theorem, we introduce the same price of risk as (4.8) to obtain $S(t)$ in the risk neutral measure.

Thus,

$$
\begin{aligned}
dS(t) &= \kappa(\mu - S(t))dt + \sigma dW(t) \\
&= \kappa(\mu - S(t))dt + \sigma dW^{\mathbb{Q}}(t) - \sigma\theta_t dt \\
&= \kappa(\mu - S(t))dt + \sigma dW^{\mathbb{Q}}(t) - \sigma\left(\frac{\kappa(\mu_2 - V(t)) - rV(t)}{\sigma_2}\right)dt \\
&= \kappa(\mu - S(t))dt + \sigma dW^{\mathbb{Q}}(t) - (\kappa(r+\kappa)(\mu_2 - V(t)) - (r+\kappa)rV(t))dt \\
&= \kappa(\mu - S(t))dt + \sigma dW^{\mathbb{Q}}(t) + \left((\kappa + r)^2 V(t) - (r+\kappa)\kappa\mu_2\right)dt \\
&= \left(\kappa\mu - \kappa S(t) + (\kappa + r)^2 V(t) - \frac{(r+\kappa)\kappa\mu}{r}\right)dt + \sigma dW^{\mathbb{Q}}(t) \\
&= \left((r+\kappa)\left(S(t) + \frac{\mu\kappa}{r}\right) - \frac{(r+\kappa)\kappa\mu}{r} + \mu\kappa - \kappa S(t)\right)dt + \sigma dW^{\mathbb{Q}}(t) \\
&= (\mu\kappa + rS(t))dt + \sigma dW^{\mathbb{Q}}(t).
\end{aligned}
$$

Now we have both dynamics in the risk neutral risk, that are given by

$$
dS(t) = (\mu\kappa + rS(t))dt + \sigma dW^{\mathbb{Q}}(t), \tag{4.35}
$$

and

$$
(r+\kappa)dV(t) = (r+\kappa)rV(t)dt + \sigma dW^{\mathbb{Q}}(t). \tag{4.36}
$$

Subtracting (4.35) from (4.36), we have

$$
dS(t) - (r+\kappa)dV(t) = \mu\kappa dt + r\left(S(t) - (r+\kappa)V(t)\right)dt. \tag{4.37}
$$

Calling $u = S(t) - (r+\kappa)V(t)$ we have the following SDE

$$
du = (\mu\kappa + ru)dt. \tag{4.38}
$$

The equation above does not have the stochastic behavior, thus we can treat it as an ODE. Noting that a particular solution is given by $-\frac{\mu\kappa}{r}$ and solving $\frac{du}{dt} = ru$, we have the general solution is given by

$$
u = c_1 e^{rt} - \frac{\mu\kappa}{r}. \tag{4.39}
$$

This gives us the relation between $V(t)$ and $S(t)$ in the risk neutral measure, that is

$$
S(t) = (r+\kappa)V(t) - \frac{\mu\kappa}{r} + c_1 e^{rt}. \tag{4.40}
$$

As $S(t)$ is a Gaussian process, with finite expectation, we have, when the time goes to infinity, c_1 must be zero otherwise $S(t)$ would not have finite expectation.Thus,

$$S(t) = (r + \kappa)V(t) - \frac{\mu\kappa}{r}. \tag{4.41}$$

Now, we define

$$D(v) = -\delta\Big((r + \kappa)v - \frac{\mu\kappa}{r}\Big), \tag{4.42}$$

and substitute the expression in the PDE in the beginning of this section. This leads us to,

$$\partial_t P + \partial_v P\big(rv + D(v)\big) + \frac{\sigma^2}{2}\partial_{vv}P - rP =$$

$$\partial_t P + \partial_v P\left(rv - \delta\left((r + \kappa)v + \frac{\mu\kappa}{r}\right)\right) + \frac{\sigma^2}{2}\partial_{vv}P - rP =$$

$$\partial_t P + \partial_v P\Big(\big(r - \delta r - \delta\kappa\big)v - \frac{\delta\mu\kappa}{r}\Big) + \frac{\sigma^2}{2}\partial_{vv}P - rP = 0 . \tag{4.43}$$

This is a well known PDE which the existence and uniqueness of the solution is guarantee, as stated by [Eva10]. For our numerical purpose, this PDE can be solved using numerical analysis techniques like Crank-Nicolson method.

Chapter 5

Numerical Results and Simulations

In this section, we solve the problems in a fixed dividend framework. We will assume that a commodity company has a project with $V_0 = 5MM$ and the required investment is $I = 10MM$. For this initial model, assume that $r = 0$ and $\delta = 0$.

Suppose, after the calibration procedure, the following process for S_t is presented to model a commodity company project cash flow.

$$dS_t = 5(0.05 - S_t)dt + 20dW_t. \tag{5.1}$$

For a two year European and an American Call Option, under zero dividend policy, we have the following prices

	European	American
$Price$	0.57MM	0.56MM

The European price is calculated with the formula developed in (4) and the American price is calculated numerically. We can use the same arguments presented in [KK01] to justify the value of both contracts must be the same as in the Black and Scholes model. The difference between them is due to numerical errors and the results can be used to check the accuracy of the implemented algorithm. Figure 5.1 shows how the Option Price varies with the NPV

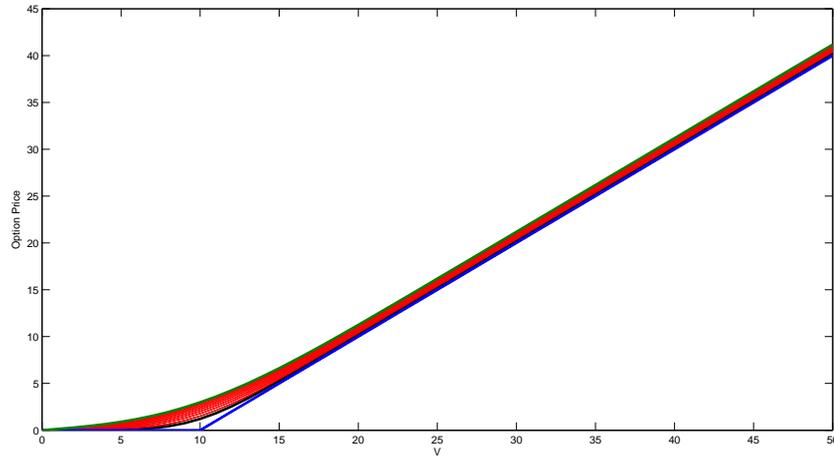Figure 5.1: Option Price for a 0% interest rate
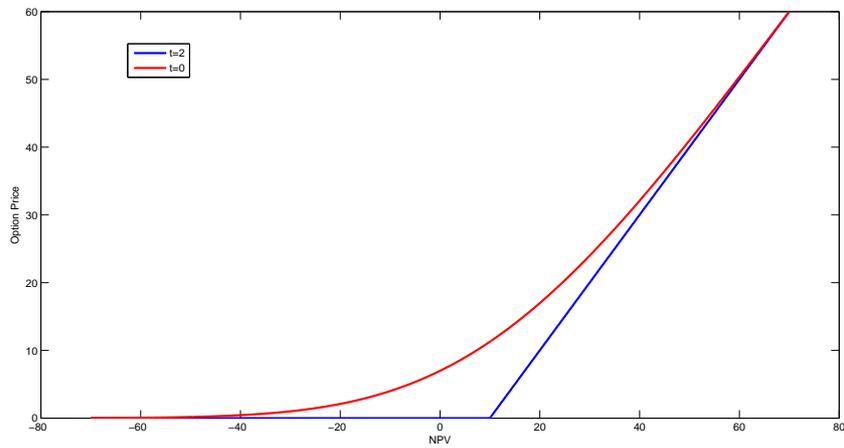
and the behavior with the Time and NPV

Figure 5.2: Option Price Surface
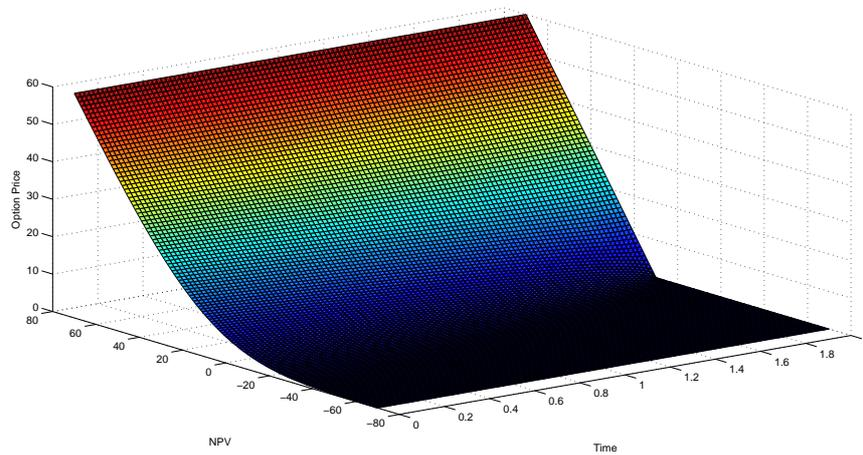
Varying the maturity in the interval $0 \leq T \leq 2.5$, we have the Figure 5.3

Figure 5.3: Option price for several maturities

Now we consider a economy with $r = 5\%$ interest rate. All other hypothesis remains the same. The prices for these contracts are
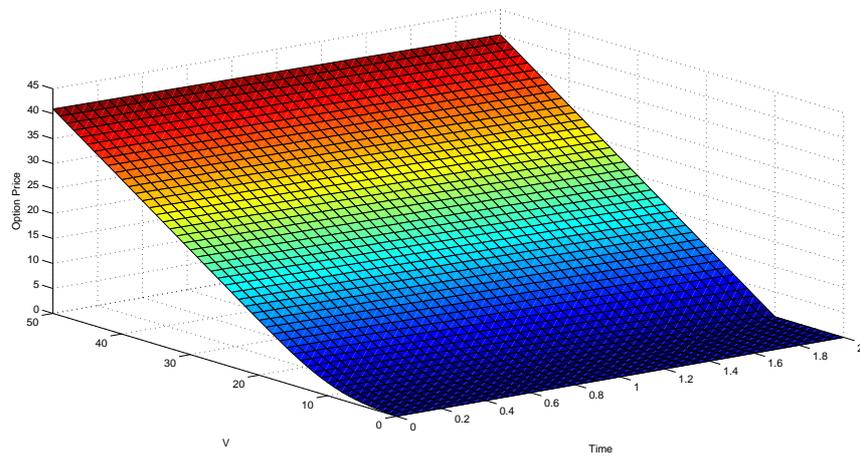
	European	American
$Price$	0.69MM	0.68MM

As expected, increasing the interest rate causes the increase in the price of the option price. The chart of the option price is

Figure 5.4: Option price for a 5% interest rate
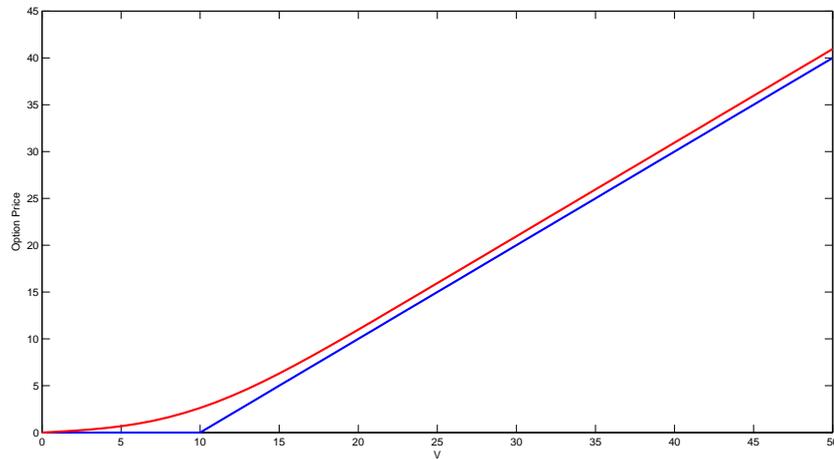
and the behavior with the Time and NPV

Figure 5.5: Option price surface for a 5% interest rate

We turn to the case of the Fixed Dividend PDE. Suppose the cash flow S_t from a project follows the stochastic process below,

$$dS_t = 5(0.2 - S_t)dt + 55dW_t. \tag{5.2}$$

The interest rate is 5%. The sunken cost is estimated as $10MM$ and the NPV at time zero is $V_0 = 5MM$. The company pays 5% of its cash-flow as dividends. We expect that, as in the Black and Scholes model, the option of the American style which does not pays dividends must be higher than another call that pays dividends. The prices are presented in the table

	Without Dividends	With Dividends
Price	4.10 MM	2.92MM

The following chart illustrates what happens with the American call option that pays dividends.

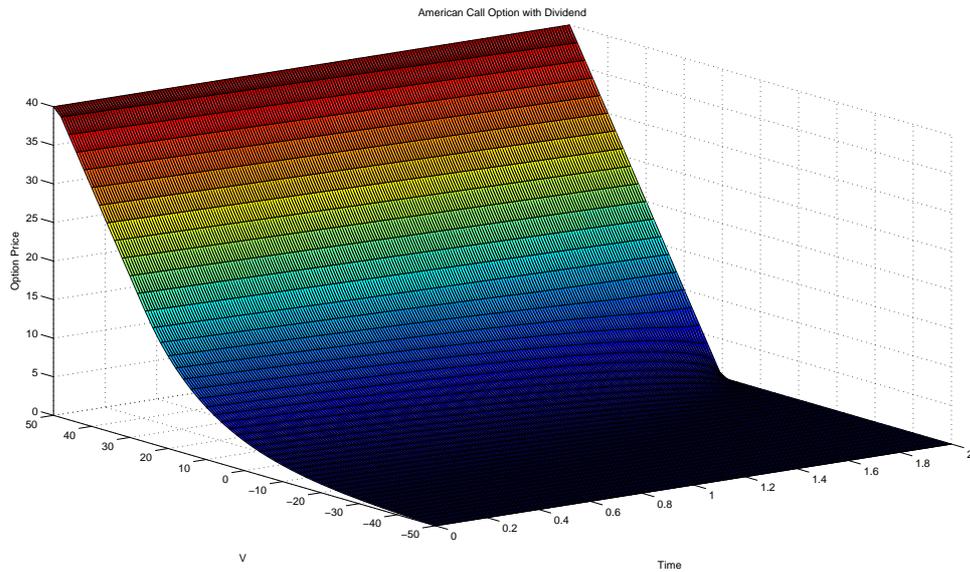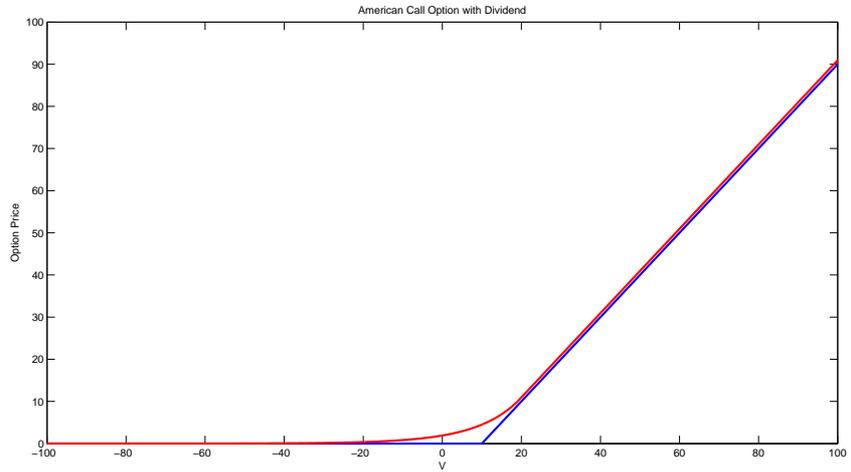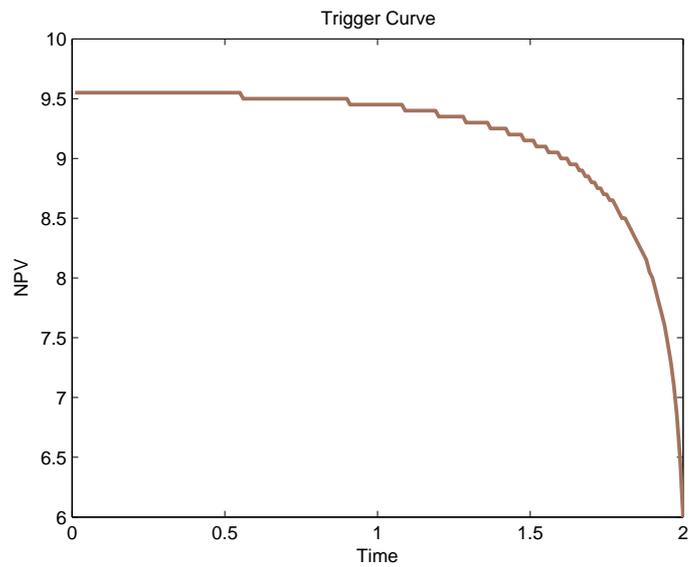Figure 5.6: American call option with dividends price surface

Figure 5.7: American call option with dividends price

American Call Option with Dividend

Another interesting chart is the threshold curve that shows the optimal
time to invest in the project.

Figure 5.8: Threshold Curve

Trigger Curve

At the same context, the price of an European call option, with the same rate of dividends, is

	Without Dividends	With Dividends
$Price$	4.10 MM	0.81MM

Note that the price of the option is naturally lower than an option that does not pay dividend but also is lower than the price of an American call option, once the later has the possibility to exercise at any time during the lifetime of the contract. We show below the chart for European Call Option with dividends.

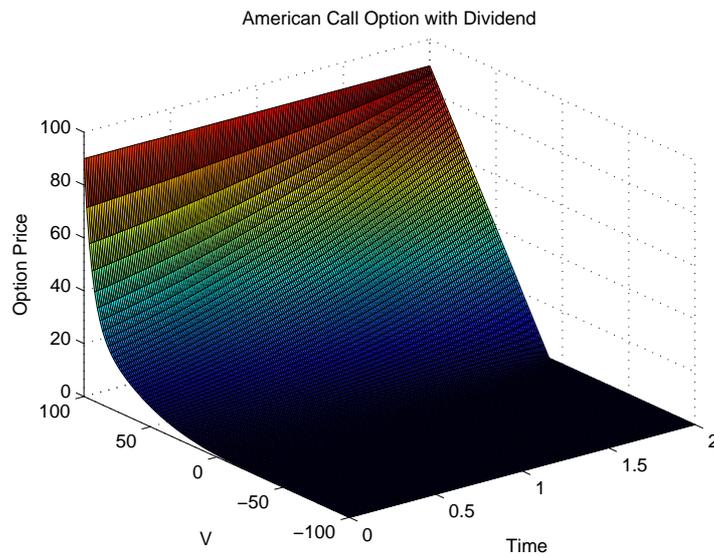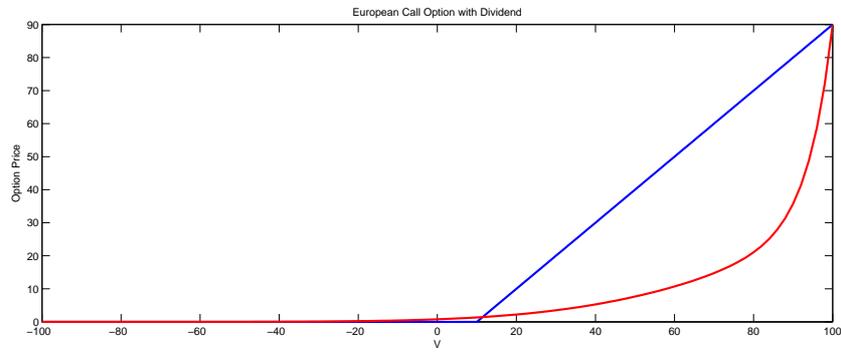Figure 5.9: European call option with dividends price surface

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Figure 5.10: European call option with dividends price

European Call Option with Dividend

To obtain the numerical results to this section, we applied the Crank Nicolson method in (4.43) PDE. The method and the algorithm are presented in the Appendix.

Chapter 6

Conclusions

This work presented the most usual approach for valuation. The most used technique between the managers is likely to be the Net Present Value. Despite having several advantages such as simplicity of calculation, it has serious limitations as described in the text. In particular the drawn back of not taking into account flexibility.

The approach of Real Options is shown as an alternative to NPV. Presenting itself as a more flexible tool for pricing, this approach incorporates more realistic assumptions and uncertainty about future realizations of cash flows of the company. In general, this method can be used to calculate the value of flexibility embedded in various types projects that managers have.

The first step to the valuation process through real options is to estimate the stochastic process of the underlying asset. Because the valuation process through real options is an adaptation of the techniques used in the capital market for derivative pricing, the first proposal was to assess a range of derivatives based on the fact that the asset followed a GBM. With the progress of studies and techniques for pricing, other stochastic processes have been suggested for the dynamics of the underlying asset.

The Exponential Mean Reverting Process is a process that is used to model derivatives prices, largely used in commodities and interest rates markets. Its extension to Real Options Analysis was done not only for projects related to commodities, but also for valuation in several areas like e-business. The techniques used in pricing which follows this dynamic are usually more complicated.

Alternatively, the work suggested a different dynamic for the cash flow process, surprisingly undeveloped in the literature. This process are the pure mean reverting process. This stochastic process can be found in some projects related to commodities. A theoretical study was developed to price derivatives for which the underlying asset has followed this dynamics. The

numerical results obtained came from algorithms developed for this specific model.

The current study confirms that flexibilities like expansion, abandonment and wait are worthy. They can often be significant in decision-making. Thus, Real Option Approach can be considered as a strong tool in helping the managers of several companies.

For future work, there are different strands to follow. The introduction of jumps in the dynamics of the underlying asset may be interesting for certain projects. The introduction of a stochastic interest rate is more realistic once, usually, projects of companies has a medium and long term duration and assume a constant interest rate over the period can be unrealistic. Also, the introduction of stochastic volatility can reflect a more realistic framework.

Appendix A

The Trinomial Tree

In this Chapter we will follow the approach of [Hul05] and [HW94].

The trinomial tree is a method used to solve problems involving exponential mean reverting process. The dynamics of the underlying is described by

$$d\mathrm{log}V = \left[\hat{V} - \mathrm{log}V \right] dt + dW_t. \tag{A.1}$$

The first step to build the tree, is to construct a symmetric tree around zero. The numerical model is

$$dX = -Xdt + dW_t. \tag{A.2}$$

Assume the spacing for the time is one year, $\Delta t = 1$. The spacing of the underlying is taken as

$$\Delta X = \sigma\sqrt{3\Delta t} = \sqrt{3} = 1.73, \tag{A.3}$$

which provide a slightly more rapid convergence, as stated by [HW].

Define (i, j) as the node where $t = i\Delta t$ and $X = j\Delta X$. For each node there are three possibilities of ramification with the probabilities of transition of each node given by

- $p_u = \frac{1}{6} + \frac{1}{2}(a^2 j^2 \Delta t^2 - aj\Delta t) = \frac{1}{6} + \frac{1}{2}(4j^2 - 2j)$

- $p_m = \frac{2}{3} - a^2 j^2 \Delta t^2 = \frac{2}{3} - 4j^2$

- $p_d = \frac{1}{6} + \frac{1}{2}(a^2 j^2 \Delta t^2 + aj\Delta t) = \frac{1}{6} + \frac{1}{2}(4j^2 + 2j)$

In most of the cases, this will be the node adopted. The sum of these probabilities must be the unity. As stated by [Hul05], for some points of
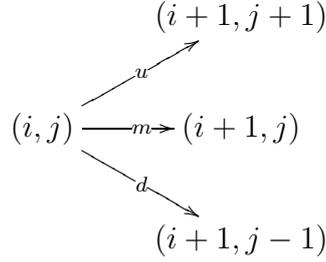
$$(i+1, j+1)$$

$$(i, j) \xrightarrow{\quad m \quad} (i+1, j)$$

$$(i+1, j-1)$$

Figure A.1: Node A

$$(i, j) \xrightarrow{\quad u \quad} (i+1, j)$$
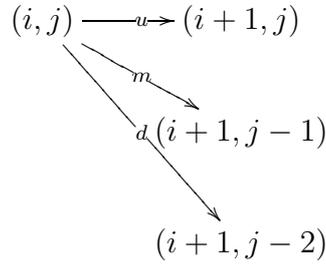
$$(i+1, j-1)$$

$$(i+1, j-2)$$

Figure A.2: Node B

the grid these probabilities can assume negative values. To avoid this inconsistence, we need two more nodes with probabilities defined below. These probabilities also pull back process when the path goes too far away from the long run mean. The second kind of node occurs when the path goes to high. It has the shape with the probabilities of transition of each node given by

- $p_u = \frac{7}{6} + \frac{1}{2}(a^2 j^2 \Delta t^2 - 3aj\Delta t) = \frac{7}{6} + \frac{1}{2}(4j^2 - 6j)$

- $p_m = -\frac{1}{3} - a^2 j^2 \Delta t^2 + 2aj\Delta t = -\frac{1}{3} - 4j^2 + 4j$

- $p_d = \frac{1}{6} + \frac{1}{2}(a^2 j^2 \Delta t^2 - aj\Delta t) = \frac{1}{6} + \frac{1}{2}(4j^2 - 2j)$

The same situation occurs when the prices are too low. The probabilities of transition of each node are given by with the probabilities of transition of each node given by

- $p_u = \frac{1}{6} + \frac{1}{2}(a^2 j^2 \Delta t^2 + aj\Delta t) = \frac{1}{6} + \frac{1}{2}(4j^2 + 2j)$

- $p_m = -\frac{1}{3} - a^2 j^2 \Delta t^2 - 2aj\Delta t = -\frac{1}{3} - 4j^2 - 4j$

- $p_d = \frac{7}{6} + \frac{1}{2}(a^2 j^2 \Delta t^2 + 3aj\Delta t) = \frac{7}{6} + \frac{1}{2}(4j^2 + 6j)$
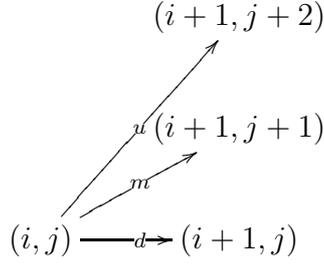
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Figure A.3: Node C

We can define a range $[j_{min}, j_{max}]$ such that the probabilities of Figure (A.1) remains positive. For j below j_{min} the node adopted is shown in Figure A.3 and for j higher than j_{max} the node as in Figure A.2. As stated in [Hul05], we can define

$$j_{max} = \lceil \frac{0.184}{a\Delta t} \rceil$$

$$j_{min} = -j_{max}$$

With all these quantities in hand, we can build a symmetric tree that represents the process (A.2).

The next step is, starting from the symmetric tree, to build a tree that represents the original process. For this, we will displace the nodes by $\alpha(t)$ where

$$\alpha(t) = V(t) - X(t) \qquad (A.4)$$

Let α_i be the difference (A.4) at time $i\Delta t$. We also define the quantity $Q(i,j)$ as the present value of a bond which pays US\$ 1 if the node (i,j) is reached and nothing otherwise. Both quantities are calculated forward in time matching the future prices. First we determine α and then evaluate the value of Q. The formulae for α_m and Q_m, at time $m\Delta$ are expressed as

$$\alpha_m = \frac{\log\left(\sum_{j=-n_m}^{n_m} Q_{m,j} e^{-j\Delta R \Delta t}\right) - \log P_{m+1}}{\Delta t}, \qquad (A.5)$$

where P_{m+1} is

$$P_{m+1} = \sum_{j=-n_m}^{n_m} Q_{m,j} e^{-(\alpha_m + j\Delta R)\Delta t}. \qquad (A.6)$$

$$Q_{m+1,j} = \sum_k Q_{m,k} q(k,j) e^{-(\alpha_m + k\Delta R)\Delta t}, \tag{A.7}$$

where $q(k,j)$ is the probability to pass from node (m,k) to node $(m+1,j)$.

These steps establish the tree for the process of the class of (A.1). Now if we want to price an American put or call on an asset which realization of the process are represented by this tree, we only need to proceed backwards on time and compare at each node the value of the option with its payoff, taking that one which maximize the option value.

Appendix B

Crank Nicolson for Fixed Dividend PDE

The Crank-Nicolson scheme is a finite difference method that can be viewed as a hybrid method between the explicit an implicit [Bra06]. It has the interesting property of being unconditionally stable. The method also guarantee the convergence to the solution of the PDE that the numerical scheme represents. These properties make the numerical method interesting to solve PDE's. Discretizing PDE (B.1)

$$\partial_t P + \partial_v P\left(\left(r - \delta r - \delta\kappa\right)v - \frac{\delta\mu\kappa}{r}\right) + \frac{\sigma^2}{2}\partial_{vv}P - rP = 0. \qquad \text{(B.1)}$$

The grid will be represented by the coordinates (v, t), where

$$v = 0, \ \Delta v, \ 2\Delta v, ..., \ M\Delta v = v_{max}$$
$$t = 0, \ \Delta t, \ 2\Delta t, ..., N\Delta t = T.$$

Let us denote $P_{i,j} = P(i\Delta v, j\Delta t)$. For a call option, we have the following boundary condition

1. $P_{0,j} = 0, \ j = 0, \ 1, \ 2, \ ..., \ N$

2. $P_{M,j} = M\Delta v - Ke^{-r(N-j)\Delta t}, \ j = 0, \ 1, \ 2, \ ..., \ N$

3. $P_{i,N} = \max[i\Delta v - K, 0], \ i = 0, \ 1, \ 2, \ ..., \ M$

where $v_{\max}$ is a large value for the project that cannot be reached by v in the time interval under consideration. It represents $+\infty$ in our grid.

Equation (B.1) can be represented as a finite difference scheme:

$$\frac{P_{i,j} - P_{i,j-1}}{\Delta t} + \frac{1}{2}\left((r - \delta r - \delta\kappa)i\Delta v - \frac{\delta\mu\kappa}{r}\right)\left(\frac{P_{i+1,j-1} - P_{i-1,j-1}}{2\Delta v}\right)$$

$$+ \left(\frac{1}{2}(r - \delta r - \delta\kappa)v - \frac{\delta\mu\kappa}{r}\right)\left(\frac{P_{i+1,j} - P_{i-1,j}}{2\Delta v}\right)$$

$$+ \frac{\sigma^2}{4}\left(\frac{P_{i+1,j-1} - 2P_{i,j-1} + P_{i-1,j-1}}{(\Delta v)^2}\right) + \frac{\sigma^2}{4}\left(\frac{P_{i+1,j} - 2P_{i,j} + P_{i-1,j}}{(\Delta v)^2}\right)$$

$$= \frac{r}{2}P_{i,j-1} + \frac{r}{2}P_{i,j}. \quad \text{(B.2)}$$

Putting all terms indexed by $j-1$ to one side of the equality and the other indexed by j to the other side, we arrived at

$$-\alpha_i P_{i-1,j-1} + (1 - \beta_i)P_{i,j-1} - \gamma_i P_{i+1,j-1} = \alpha_i P_{i-1,j} + (1 + \beta_i)P_{i,j} + \gamma_i P_{i+1,j}, \quad \text{(B.3)}$$

where

$$\alpha = \frac{\Delta t}{4}\left(\frac{\sigma^2}{(\Delta v)^2} - (r - \delta r - \delta\kappa)i - \frac{\delta\mu\kappa}{r\Delta v}\right)$$

$$\beta_i = -\frac{\Delta t}{2}\left(\frac{\sigma^2}{(\Delta v)^2} + r\right)$$

$$\gamma_i = \frac{\Delta t}{4}\left(\frac{\sigma^2}{(\Delta v)^2} + (r - \delta r - \delta\kappa)i + \frac{\delta\mu\kappa}{r\Delta v}\right). \quad \text{(B.4)}$$

In matrix form, we have

$$M_1 P_{j-1} = M_2 P_j, \quad \text{(B.5)}$$

where

$$\mathbf{M_1} = \begin{pmatrix} 1 - \beta_1 & -\gamma_1 & 0 & 0 & \ldots \\ -\alpha_2 & 1 - \beta_2 & -\gamma_2 & 0 & \ldots \\ 0 & -\alpha_2 & 1 - \beta_2 & -\gamma_2 & \ldots \\ 0 & \ddots & \ddots & \ddots & \ldots \\ 0 & \ldots & \ldots & -\alpha_{M-1} & 1 - \beta_{M-1} \end{pmatrix}$$

$$\mathbf{M_2} = \begin{pmatrix} 1 + \beta_1 & \gamma_1 & 0 & 0 & \ldots \\ \alpha_2 & 1 + \beta_2 & \gamma_2 & 0 & \ldots \\ 0 & \alpha_2 & 1 + \beta_2 & \gamma_2 & \ldots \\ 0 & \ddots & \ddots & \ddots & \ldots \\ 0 & \ldots & \ldots & \alpha_{M-1} & 1 + \beta_{M-1} \end{pmatrix}$$

$$P_j = [P_{1,j}, P_{2,j}, ..., P_{M-1,j}]^T$$

To solve this system of equations, one can use the LU decomposition or SOR method to provide a better computational performance.

Another important issue is the fact that we are interested on American options. Thus, at each iteration, the value of the option is given by the maximum of the payoff and the value of the contract.

After the implementation of the algorithm, we test it against the closed formula for an European call option.

We already know that the closed formula solution for an European call is given by

$$P(0, V_0) = \left(V_0 - Ke^{-rT}\right)\Phi(d_1) + var(0,T)\varphi(d_1),$$

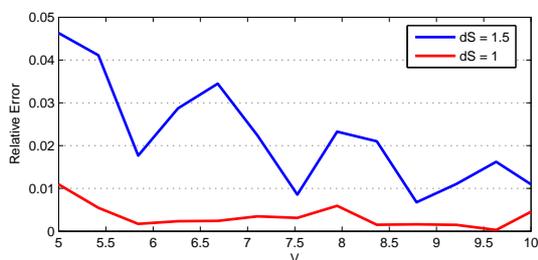where

$$d_1 := \frac{V_0 - Ke^{-rT}}{var(0,T)},$$

and

$$var^2(0, t) = \frac{\sigma^2}{2r}\left(1 - e^{-2rt}\right). \tag{B.6}$$

If we adopt the model (5.1), with the following parameters

- $V_0 = 5$

- $I = 10$

- $r = 0$

- $T = 2$

- $\sigma = 20$

we have the relative error presented in Figure (B.1) when we vary V_0. It is verified that the rate of convergence is slow when we make $V \to 0$ or $V \to \infty$. The result is presented below for the range of our example.

Figure B.1: Relative Error

We also analyze the behavior of the solution against the grid size, shown in Figure (B.3) and (B.5). We also plot a zoom of this sensitivities.
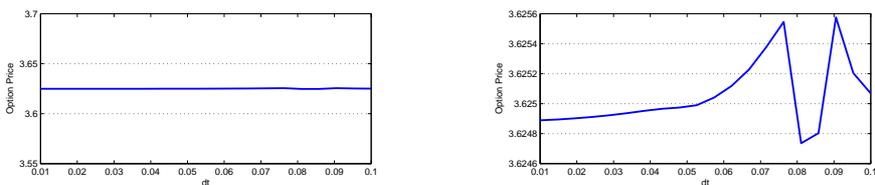
Figure B.2: Sensitivities

Figure B.3: Sensitivity of price with respect to dt
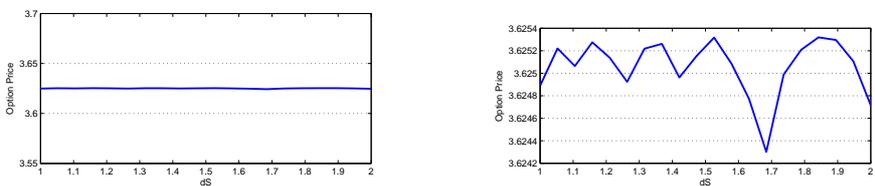
Figure B.4: sensitivities

Figure B.5: Sensitivity of price with respect to dS

Appendix C

Algorithms

This Chapter presents the Matlab Code used to compute the prices in the previous sections.

C.1 Bachelier

The first algorithm presented is the Bachelier code. We use the Crank-Nicolson method to solve the Bachelier PDE presented in Appendix B and also calculate the price with the closed formula develop in Chapter 4. The output of the function are the price calculated by the two methods. The inputs of the function are the initial Net Present Value, the strike price of the option, the interest rate, the time to maturity, the maximum value that the underlying can't achieve (it will represent the infinitum), the increment of the asset and the increment of time.

Bachelier Function

```
    function [AmPrice,EuPrice]=bachelier(S0, K, r, T, sigma, Smax, dS, dt)
tic

    %Define the length of dS and dt
M=round(Smax/dS);
dS=Smax/M;
N=round(T/dt);
dt=T/N;

    %Vector that indicates the grid position
vetS=linspace(0,Smax,M+1)';
veti=0:M;
vetj=0:N;
matval=ones(M+1,N+1);%matrix that store the values of the contract
```

```
%Matrix with Payoff of the contract in every time for all de S_t
payoff=zeros(M+1,N+1);
for k = 1:M+1
for j=1:N+1
payoff(k,j)=vetS(k)-K*exp(-r*dt*(N-vetj(j)));
end
end

    %Coefficients
alpha=0.25*dt*((sigma∧/(dS.2̂))-r*veti);
beta=-(dt/2)*((sigma∧2/(dS.2̂))+r*ones(size(veti)));
gamma=0.25*dt*((sigma∧2/(dS.2̂))+r*veti);

    %Matrix with coefficients
M1=diag(-alpha(3:M),-1)+diag(1-beta(2:M))+diag(-gamma(2:M-1), 1);
M2=diag(alpha(3:M),-1)+diag(1+beta(2:M))+diag(gamma(2:M-1), 1);

    %LU Decomposition to speed up the resolution of the system
    [L,U]=lu(M1);
    %Boundary Conditions
matval(:,N+1)=max(vetS-K,0);%At maturity, the payoff is S-K
matval(1,:)=zeros(size(vetj));%If S=-Inf then P_t=0
matval(M+1,:)= Smax*ones(size(vetj))-K*exp(-r*dt*(N-vetj));%If S=Inf=¿P_t=S-
K*epx

    %Solve the system aux=zeros(M-1,1);
for j=N:-1:1
aux(1)=alpha(2)*(matval(1, j)+matval(1, j+1));
aux(M-1)=gamma(M)*(matval(M+1,j)+matval(M+1,j+1));
matval(2:M,j) = U\(L\(M2*matval(2:M,j+1) + aux));
matval(2:M,j) = max(matval(2:M,j),payoff(2:M,j));
end

    %Price at t=0
AmPrice= interp1(vetS, matval(:,1), S0);

    %Plot the Option Price versus S_t at the maturity and in t=0
figure
plot(vetS,matval(:,end))
hold on
plot(vetS,matval(:,1),'r')
```

```
    %Plot t x S_t x P_t
    [x,y]=meshgrid(dt*vetj,vetS);
figure
surf(x,y,matval)

    %Closed Formula to the Price
if r==0
EuPrice=sigma*sqrt(T)*normpdf((((S0-K*exp(-r*T))/(sigma*sqrt(T))),0,1)+...
(S0-K*exp(-r*T))*normcdf((((S0-K*exp(-r*T))/(sigma*sqrt(T))),0,1);
else
var0T=sqrt(0.5*sigma^2*(1/r)*(1-exp(-2*r*T)));
EuPrice=var0T*normpdf((S0-K*exp(-r*T))/var0T,0,1)+...
(S0-K*exp(-r*T))*normcdf((S0-K*exp(-r*T))/var0T,0,1);
end
toc
end
```

The charts presented were produced calling

$$[p1,p2]=bachee(5, 10, 0, 2, 20, 100, 1, 1/50)$$

If we set the third input as 0.05, the code return the prices and the charts for an interest rate of 5%.

C.2   Fixed Dividend Algorithm

Here, the core of the algorithm is almost identical to the previous one. We implemented the Crank Nicolson scheme discussed in Appendix B. The output of the functions are the price of an American call options without dividends and the price of the contract which pays dividends. All the previous inputs are necessary and also the mean reverting rate, long mean rate and the dividend rate.
Fixed Dividend Function

```
    function [AmPrice,EuPrice]=fixeddiv(S0, K, r, T, sigma,mu,kappa,div,Smax,
dS, dt)
    tic
termo = mu*kappa*div/r;
newr = r-r*(kappa+div);
    %Define the length of dS and dt
```

```matlab
    M=round(Smax/dS);
dS=Smax/M;
N=round(T/dt);
dt=T/N;

    %Vector that indicates the grid position
vetS=linspace(0,Smax,M+1)';
veti=0:M;
vetj=0:N;
matval=ones(M+1,N+1);%matrix that store the values of the contract

    %Coefficients

    alpha=0.25*dt*((sigma^2*ones(size(vetS))./(dS.^2))'-newr*veti-termo./dS );
beta=-(dt/2)*((sigma^2*ones(size(vetS))./(dS.^2))'+r*ones(size(veti)));
gamma=0.25*dt*((sigma^2*ones(size(vetS))./(dS.^2))'+newr*veti +termo./dS);

    %Matrix with coefficients
M2=diag(alpha(3:M),-1)+diag(1+beta(2:M))+diag(gamma(2:M-1), 1);
M1=diag(-alpha(3:M),-1)+diag(1-beta(2:M))+diag(-gamma(2:M-1), 1);

    %LU Decomposition to speed up the resolution of the system
    [L,U]=lu(M1);

    %Boundary Conditions
matval(:,N+1)=max(vetS-K,0);%At maturity, the payoff is S-K
matval(1,:)=zeros(size(vetj));%If S=-Inf then P_t=0
matval(M+1,:)= Smax*ones(size(vetj))-K;%If S=Inf=¿P_t=S-K*epx

    %Matriz com os Payoff em todos os instantes, pra todos valores de S_t
payoff=zeros(M+1,N+1);
for k = 1:M+1
for j=1:N+1
payoff(k,j)=vetS(k)-K*exp(-r*dt*(N-vetj(j)));
end
end

    %Solve the system aux=zeros(M-1,1);
trigger=zeros(M+1,N);%set a matrix to see where the contract is exercised
for j=N:-1:1
aux(1)=alpha(2)*(matval(1, j)+matval(1, j+1));
aux(M-1)=gamma(M)*(matval(M+1, j)+matval(M+1,j+1));
matval(2:M,j) = U\(L\(M2*matval(2:M,j+1) + aux));
```

```
matval(2:M,j) = max(matval(2:M,j),payoff(2:M,j));
trigger(2:M,j)=(matval(2:M,j) =payoff(2:M,j));
end

    %Price at t=0
AmercianCallPrice= interp1(vetS, matval(:,1), S0);

    %Plot the trigger curve trigger(M+1,:)=(matval(M+1,1:N) =payoff(M+1,1:N));
indice=sum(trigger);
front = dS*indice-Smax/2;
figure
gr=plot(dt*(1:length(front)),front);
set(gr,'LineWidth',2,'Color',[0.64 .45 0.37])
title('Trigger Curve')
xlabel('Time')
ylabel('NPV')

    %Plot Option value vs. V_t at the maturity and at t=0
    figure
plot(vetS,matval(:,end))
hold on
plot(vetS,matval(:,1),'r')
title('American Call Option with Dividend')
xlabel('V')
ylabel('Option Price')

    %Plot t x S_t x P_t
    [x,y]=meshgrid(dt*vetj,vetS);
figure
surf(x,y,matval)
title('American Call Option with Dividend')
xlabel('Time')
ylabel('V')
zlabel('Option Price')
    var0T=sqrt(0.5*sigma^2*(1/r)*(1-exp(-2*r*T)));
EuroCallPrice=var0T*normpdf((S0-K*exp(-r*T))/var0T,0,1)+...
(S0-K*exp(-r*T))*normcdf((S0-K*exp(-r*T))/var0T,0,1);
toc
end
```

The charts presented were produced calling

[p2,p1]=fixeddiv(5, 10, 0.05, 2, 11,0.2,5,0.05,100,1, 1/100)

To produce the European option solution, it is enough to comment one of the lines

matval(2:M,j) = max(matval(2:M,j),payoff(2:M,j));

in the code.

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