

# **Simple cocycles over Lorenz attractors**

Mohammad Fanaee

# **Livros Grátis**

<http://www.livrosgratis.com.br>

Milhares de livros grátis para download.

ABSTRACT. We show that Lyapunov exponents of typical Hölder continuous fiber bunched linear cocycles over Lorenz attractor have multiplicity one: the complement has infinite codimension. It is described in terms of rather explicit geometric conditions on sufficient simplicity criterion exhibited in [AV07] for cocycles over complete shift.

# Contents

Chapter 1. Introduction	5
1. Linear cocycles	6
2. Suspension flows	7
Chapter 2. Lorenz Attractors	9
1. Lorenz equations	9
2. Geometric model	10
3. Physical probability	13
Chapter 3. Symbolic Structure	15
1. Shift map	15
2. Markov structure in dimension 1	16
3. Markov structure in dimension 2	16
4. Lifting the absolutely continuous probability	17
5. Suspending the bi-lateral shift by the flow	19
Chapter 4. Fiber Bunched Cocycles	21
1. Holonomy maps	22
2. Dependence on cocycle	23
Chapter 5. Perturbation Tools	25
1. Periodic orbits	26
2. Homoclinic orbits	26
3. The main perturbation	27
Chapter 6. Proof of the Main Results	29
1. Proof of Theorem 2	29
2. Proof of Theorem 1	30
Chapter 7. Final Remarks	33
Bibliography	35



## CHAPTER 1

# Introduction

This work is dedicated to the study of linear cocycles over geometric Lorenz attractors. We prove that, in most cases, their Lyapunov exponents are all distinct.

A linear cocycle over a flow  $X^t : M \rightarrow M$  is a flow  $F^t : M \times \mathbb{C} \rightarrow M \times \mathbb{C}$  of the form

$$F^t(x, v) = (X^t(x), A^t(x)v)$$

where each  $A^t(x) : \mathbb{C}^d \rightarrow \mathbb{C}^d$  is a linear isomorphism. The Lyapunov exponents are the exponential rates of growth/decay

$$\lambda(x, v) = \lim_{t \rightarrow \infty} \frac{1}{t} \log \|A^t(x)v\|$$

of the orbits of non-zero vectors  $v \in \mathbb{C}^d$ . By Oseledets [O68] this limit exists for every  $v \in \mathbb{C}^d \setminus \{0\}$  and almost every  $x \in M$ , relative to any invariant probability.

One problem is to characterize when these exponents are different from zero. Another main problem is to know when all exponents are distinct meaning that the subspace of vectors  $v \in \mathbb{C}$  that share the same value of  $\lambda(x, v)$  has dimension 1. In this case, we say that the Lyapunov spectrum is simple.

There has been much recent progress on these problems, specially when the base dynamical system is hyperbolic. See [BGV03,V08] for the first problem and [BV04,AV07] for the second one.

Here, we extend the theory to the case when the base dynamics is a geometric Lorenz attractor. This class of systems was introduced in [W79,GW79] as a model for the behaviour of the famous Lorenz equations [L63]. Recently, it was shown by Tucker [T98] that these equations have the main feature predicted by the geometric models.

A geometric Lorenz flow in 3-dimensions admits a cross-section  $S$  and a Poincaré return transformation  $P : S \setminus \Gamma \rightarrow S$  defined outside a curve  $\Gamma$  which is contained in the intersection of  $S$  with the stable manifold of some hyperbolic equilibrium. Trajectories through  $\Gamma$  just converge to the equilibrium. The other trajectories through  $S$  eventually return to  $S$ . Their accumulation set is the so-called geometric Lorenz strange attractor  $\Lambda$ . It is a special case of the class of singular hyperbolic attractors introduced in [MPP04]. Singular hyperbolic sets are characterized by the existence of an invariant splitting

$$T_\Lambda M = E^s \oplus E^{cu}$$

of the tangent bundle where uniformly contracting bundle  $E^s$  has dimension 1, and volume-expanding bundle  $E^{cu}$  which contains the flow direction has dimension 2. Another important feature is that the Poincaré transformation  $P : S \setminus \Gamma \rightarrow S$  admits an invariant contracting foliation  $\mathcal{F}$  through which the dynamics can be reduced to that of a map on the interval (leaf space of  $\mathcal{F}$ ).

We consider linear cocycles that satisfy a partial hyperbolicity condition called “fiber bunching” [AV07,ASV,AV10] (The same notion was called “domination” in [BGV03,BV04]). Our main result states that for such cocycles the Lyapunov spectrum is simple as long as some extremely conditions are satisfied. The exceptional set of cocycles that do not meet these conditions is nowhere dense and, in fact, has infinite codimension in a sense that will be made precise in a while.

## 1. Linear cocycles

**1.1. Discrete time systems.** A *linear cocycle* over a diffeomorphism  $f : N \rightarrow N$  is a transformation  $F : N \times \mathbb{C}^d \rightarrow N \times \mathbb{C}^d$  satisfying  $f \circ \pi = \pi \circ F$  which acts by a linear isomorphisms  $A(x)$  on fibers. So, the cocycle has the form

$$F(x, v) = (f(x), A(x)v)$$

where

$$A : N \rightarrow \mathrm{GL}(d, \mathbb{C}).$$

Conversely, any  $A : N \rightarrow \mathrm{GL}(d, \mathbb{C})$  defines a linear cocycle over  $f$ . Note that  $F^n(x, v) = (f^n(x), A^n(x)v)$ , where

$$\begin{aligned} A^n(x) &= A(f^{n-1}(x)) \dots A(f(x))A(x), \\ A^{-n}(x) &= (A^n(f^{-n}(x)))^{-1}, \end{aligned}$$

for any  $n \geq 1$ , and we define  $A^0(x) = \mathrm{id}$ .

Let  $\mu_f$  be a probability measure invariant by  $f$ . If  $w \mapsto \max\{0, \log \|A(x)\|\}$  is  $\mu$ -integrable then Oseledets Theorem [O68] implies that there exists a *Lyapunov splitting*

$$E_1(x) \oplus \dots \oplus E_k(x), \quad 1 \leq k = k(x) \leq d,$$

and *Lyapunov exponents*  $\lambda_1(x) > \dots > \lambda_k(x)$ ,

$$\lambda_i(x) = \lim_{n \rightarrow +\infty} \frac{1}{n} \log \|A^n(x)v_i\|, \quad v_i \in E_i(x), \quad 1 \leq i \leq k,$$

at  $\mu$ -almost every point. Thus, Lyapunov exponents are constant when  $\mu_f$  is ergodic, and then we denote these constants by  $\lambda_i(A, \mu_f)$ . We call

$$\mathrm{Spec}(A, \mu_f) = \{\lambda_1(A, \mu_f), \dots, \lambda_k(A, \mu_f)\}$$

*Lyapunov spectrum* of the cocycle and, we say that Lyapunov spectrum of  $A$  is *simple* if  $|\mathrm{Spec}(A, \mu_f)| = d$ .

We recall that, for any  $r \in \mathbb{N} \cup \{0\}$  and  $0 \leq \rho \leq 1$ , the  $C^{r,\rho}$  topology is defined by

$$\|A\|_{r,\rho} = \max_{0 \leq i \leq r} \sup_x \|D^i A_P(x)\| + \sup_{x \neq y} \frac{\|D^r A_P(x) - D^r A_P(y)\|}{d(x, y)^\rho}$$

(for  $\rho = 0$  omit  $\|A\|_\rho$ ). The assumption  $r + \rho > 0$  implies  $\eta$ -Hölder continuity:

$$\|A(x) - A(y)\| \leq \|A\|_{0,\eta} d(x, y)^\eta,$$

with

$$\eta = \begin{cases} \rho & r = 0 \\ 1 & r \geq 1. \end{cases}$$

**1.2. Continuous time systems.** A linear cocycle over a flow  $X^t : M \rightarrow M$  is defined as a flow

$$F^t : M \times \mathbb{C}^d \rightarrow M \times \mathbb{C}^d$$

$$F^t(x, v) = (X^t(x), A^t(x)v)$$

where  $A^t(x) \in \mathrm{GL}(d, \mathbb{C})$ , for all  $w$  and  $t \in \mathbb{R}$  with  $A^0(x) = \mathrm{id}$ . It satisfies  $X^t \circ \pi = \pi \circ F^t$  where  $\pi : M \times \mathbb{C}^d \rightarrow M$  is the natural projection. We represent this cocycle by  $A^t$  when there is no ambiguous.

Oseledets Theorem [O68] states that, for any invariant probability measure  $\mu_X$ , almost every  $x \in M$  admits a *Lyapunov splitting*

$$E_1(x) \oplus \dots \oplus E_k(x), \quad 1 \leq k = k(x) \leq d,$$

and corresponding *Lyapunov exponents*  $\lambda_1(x) > \dots > \lambda_k(x)$ ,

$$\lambda_i(x) = \lim_{t \rightarrow +\infty} \frac{1}{t} \log \|A^t(x)v_i\|, \quad \mathbf{0} \neq v_i \in E_i(x), \quad 1 \leq i \leq k.$$

Lyapunov exponents and Oseledets' subspaces are uniquely defined at almost every  $x$  and vary measurably with the base point  $x$ .

These exponents are invariant by the flow  $X^t$ , and so, they are constant in a full measure subset of  $\Sigma$  when  $\mu_X$  is ergodic. We denote these constants by  $\lambda_i(A^t, \mu_X)$ , and

$$\mathrm{Spec}(A^t, \mu_X) = \{\lambda_1(A^t, \mu_X), \dots, \lambda_k(A^t, \mu_X)\}$$

is called the *Lyapunov spectrum* of  $A^t$ . We say that the spectrum of  $A^t$  is *simple* if Lyapunov exponents have multiplicity one, or equivalently Oseledets' subspaces have dimension one, i.e. when  $k = d$ , almost every where:  $|\mathrm{Spec}(A^t, \mu_X)| = d$ .

## 2. Suspension flows

Let for  $f : N \rightarrow N$  there exist an invariant splitting

$$TN = E^s \oplus E^u,$$

and  $\theta(x) \in (0, 1)$  such that

$$\|Df(x)|_{E^s}\| < \theta(x)$$

and

$$\|Df(x)|_{E^u}\| > \theta(x)^{-1},$$

for any  $x \in N$ .

Consider the corresponding suspension flow  $X^t : M \rightarrow M$  and let  $R : N \rightarrow \mathbb{R}$  be the corresponding return time to  $M$ . Assume that  $A^t : M \times \mathbb{C}^d \rightarrow M \times \mathbb{C}^d$  is a linear cocycle over the flow  $X^t$ , and define

$$A_f(x) = A^{R(x)}(x),$$

for any  $x \in N$ . Note that  $A_f : N \times \mathbb{C}^d \rightarrow N \times \mathbb{C}^d$  is a linear cocycle over  $f$ .

Then, we define, for any  $r \in \mathbb{N} \cup \{0\}$  and  $0 \leq \rho \leq 1$  with  $r + \rho > 0$ ,

$$\mathcal{C}^{r,\rho}(M, d, \mathbb{C}) = \{A^t : M \rightarrow \mathrm{GL}(d, \mathbb{C}) : \|A_f\|_{r,\rho} < +\infty\}.$$

This means that,  $A^t$  is  $\eta$ -Hölder continuous if the corresponding discrete time linear cocycle  $A_P$  is Hölder.

**2.1. Fiber bunching.** Let  $A^t$  be an  $\eta$ -Hölder continuous linear cocycle over  $f^t$ .

**Definition 1.1.**  $A^t$  is fiber bunched if there exists some constant  $\tau \in (0, 1)$  such that

$$\|A_f(x)\| \|A_f(x)^{-1}\| \theta(x)^\eta < \tau,$$

for any  $x \in N$ .

**Theorem 1.** Typical Hölder continuous fiber bunched linear cocycles over Lorenz attractor have simple spectrum: the set of exceptional cocycles is locally contained in finite unions of closed submanifolds with arbitrarily high codimension.

**2.2. Typical cocycles.** Bonatti and Viana [BV04] showed that generic dominated linear cocycles over any hyperbolic transformation have simple spectrum. Avila and Viana [AV07] proved a sufficient condition to simplicity of Lyapunov spectrum over complete shift structure when the corresponding invariant measure has product structure. Applying certain perturbation arguments on simplicity sufficient criterion (pinching and twisting),

**Theorem 2.** Typical Hölder continuous fiber bunched linear cocycle over complete shift map have simple spectrum: the set of exceptional cocycles is locally contained in finite unions of closed submanifolds with arbitrarily high codimension.

We will show that the Lorenz attractors admit the structure of complete shift, and then Theorem 1 gets carrying out Theorem 2, by corresponding suspension.

**Acknowledgment:** I would like to thanks advisor of my PhD thesis, prof. Marcelo Viana for all supports during my studies at IMPA. I am grateful also to Jimmy Santamaria and Alien Herera for several useful conversations on this work.

This work is supported by a doctoral grant from CNPq-TWAS.

## CHAPTER 2

### Lorenz Attractors

Attractors of flows present important features with respect to the discrete time case when they involve singularities interacting with regular orbits.

The geometric model of Lorenz attractor is historically the first example of a robust singular attractor as a rigorous model for the behavior of the Lorenz attractor, near the classical parameters. There is now a vast literature on the geometric and ergodic properties of these Lorenz like attractors.

An *attractor* for a smooth flow  $X^t$  is an invariant transitive set  $\Lambda$  admitting an open neighborhood  $U$  such that  $X^t(\bar{U}) \subset U$ , for all  $t > 0$ , and

$$\Lambda = \bigcap_{t>0} X^t(U).$$

$\Lambda$  is *robust* if  $\bigcap_{t>0} Y^t(U)$  is also an attractor, for any smooth vector field  $Y$  in a neighborhood of  $X$ . The attractor is *singular* if it contains some singularity of the vector field.

The *topological basin* of  $\Lambda$  is the set

$$B(\Lambda) = \{x : \lim_{t \rightarrow +\infty} d(X^t(x), \Lambda) = 0\},$$

and it is clear that  $U \subset B(\Lambda)$ .

An invariant probability  $\mu$  is *physical* if the set  $B(\mu)$  of points  $x$  satisfying

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \varphi(X^t(x)) dt = \int \varphi d\mu,$$

for all continuous function  $\varphi$ , has positive Lebesgue probability.  $B(\mu)$  is called the *basin* of  $\mu$ .

The Lorenz attractor supports a unique physical probability measure  $\mu_X$  which is ergodic, and  $B(\mu) = B(\Lambda)$ , Leb. mod 0 (see 2.4, and [APPV09] for more details in a more general case).

#### 1. Lorenz equations

The smooth system of equations

$$(1) \quad \begin{aligned} \dot{x} &= a(y - x), \\ \dot{y} &= bx - y - xz, \\ \dot{z} &= xy - cx, \end{aligned}$$

proposed by Lorenz [L63] loosely related to fluid convection and weather prediction. W. Tucker [T98] showed that the solutions of (1) supports a robust transitive

attractor  $\Lambda$ , for *classical parameters*  $a = 10$ ,  $b = 28$ ,  $c = 8/3$ .

From measure theoretic view point, Lorenz flow provides a robust class of expansive attractors that are not hyperbolic sets but exhibit some non-uniformly hyperbolic behavior: the attractor supports a unique physical probability measure for which the tangent bundle splits into three 1-dimensional invariant subspaces

$$E^s \oplus E^X \oplus E^u,$$

at almost every point depending measurably on the base point, where  $E^s$  is stable direction,  $E^X$  is the direction of Lorenz flow  $X^t$ , and

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log \|DX^t|_{E^u}\| > 0.$$

The system of equations (1) is symmetric with respect to the  $z$ -axis and has a singularity in  $\mathbf{0}$  where  $DX(\mathbf{0})$  has real eigen-values  $\lambda_2 < \lambda_3 < 0 < \lambda_1$  with  $\lambda_1 + \lambda_3 > 0$ . There are also two symmetric saddles  $\sigma_1, \sigma_2$  with a real negative and two conjugate complex eigen-values where the complex eigen-values have positive real parts.

Since  $\text{div}(X) < 0$  the character of this flow is strongly dissipative, in particular, any maximally positively invariant subset under  $X^t$  has zero volume.

Numerical simulations show that there exists an open set  $U$  homeomorphic to a 2-torus where  $\bigcap_{t>0} X^t(U)$  is an attracting set and the origin is the only singularity contained in  $U$ . Indeed, a very general view of the orbit of a generic point in  $U$  is that the trajectory starts spiraling around one of the singularities, say  $\sigma_2$ , and suddenly jumps to the other singularity,  $\sigma_1$ , and starts spiraling around the other. This process repeats endlessly and then implies the "butterfly" appearance of Lorenz flow.

## 2. Geometric model

To construct the geometric Lorenz attractor, we should analyze the dynamics in a neighborhood of  $\mathbf{0}$ , by Hartman-Grobman Theorem, imitating the effect of the pair of saddles.

**2.1. Poincaré transformation.** By construction, there is a cross-section  $S$  intersecting the stable manifold of  $\mathbf{0}$  along a curve  $\Gamma$  that separates  $S$  into 2 connected components. We denote the corresponding Poincaré first return transformation

$$P : S \setminus \Gamma \rightarrow S.$$

Note that the future trajectories of points in  $\Gamma$  do not come back to  $S$ .

**2.1.1. Hyperbolicity constant.** We consider the smooth foliation  $\mathcal{F}$  of  $S$  into curves having  $\Gamma$  as a leaf which are invariant and uniformly contracted by forward iterates of  $P$ . Indeed,

- (i) every leaf  $\mathcal{F}_w$  is mapped by  $P$  completely inside the leaf  $\mathcal{F}_{P(w)}$ , and  $P|_{\mathcal{F}_w}$  is a uniform contraction,
- (ii) the map  $g$  induced by  $P$  is uniformly expanding with derivative tending to infinity as one approaches to  $\Gamma$ .

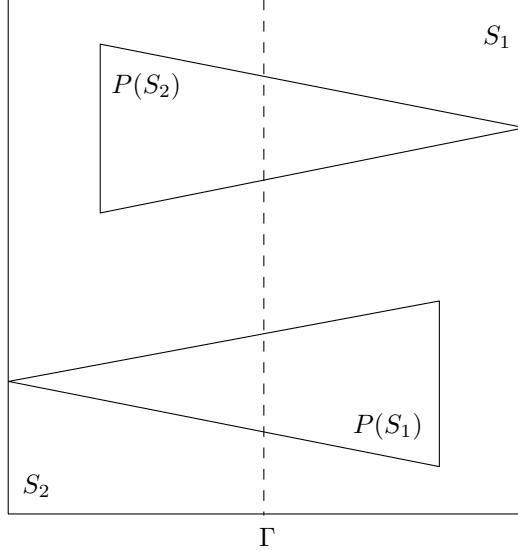


FIGURE 2.1. Poincaré transformation

Henceforth,  $P$  must have the form

$$P(x, y) = (g(x), h(x, y))$$

which by effect of saddles and singularity, we can assume that  $h$  is a contraction along its second coordinate.

We can assume that  $|g'| \geq \theta^{-1} > 1$  and since the rate of contraction of  $h$  on the second coordinate should be much higher than the expansion of  $g$ , we can take  $|\partial_y h| \leq \theta < 1$ .

More precisely, the splitting  $E^s \oplus E^{cu}$  of tangent bundle of Lorenz flow induces a continuous splitting  $E^s(S) \oplus E^u(S)$  of  $TS$  where

$$E^s(S) = E^s \cap TS$$

and

$$E^u(S) = E^{cu} \cap TS.$$

**Theorem 2.1.** [APPV09]  $E^s(S)$  and  $E^u(S)$  are invariant by  $P$ , and for any  $\theta \in (0, 1)$  there exists  $R_0$  such that if  $R(w) > R_0$  at any  $w$  then

$$\|DP|_{E^s(S)}\| < \theta$$

and

$$\|DP|_{E^u(S)}\| > \theta^{-1}.$$

**2.2. Lorenz map.** Let  $\pi$  be the canonical projection of section  $S$  into  $\mathcal{F}$ , i.e.  $\pi$  assigns to each point  $w \in S$  the leaf that contained it.

By invariance of  $\mathcal{F}$ , one dimensional *Lorenz map*

$$g : (\mathcal{F} \setminus \Gamma) \rightarrow \mathcal{F}$$

is uniquely defined so that the next diagram

$$\begin{array}{ccc} S \setminus \Gamma & \xrightarrow{P} & S \\ \pi \downarrow & & \downarrow \pi \\ \mathcal{F} \setminus \Gamma & \xrightarrow{g} & \mathcal{F} \end{array}$$

commutes, i.e.  $g \circ \pi = \pi \circ P$  on  $S \setminus \Gamma$ .

One may identify quotient space  $S/\mathcal{F}$  with a compact interval as  $I = [-1, 1]$ , and so

$$g : [-1, 1] \setminus \{0\} \rightarrow [-1, 1]$$

is smooth on  $I \setminus \{0\}$  with a discontinuity and infinite left and right derivatives at 0. Note that the symmetry of the Lorenz equations implies  $g(-x) = -g(x)$ .

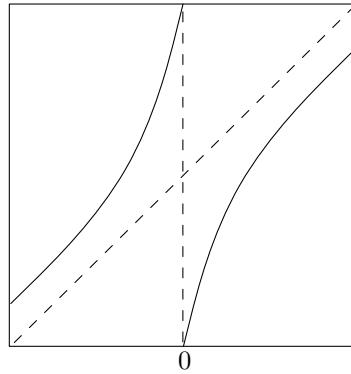


FIGURE 2.2. Lorenz map  $g$

**2.3. The attractor.** The geometric Lorenz attractor  $\Lambda$  is characterized as follows. Note that the restriction of  $h$  to both  $\{x < 0\}$  and  $\{x > 0\}$  admits continuous extensions to the point 0. Hence,  $h$  may be considered as an extention to a 2-valued map at 0 and continuous on both  $\{x \leq 0\}$  and  $\{x \geq 0\}$ . Correspondingly, the restriction of the Poincaré transformation to each of the connected components of  $S \setminus \Gamma$  admits a continuous extension to the closure, each one collapsing the curve  $\Gamma$  to a single point. Thus,  $P$  may also be considered as a 2-valued transformation defined on the whole cross-section and continuous on the closure of each of the connected components. Let

$$\Lambda_P = \bigcap_{n \geq 0} P^n(S) \subset S.$$

We define  $\Lambda$  to be the saturation of  $\Lambda_P$  by the flow of  $X$ , that is, the orbits of its points. Therefore, orbits in  $\Lambda$  intersect the cross-section infinitely often, both forward and backward.

This attractor has a complicated fractal structure that can be described as "a cantor book with uncountably many pages" joined along a spine corresponding to

the unstable manifold of the singularity point  $\mathbf{0}$  (see figure 4). Notice that the unstable manifold accumulates on itself, and so the geometry of  $\Lambda$  is indeed very complex.

Dynamical properties of  $\Lambda$  may be deduced from corresponding properties for the quotient map  $h$ . More important, a quotient map with similar properties exists for all nearby vector fields, and so such properties are robust for these flows.

### 3. Physical probability

In this section, we study the construction of the unique physical probability measures invariant by Lorenz attractor.

The existence of a unique absolutely continuous invariant probability  $\mu_g$  which is ergodic and  $0 < \frac{d\mu_g}{dm} < +\infty$  for Lorenz one-dimensional map  $g$  is well-known (see [V97]).

Now, One may construct an invariant probability measure  $\mu_P$  on  $\Lambda_P$ , as the lifting of  $\mu_g$ . Indeed, we may think of  $\mu_g$  as a probability measure on Borel subsets of  $\mathcal{F}$ . Since  $P$  is uniformly contracting on leaves of  $\mathcal{F}$ , one concludes that the sequence

$$(P_*^n \mu_g)_{n \geq 1},$$

of push-forwards is weak\*-Cauchy: given any continuous  $\varphi : S \rightarrow \mathbb{R}$ ,

$$\int \varphi \, d(P_*^n \mu_g) = \int (\varphi \circ P^n) \, d\mu_g, \quad n \geq 1,$$

is a Cauchy sequence in  $\mathbb{R}$ . Define the probability measure  $\mu_P$  as the weak\*-limit of this sequence that is

$$\int \varphi \, d\mu_P = \lim_{n \rightarrow +\infty} \int \varphi \, d(P_*^n \mu_g),$$

for each continuous  $\varphi$ . Thus  $\mu_P$  is invariant under  $P$ , and it is a physical probability measure on Borel subsets of  $\Lambda_P$  which is ergodic.

Later, as the Poincaré  $\frac{1}{2}$  transformation maybe extended to the Lorenz flow through a suspension construction, the invariant probability  $\mu_P$  corresponds to an ergodic physical probability measure on  $\Lambda$ : Denote by  $R : S \setminus \Gamma \rightarrow (0, +\infty)$  the *first return time* to  $S$  defined by

$$P(w) = X_{R(w)}(w).$$

The first return time  $R$  is Lebesgue integrable, since  $P(w) \approx |\log(d(w, \Gamma))|$ , for  $w$  close to  $\Gamma$ . This follows that

$$\int R \, d\mu_P < +\infty.$$

Let  $\sim$  be an equivalence relation on  $S \times \mathbb{R}$  by  $(w, R(w)) \sim (P(w), 0)$ . Set  $N = (S \times \mathbb{R}) / \sim$  and define the finite measure

$$\tilde{\mu} = \pi_*(\mu_P \times dt)$$

where  $\pi : S \times \mathbb{R} \rightarrow N$  is the quotient map and  $dt$  is Lebesgue measure in  $\mathbb{R}$ . Define  $\phi : N \rightarrow M$  as  $\phi(w, t) = X^t(w)$ , and let

$$\mu_X = \phi_* \tilde{\mu}.$$

One may check also that

$$\frac{1}{T} \int_0^T \varphi(X^t(w)) dt \rightarrow \int \varphi d\mu_X$$

as  $T \rightarrow +\infty$ , for any continuous  $\varphi : M \rightarrow \mathbb{R}$ , and Lebesgue almost every  $w \in \phi(N)$ .  
For more details see [V97,APPV09]

## CHAPTER 3

# Symbolic Structure

In this chapter, we introduce a symbolic structure on the Lorenz attractor. More precisely, we will get a full measure subset of  $\Lambda$  which has the structure of complete shift with respect to some return map.

### 1. Shift map

Suppose that  $\Sigma = \mathbb{N}^{\mathbb{Z}}$ , the full shift space with countably many symbols, and  $f : \Sigma \rightarrow \Sigma$  is the shift map

$$f((x_n)_{n \in \mathbb{Z}}) = (x_{n+1})_{n \in \mathbb{Z}}.$$

A *cylinder* of  $\Sigma$  is any subset

$$[\iota_k, \dots; \iota_0; \dots \iota_l] = \{x : x_j = \iota_j, j = k, \dots, l\}$$

of  $\Sigma$ . We endowed  $\Sigma$  with topology generated by cylinders. The local stable and local unstable sets of any  $x \in \Sigma$  are defined as

$$W_{\text{loc}}^s(x) = \{y : x_n = y_n, n \geq 0\}$$

and

$$W_{\text{loc}}^u(x) = \{y : x_n = y_n, n < 0\}.$$

Let  $\Sigma_u = \mathbb{N}^{\{n \geq 0\}}$  and  $\Sigma_s = \mathbb{N}^{\{n < 0\}}$ . The map

$$x \mapsto (x_s, x_u)$$

is a homeomorphism from  $\Sigma$  onto  $\Sigma_s \times \Sigma_u$  where  $x_s = \pi_s(x)$  and  $x_u = \pi_u(x)$ , for natural projections  $\pi_s : \Sigma \rightarrow \Sigma_s$  and  $\pi_u : \Sigma \rightarrow \Sigma_u$ . We also consider the maps  $f_s : \Sigma_s \rightarrow \Sigma_s$  and  $f_u : \Sigma_u \rightarrow \Sigma_u$  defined by

$$f_u \circ \pi_u = \pi_u \circ f,$$

$$f_s \circ \pi_s = \pi_s \circ f^{-1}.$$

Assume that  $\mu_f$  is an ergodic probability measure for  $f$ . Let  $\mu_s = (\pi_s)_* \mu_f$  and  $\mu_u = (\pi_u)_* \mu_f$  be the images of  $\mu_f$  under the natural projections. It is easy to see that  $\mu_s$  and  $\mu_u$  are ergodic probabilities for  $f_s$  and  $f_u$ , respectively. Notice that  $\mu_s$  and  $\mu_u$  are positive on cylinders, by definition.

We say that  $\mu_f$  has product structure if there exists a measurable density function  $\omega : \Sigma \rightarrow (0, +\infty)$  such that

$$\mu_f = \omega(x)(\mu_s \times \mu_u).$$

**Notation 3.1.** For simplicity, we omit  $u$  in the notation of  $f_u$ ,  $\pi_u$ ,  $\Sigma_u$  and  $\mu_u$  and represent these objects by  $\hat{f}$ ,  $\hat{\pi}$ ,  $\hat{\Sigma}$  and  $\hat{\mu}$ .

## 2. Markov structure in dimension 1

We now consider the Lorenz map  $g : I \setminus \{0\} \rightarrow I$ . A return map  $\hat{g}$  to some domain  $\hat{I} \subset I$  is a *Markov map*, if there is a countable *Markov partition*  $\{\hat{I}(l) : l \in \mathbb{N}\}$  of  $\hat{I}$ , Lebesgue mod 0, such that

- (i)  $\hat{g}$  maps any  $\hat{I}(l)$  bijectively on the whole domain  $\hat{I}$ , and
- (ii) for any sequence  $(l_n)_{n \geq 0}$ , the intersection of  $\hat{g}^{-n}(\hat{I}(l_n))$  over all  $n \geq 0$  consists of exactly one point.

Indeed, we can define a return time function  $r : \hat{I} \rightarrow \mathbb{N}$  which is constant on each  $\hat{I}(l)$  such that the Markov map  $\hat{g}$  is defined by

$$\hat{g}|_{\hat{I}(l)} = g^{r(\hat{I}(l))}$$

as a bijection from  $\hat{I}(l)$  onto  $\hat{I}$ .

In this case,  $\hat{g}$  maybe seen as the shift map on  $\hat{\Sigma}$ : there exists a conjugation between  $\hat{f}$  and  $\hat{g}$  presented by the next commuting diagram

$$\begin{array}{ccc} \hat{\Sigma} & \xrightarrow{\hat{f}} & \hat{\Sigma} \\ \hat{\phi} \downarrow & & \downarrow \hat{\phi} \\ \hat{I} & \xrightarrow{\hat{g}} & \hat{I} \end{array}$$

where the bijection  $\hat{\phi}$  maybe defined as

$$\hat{\phi} : (l_n)_{n \geq 0} \mapsto \bigcap_{n \geq 0} \hat{g}^{-n}(\hat{I}(l_n)).$$

Now, we recall some results of [D06] which let us to obtain a Markov map  $\hat{g}$ .

**Theorem 3.1.** [D06] *There exists a return map  $\hat{g}$ , an interval  $\hat{I} = (-\delta, \delta)$ ,  $0 < \delta < 1$ , and a partition  $\{\hat{I}(l) : l \in \mathbb{N}\}$  to subintervals of  $\hat{I}$ , Lebesgue mod 0, for which  $\hat{g}$  maps any  $\hat{I}(l)$  diffeomorphically onto  $\hat{I}$ , and the return time  $r$  is Lebesgue integrable. Moreover, there exists a constant  $0 < c < 1$  such that, for all  $x, y$  in any  $\hat{I}(l)$ ,*

$$\log \frac{|\hat{g}'(x)|}{|\hat{g}'(y)|} \leq c^{n(x,y)}$$

where  $n(x, y) = \min\{n : \hat{g}^n(x) \in \hat{I}(l_i), \hat{g}^n(y) \in \hat{I}(l_j), i \neq j\}$ .

Note that, as Lorenz map  $g$  is uniformly expanding, the intesection of  $(\hat{g}^{-n}(J(l_n)))$  over all  $n \geq 0$  consists of exactly one point.

## 3. Markov structure in dimension 2

Now, we consider the domain  $\hat{S} = \pi^{-1}(\hat{I})$  and corresponding to the Markov partition of  $\hat{I}$  define a Markov partition  $\{\hat{S}(l) = \pi^{-1}(\hat{I}(l)) : l \in \mathbb{N}\}$  of  $\hat{S}$ . The return time is defined as

$$r(w) = r(\pi(w)).$$

Hence, there exists a return map  $\hat{P}$  to  $\hat{S}$  induced by  $\hat{g}$  as:

$$\hat{P}(w) = P^{r(w)}(w),$$

for any  $w \in \hat{S}$ . Indeed

$$\hat{g} \circ \pi = \pi \circ \hat{P}.$$

Suppose that

$$\Lambda_{\hat{P}} = \bigcap_{n \geq 0} \hat{P}^n(\hat{S})$$

the maximal  $\hat{P}$ -invariant subset of  $\hat{S}$ . So  $\hat{P}|_{\Lambda_{\hat{P}}}$  is homeomorphically conjugate to the complete shift  $f$ .

It is enough to define a bijection as follows: to any  $w$ , we assign  $(l_n)_{n \in \mathbb{Z}}$  where  $\hat{P}^n(w) \in \hat{S}_{l_n}$ . In other words, since  $\bigcap_{n \in \mathbb{Z}} \hat{P}^{-n}(\hat{S}(l_n))$  consists of exactly one point, one may define a bijection  $\phi : \Sigma \rightarrow \Lambda_{\hat{P}}$  as

$$\phi : (l_n)_{n \in \mathbb{Z}} \mapsto \bigcap_{n \in \mathbb{Z}} \hat{P}^{-n}(\hat{S}(l_n)).$$

The following commuting diagram

$$\begin{array}{ccc} \Sigma & \xrightarrow{f} & \Sigma \\ \phi \downarrow & & \downarrow \phi \\ \Lambda_{\hat{P}} & \xrightarrow[\hat{P}]{} & \Lambda_{\hat{P}} \end{array}$$

explains this observation.

#### 4. Lifting the absolutely continuous probability

The normalized restriction  $\hat{\mu}$  of  $\mu_g$  to the domain of  $\hat{g}$  is an absolutely continuous ergodic probability for  $\hat{g}$  and then for  $\hat{f}$ , for conjugacy.

The natural extension of  $\hat{f}$  maybe realized as the complete shift map  $f$  on  $\Sigma$ , and the *lift*  $\mu$  of  $\hat{\mu}$  is the unique  $f$ -invariant probability measure on  $\Sigma$  such that

$$\hat{\pi}_* \mu = \hat{\mu}.$$

Indeed, let  $\mu(E) = \hat{\mu}(\hat{\pi}(E))$ , for any cylinder  $E \subset \Sigma$ . Using properties of  $\hat{\mu}$  and applying Approximation Theorem, one may define  $\mu$  on any Borelian subset of  $\Sigma$ . Finally, Extension Theorem for probabilities concludes a unique extended probability  $\mu$  on Borelians of  $\Sigma$  for which, by construction,  $\hat{\pi}_* \mu = \hat{\mu}$ .  $\mu$  is an invariant ergodic probability for  $f$ .

**Proposition 3.1.** *The lift probability  $\mu$  has product structure.*

PROOF. Note that the second part of Theorem 3.1 states that there exists constant  $0 < c < 1$  such that, for all  $\hat{x}, \hat{y}$  in the same cylinder,

$$\log \frac{J\hat{f}(\hat{x})}{J\hat{f}(\hat{y})} \leq c^{n(x,y)}.$$

The rest of the proof is based on 4 main steps stated with more details in Appendix A of [AV07]:

*Step 1.* If  $\hat{x}, \hat{y} \in \hat{\Sigma}$  then for any  $x \in W_{\text{loc}}^s(\hat{x})$  and  $y \in W_{\text{loc}}^u(x) \cap W_{\text{loc}}^s(\hat{y})$ , the limit

$$J_{\hat{x}, \hat{y}}(x) = \lim_{n \rightarrow \infty} \frac{J\hat{f}^n(\hat{x}^n)}{J\hat{f}^n(\hat{y}^n)},$$

where  $\hat{x}^n = \hat{\pi}(f^{-n}(x))$ ,  $\hat{y}^n = \hat{\pi}(f^{-n}(y))$ , exists uniformly on  $\hat{x}, \hat{y}, x$ . Moreover,

$$(\hat{x}, \hat{y}, x) \mapsto J_{\hat{x}, \hat{y}}(x)$$

is continuous and uniformly bounded from zero and infinity.

Indeed, we observe that

$$\log \frac{J\hat{f}^n(\hat{x}^n)}{J\hat{f}^n(\hat{y}^n)} \leq \sum_{i=1}^n \log \frac{J\hat{f}(\hat{x}^i)}{J\hat{f}(\hat{y}^i)}.$$

Since  $\hat{x}^i$  and  $\hat{y}^i$  are in the same cylinder, the series is uniformly bounded by  $\sum_i c^n(\hat{x}^i, \hat{y}^i)$ . But  $n(\hat{x}^i, \hat{y}^i)$  is strictly increasing that implies uniform convergence of the series.

*Step 2.* If  $\{\mu_{\hat{x}} : \hat{x} \in \hat{\Sigma}\}$  be an integration of  $\mu$  then, for  $\mu$ -almost every  $\hat{x} \in \hat{\Sigma}$ ,

$$\mu_{\hat{x}}(\xi_n) = \frac{1}{J\hat{f}^n(\hat{x}^n)},$$

for every cylinder  $\xi_n = [x_{-n}, \dots, x_{-1}]$ ,  $n \geq 1$ , and any  $x \in \xi_n \times \{\hat{x}\}$ .

*Step 3.* Given any disintegration, by the last step, one may find a disintegration  $\{\mu_{\hat{x}} : \hat{x} \in \hat{\Sigma}\}$  of  $\mu$  so that

$$\mu_{\hat{y}} = J_{\hat{x}, \hat{y}} \mu_{\hat{x}}.$$

*Step 4.* Fixing any point  $\hat{x}_0 \in \hat{\Sigma}$ , one may define

$$\hat{\omega}(x_s, x_u) = J_{\hat{x}_0, x_u}(x_s, x_u),$$

for every  $x = (x_s, x_u) \in \Sigma$ . By Step 2,  $\mu_{x_u} = \hat{\omega}(x_s, x_u)$ , for any  $x_u \in \hat{\Sigma}$ .

The lift measure  $\mu$  projects to  $\hat{\mu} = \mu_{\hat{u}}$ , but the projection  $\mu_s$  to  $\Sigma_s$  is given by

$$\mu_s = \mu_{\hat{x}_0} \int_{\hat{\Sigma}} \hat{\omega}(x_s, x_u) d\hat{\mu}.$$

Therefore

$$\mu = \omega(x_s, x_u) \mu_s \times \mu_u$$

where

$$\omega(x_s, x_u) = \frac{1}{\int_{\hat{\Sigma}} \hat{\omega}(x_s, x_u) d\hat{\mu}} \hat{\omega}(x_s, x_u).$$

□

**Remark 3.1.** As conditional probabilities in the proof of the last proposition vary continuously with the base point so the density function  $\omega$  is continuous. Also,  $\omega$  is bounded from zero and infinity.

## 5. Suspending the bi-lateral shift by the flow

We define  $\hat{\Lambda}$  to be the saturation of  $\Lambda_{\hat{P}}$  by the Lorenz flow  $X^t$ , that is, the orbits of its points by  $X^t$ . Hence

$$\hat{\Lambda} \subset \Lambda$$

where, by ergodicity of  $\mu_X$ , we observe that  $\hat{\Lambda}$  is of  $\mu_X$  full measure since it is invariant and has positive measure.

**Notation 3.2.** As  $\mu_X(\hat{\Lambda}) = 1$ , it is sufficient to consider  $\hat{\Lambda}$  in the place of  $\Lambda$ . Notice that  $\hat{\Lambda}$  is not necessarily compact.

A return time to  $\Lambda_{\hat{P}}$  is defined as

$$\begin{aligned} \hat{R} : \Lambda_{\hat{P}} &\rightarrow \mathbb{R} \\ \hat{R}(x) &= \sum_{j=0}^{r(x)-1} R(P^j(x)), \end{aligned}$$

for any  $x \in \Lambda_{\hat{P}}$ , and

$$\int \hat{R}(x) dm = \int r(x) \left[ \frac{1}{r(x)} \sum_{j=0}^{r(x)-1} R(P^j(x)) \right] dm \leq \int r(x) \left( \int R dm \right) dm < +\infty$$

which implies

$$\int \hat{R} d\mu < +\infty.$$

**5.1. Cocycle reduction.** Corresponding to any linear cocycle  $A^t$  over  $\hat{\Lambda}$  one may define the linear cocycle  $A_{\hat{P}}$  on  $\Lambda_{\hat{P}}$  by

$$A_{\hat{P}}(x) = A^{\hat{R}(x)}(x),$$

for any  $x \in \Lambda_{\hat{P}}$ .

**Lemma 3.1.** Lyapunov spectrum of  $A^t$  is simple if and only if Lyapunov spectrum of  $A_{\hat{P}}$  is simple.

PROOF. The Lyapunov exponents of  $A_{\hat{P}}$  are obtained by multiplying those of  $A^t$  by the average return time

$$s_n(x) = \sum_{j=0}^{n-1} \hat{R}(P^j(x)), \quad w \in \Lambda_{\hat{P}}.$$

Indeed, given any non zero vector  $v$ ,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \|A_{\hat{P}}^n(x)v\| = \lim_{n \rightarrow \infty} \frac{1}{n} \log \|A^{s_n(x)}(x)v\|$$

which, for  $\mu$ -almost every  $x$ , this is equal to

$$\lim_{n \rightarrow \infty} \frac{1}{n} s_n(x) \lim_{m \rightarrow \infty} \frac{1}{m} \log \|A^m(x)v\|.$$

But  $\frac{1}{n}s_n(x)$  converges to  $\int \hat{R} d\mu < +\infty$ . The proof is now completed.  $\square$



## CHAPTER 4

### Fiber Bunched Cocycles

In this chapter, we study fiber bunched linear cocycles over complete shift map, and show that fiber bunched cocycles admit holonomy maps as a family of linear maps on fiber bundles which are smooth maps with respect to the cocycle.

Assume that  $A^t$  is a Hölder continuous and fiber bunched linear cocycle over  $\hat{\Lambda}$ . Then, by definition,

$$\|A_{\hat{P}}(x)\| \|A_{\hat{P}}(x)^{-1}\| \theta^{r(x)\eta} < \tau,$$

for any  $x \in \Lambda_{\hat{P}}$ . We induce the differentiable structure of  $\Lambda_{\hat{P}}$  on  $\Sigma$  by the bijection  $\phi$  defined in the last chapter.

In general, suppose that  $\Sigma$  is endowed with a metric  $d$  for which

- (i)  $d(f(y), f(z)) \leq \theta(x)d(y, z)$ , for all  $y, z \in W_{\text{loc}}^s(x)$ ,
  - (ii)  $d(f^{-1}(y), f^{-1}(z)) \leq \theta(x)d(y, z)$ , for all  $y, z \in W_{\text{loc}}^u(x)$ ,
- where  $0 < \theta(x) \leq \theta < 1$ , for all  $x \in \Sigma$ .

Let  $A$  be an  $\eta$ -Hölder linear cocycle over  $f$ .

**Definition 4.1.** *We say that  $A$  is fiber bunched if there exists some  $\tau \in (0, 1)$ , for which*

$$\|A(x)\| \|A(x)^{-1}\| \theta(x)^\eta < \tau,$$

for any  $x \in \Sigma$ .

**Remark 4.1.** *Fiber bunching is an open condition: given any  $\bar{\tau} \in (\tau, 1)$ , there exists a  $C^0$  neighborhood  $\mathcal{U}$  of  $A$  so that, for any  $B \in \mathcal{U}$ ,  $B$  is also fiber bunched, by definition. However, for simplicity, we will denote by  $\tau$  the fiber bunching constant with respect to  $\mathcal{U}$ .*

The difference between this definition and definition of fiber bunching in [ASV] is the lack of compactness of ambient space  $\Sigma$ . In fact, in [ASV], one may take fiber bunching constant  $\tau = 1$ , just by compactness.

The next lemma asserts that fiber bunching induces a weaker fiber bunching notion, rather a uniform constant, on points in the local stable set. Note that all results hold, up to appropriate adjustments, under weaker condition that the last definition expressed for some power  $N \geq 1$ .

**Notation 4.1.** Set

$$\theta^n(x) = \theta(f^{n-1}(x)) \dots \theta(x).$$

**Lemma 4.1.** *If  $A$  is fiber bunched then, there exists some constant  $C > 0$  such that*

$$\|A^n(y)\| \|A^n(z)^{-1}\| \theta^n(x)^\eta \leq C\tau^n,$$

for any  $y, z \in W_{\text{loc}}^s(x)$ , and all  $n \geq 1$ .

PROOF. Sub-multiplicativity of norms implies that

$$\| A^n(y) \| \| A^n(z)^{-1} \| \leq \prod_{j=0}^{n-1} \| A(f^j(y)) \| \| A(f^j(z))^{-1} \|.$$

By regularity of cocycle  $A$ , there is  $C_1 > 0$  such that

$$\| A(f^j(y)) \| / \| A(f^j(x)) \| \leq \exp(C_1 d(f^j(x), f^j(y))^\eta) \leq \exp(C_1 \theta^j(x)^\eta d(x, y)^\eta).$$

It is similar for  $\| A(f^j(z))^{-1} \| / \| A(f^j(x))^{-1} \|$ . So, the right hand side in lemma is bounded above by

$$\exp[C_1 \sum_{j=0}^{n-1} \theta^j(x)^\eta (d(x, y)^\eta + d(x, z)^\eta)] \prod_{j=0}^{n-1} \| A(f^j(x)) \| \| A(f^j(x))^{-1} \| \theta^{n\eta}.$$

Since  $\theta(x) < \theta < 1$ , the first factor is bounded by some uniform constant  $C > 0$ , and fiber bunching implies that the second one is bounded by  $\tau^n$ . The proof is now completed.  $\square$

## 1. Holonomy maps

Let  $H_{x,y}^n = A^n(y)^{-1} A^n(x)$ .

**Definition 4.2.** *A cocycle  $A$  admits  $s$ -holonomy if*

$$H_{x,y}^s = \lim_{n \rightarrow +\infty} H_{x,y}^n$$

*exists for any pair of points  $x, y$  in the same local stable set.  $u$ -holonomy is defined in a similar way, when  $n \rightarrow -\infty$ , for pairs of points in the same local unstable set.*

**Proposition 4.1.** *If  $A$  is fiber bunched then, for all  $x$  and any  $y \in W_{\text{loc}}^s(x)$ ,  $s$ -holonomy  $H_{x,y}^s$  exists, where*

- (a)  $H_{x,y}^s = H_{z,y}^s \cdot H_{x,z}^s$ , for any  $z \in W_{\text{loc}}^s(x)$ , and  $H_{y,x}^s \cdot H_{x,y}^s = \text{id}$ ,
- (b)  $H_{f^j(x), f^j(y)}^s = A^j(y) \circ H_{x,y}^s \circ A^j(x)^{-1}$ , for all  $j \geq 1$ .

PROOF. We have

$$\| H_{x,y}^{n+1} - H_{x,y}^n \| \leq \| A^n(x)^{-1} \| \| A(f^n(x))^{-1} A(f^n(y)) - \text{id} \| \| A^n(y) \|.$$

By regularity of  $A$ , there is  $C_2 > 0$  such that the middle factor is bounded by

$$C_2 d(f^n(x), f^n(y))^\eta \leq C_2 \theta^n(x)^\eta d(x, y)^\eta,$$

and hence, by the last lemma

$$(2) \quad \| H_{x,y}^{n+1} - H_{x,y}^n \| \leq C C_2 \tau^n d(x, y)^\eta.$$

As  $\tau < 1$ , this implies that  $H_n(x, y)$  is a cauchy sequence, uniformly on  $x, y$ , and therefore, it is uniformly convergent. This proves the first part of proposition. (a) follows immediately from definition, and

$$(3) \quad A^n(f^j(y))^{-1} A^n(f^j(x)) = A^j(y) A^{n+j}(y)^{-1} A^{n+j}(x) A^j(x)^{-1}$$

proves (b). The proof is now completed.  $\square$

**Remark 4.2.** The  $s$ -holonomy  $H_{x,y}^s$  vary continuously on  $(x,y)$  in the sense that the map

$$(x,y) \rightarrow H_{x,y}^s$$

is continuous on  $W_n^s = \{(x,y) : f^n(y) \in W_{\text{loc}}^s(x)\}$ , for every  $n \geq 0$ . It is, in fact, a direct consequence of the uniform limit on (3) when  $(x,y) \in W_0^s$ , for instance. The general case  $n > 0$  follows immediately, by (b) of the last proposition .

Indeed, as the constants  $C, \bar{C}$  may be taken uniformly on  $\mathcal{U}$ , the Cauchy estimate in (3) is also locally uniform on  $A$ . Therefore, one may consider this notion of dependence:

$$(A,x,y) \rightarrow H_{A,x,y}^s$$

is continuous on  $\mathcal{C}^{r,\rho}(M,d,\mathbb{C}) \times W_n^s$ , for all  $n \geq 0$ .

There exist dual expressions of last results for u-holonomies, for points in  $W_{\text{loc}}^u(x)$ .

## 2. Dependence on cocycle

Notice that  $\mathcal{C}^{r,\rho}(\Sigma, d, \mathbb{C})$  is the Banach space of all  $C^{r,\rho}$  maps from  $\Sigma$  to the space of all  $d \times d$  invertible matrices, and so the tangent space at each point  $A \in \mathcal{C}^{r,\rho}(\Sigma, d, \mathbb{C})$  is naturally identified with that Banach space.

**Proposition 4.2.** If  $A$  is fiber bunched then the map

$$B \mapsto h_{B,x,y}^s$$

is of class  $C^1$  on  $\mathcal{U}$ , for any  $y \in W_{\text{loc}}^s(x)$ , and

$$\begin{aligned} \partial_B h_{B,x,y}^s(\dot{B}) = \sum_{i=0}^{+\infty} B^i(y)^{-1} [H_{B,f^i(x),f^i(y)}^s B(f^i(x))^{-1} \dot{B}(f^i(x)) - \\ B(f^i(y))^{-1} \dot{B}(f^i(y))] H_{B,f^i(x),f^i(y)}^s B^i(x). \end{aligned}$$

PROOF. First, we show that the expression of  $\partial_B H_{B,x,p}^s$  is well-defined. Let  $i \geq 0$ .

$$(4) \quad H_{B,f^i(x),f^i(y)}^s B(f^i(x))^{-1} \dot{B}(f^i(x)) - B(f^i(y))^{-1} \dot{B}(f^i(y)) H_{B,f^i(x),f^i(y)}^s$$

may be written as

$$\begin{aligned} (H_{B,f^i(x),f^i(y)}^s - \text{Id}) B(f^i(x))^{-1} \dot{B}(f^i(x)) + B(f^i(y))^{-1} \dot{B}(f^i(y)) (\text{Id} - H_{B,f^i(x),f^i(y)}^s) \\ + B(f^i(x))^{-1} \dot{B}(f^i(x)) - B(f^i(y))^{-1} \dot{B}(f^i(y)). \end{aligned}$$

By last proposition and Remark , there is some uniform  $\bar{C} > o$  such that the first term is bounded by

$$\bar{C} d(f^i(x), f^i(y))^\eta \| B(f^i(x))^{-1} \| \| \dot{B}(f^i(x)) \|.$$

It is the same for second term. The third one is equal to

$$B(f^i(x))^{-1} [\dot{B}(f^i(x)) - \dot{B}(f^i(y))] + [B(f^i(x))^{-1} - B(f^i(y))^{-1}] \dot{B}(f^i(y)),$$

and since  $B^{-1}$  and  $\dot{B}$  are Hölder continuous, using (3), it is bounded by

$$\| \|B^{-1}\|_{0,0} \eta(\dot{B}) + \eta(B^{-1}) \| \dot{B} \|_{0,0} \| d(f^i(x), f^i(y))^\eta \| \leq \| B^{-1} \|_{0,\eta} \| \dot{B} \|_{0,\eta} d(f^i(x), f^i(y))^\eta.$$

Hence (5) is bounded by

$$(2\bar{C} + 1) C_3 \| \dot{B} \|_{0,\eta} d(f^i(x), f^i(y))^\eta \leq (2\bar{C} + 1) C_3 \| \dot{B} \|_{0,\eta} \theta^i(x)^\eta d(x, y)^\eta$$

where  $C_3 = \sup\{\|B^{-1}\|_{0,\eta}, B \in \mathcal{U}\}$ . So, the  $i$ th term in the expression of  $\partial_B h_{B,y,z}^s(\dot{B})$  is bounded by

$$(5) \quad C_4 \|\dot{B}\|_{0,\eta} \theta^i(x)^\eta d(x,y)^\eta \|B^i(p)^{-1}\| \|B^i(x)\| \leq C_4 \tau^i d(x,y)^\eta \|\dot{B}\|_{0,\eta},$$

by fiber bunching hypothesis where  $C_4 = (2\bar{C} + 1)C_3$ . Therefore, as  $\tau < 1$ , the series (5) does converge, uniformly.

Now, we count the derivative of  $H_{B,x,y}^s$ . By definition,  $H_{B,x,y}^s$  is the uniform limit of  $H_{B,x,y}^n = B^n(y)^{-1}B^n(x)$  when  $n \rightarrow \infty$ . Indeed,  $H_{B,x,y}^n$  is a differentiable function of  $B$  with derivative  $\partial_B H_{B,x,y}^n(\dot{B})$  equal to

$$\sum_{i=0}^{n-1} B^i(y)^{-1} [H_{B,f^i(x),f^i(y)}^{n-i} B(f^i(x))^{-1} \dot{B}(f^i(x)) - B(f^i(y))^{-1} \dot{B}(f^i(y)) H_{B,f^i(x),f^i(y)}^{n-i}] B^i(x),$$

for all  $\dot{B} \in T_B \mathcal{C}^{r,\rho}(\Sigma, d, \mathbb{C})$  and any  $n \geq 1$ .

It suffices to show that  $\partial_B H_{B,x,y}^n$  converges uniformly to  $\partial_B H_{B,x,y}^s$  as  $n \rightarrow \infty$ . By (2), for any  $\tau_0 \in (\tau, 1)$ ,

$$\begin{aligned} \|H_{B,x,y}^s - H_{B,x,y}^n\| &\leq CC_2 \sum_{i=n}^{\infty} \tau^i d(x,y)^\eta \\ &\leq C_5 \tau^n d(x,y)^\eta \\ &\leq C_5 \tau_0^n d(x,y)^\eta, \end{aligned}$$

for some uniform constant  $C_5 > 0$ . Then, for all  $0 \leq i \leq n$ ,

$$\begin{aligned} \|H_{B,f^i(x),f^i(y)}^s - H_{B,f^i(x),f^i(y)}^{n-i}\| &\leq C_5 \tau_0^{(n-i)} d(f^i(x), f^i(y))^\eta \\ &\leq C_5 \tau_0^{(n-i)} \theta^i(x)^\eta d(x,y)^\eta. \end{aligned}$$

It follows, by Lemma 4.1, that the difference between the  $i$ th terms in the expressions of  $\partial_B H_{B,x,y}^s$  and  $\partial_B H_{B,x,y}^n$  is bounded by

$$2C_3 C_5 \tau_0^{n-i} \theta^i(x)^\eta d(x,y)^\eta \|B^i(y)^{-1}\| \|B^i(x)\| \leq 2C_3 C_5 \tau_0^{n-i} \tau^i d(x,y)^\eta.$$

Combining with,  $\|\partial_B H_{B,x,y}^s - \partial_B H_{B,x,y}^n\|$  is bounded by

$$\{2C_3 C_5 \tau_0^n \sum_{i=0}^{n-1} (\tau_0^{-1} \tau)^i + C_4 \sum_{i=n}^{+\infty} \tau^i\} d(x,y)^\eta \|\dot{B}\|_{0,\eta}.$$

Since  $\tau, \tau_0$  and  $(\tau_0^{-1} \tau)$  are strictly less than 1, therefore the series tends uniformly to 0 as  $n \rightarrow \infty$ . The proof is now completed.  $\square$

The dual result of the last proposition is the following.

**Proposition 4.3.** *If  $A$  is fiber bunched then*

$$\mathcal{U} \ni B \mapsto h_{B,x,y}^u$$

is of class  $C^1$ , and, for any  $y \in W_{\text{loc}}^u(x)$ ,

$$\begin{aligned} \partial_B h_{B,x,y}^u(\dot{B}) &= - \sum_{i=1}^{+\infty} B^{-i}(y)^{-1} [h_{B,f^{-i}(x),f^{-i}(y)}^s B(f^{-i}(x))^{-1} \dot{B}(f^{-i}(x)) \\ &\quad - B(f^{-i}(y))^{-1} \dot{B}(f^{-i}(y)) h_{B,f^{-i}(x),f^{-i}(y)}^s] B^{-i}(x). \end{aligned}$$

## CHAPTER 5

### Perturbation Tools

The proof of Theorem 2 and then Theorem 1 are based on a criterion for simplicity of the Lyapunov spectrum due to Avila and Viana [AV07]. This criterion requires the cocycle to satisfy two transversality conditions, called pinching and twisting, that will be recalled later. In the present chapter, we introduce the technical tools that we will use to show that most fiber bunched cocycles satisfy those conditions.

Suppose that  $p$  is a periodic point of  $f$ , and  $q$  a homoclinic point of  $p$ , i.e.  $q \in W_{\text{loc}}^u(p)$  and there is some multiple  $m \geq 1$  of  $\text{per}(p)$  such that  $f^m(q) \in W_{\text{loc}}^s(p)$ . We define the *transition map*

$$\Psi_{A,p,q} : \mathbb{C}_p^d \rightarrow \mathbb{C}_p^d$$

by

$$\Psi_{A,p,q} = H_{f^m(q),p}^s \cdot A^m(q) H_{p,q}^u \in \text{GL}(d, \mathbb{C}).$$

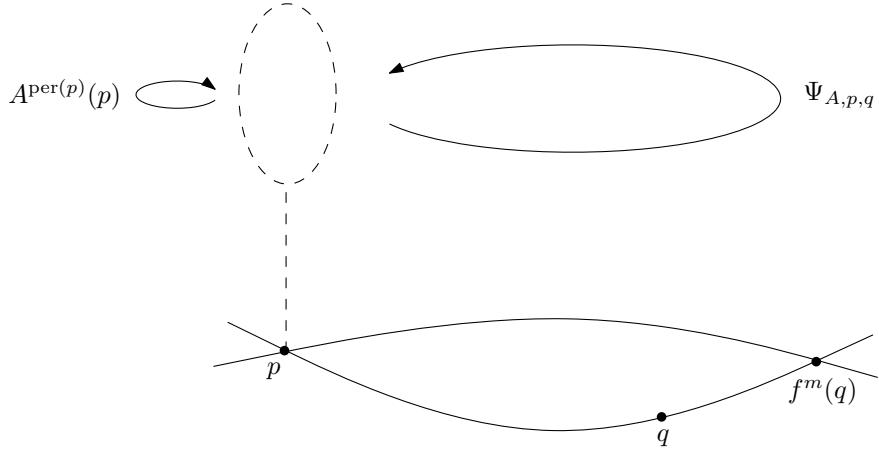


FIGURE 5.1. Perturbation along periodic orbit and homoclinic orbit

We remember that a submersion  $\mathcal{S}_1 \rightarrow \mathcal{S}_2$  means that all elements of  $\mathcal{S}_2$  are regular values which implies that every non-empty pre-image of any element of  $\mathcal{S}_2$  is a submanifold of  $\mathcal{S}_1$  with codimension equal to the dimension of  $\mathcal{S}_2$ . In a more general case, the pre-image of any submanifold of  $\mathcal{S}_2$  is a submanifold of  $\mathcal{S}_1$  with the same codimension.

## 1. Periodic orbits

As we mentioned before, the tangent space at any  $B \in \mathcal{C}^{r,\rho}(\Sigma, d, \mathbb{C})$  is identified naturally with the space of all  $C^{r,\rho}$  maps on  $\Sigma$  into the space of linear maps in  $\mathbb{C}^d$ . Indeed, we may give the tangent vectors  $\dot{B}$  as  $C^{r,\rho}$  maps which assign to every point of  $\Sigma$  a linear map on  $\mathbb{C}^d$ .

**Proposition 5.1.** *Let  $p$  be a periodic point of  $f$  then the application*

$$A \mapsto A^{\text{per}(p)}(p) \in \text{GL}(d, \mathbb{C}),$$

*is a submersion at any  $A \in \mathcal{C}^{r,\rho}(\Sigma, d, \mathbb{C})$ , even restricted to tangent vectors supported in some neighborhood of  $p$ .*

PROOF. Assume that  $\text{per}(p) = 1$ . It is easy to see that

$$\partial_A A(p)(\dot{A}) = \dot{A}(p).$$

Fix a neighborhood  $U_p$  of  $p$  such that  $p$  is the unique point of its orbit in  $U_p$ . Let  $\alpha : \Sigma \rightarrow [0, 1]$  be a  $C^{r,\rho}$  function vanishing outside  $U_p$ , and  $\alpha(p) = 1$ . For any  $\mathcal{A} \in \text{GL}(d, \mathbb{C})$ , define  $\dot{\mathcal{A}} \in T_B \mathcal{C}^{r,\rho}(\Sigma, d, \mathbb{C})$  as

$$\dot{\mathcal{A}}(w) = \mathcal{A}.A(p)^{-1}.\alpha(w)A(w).$$

Note that  $\dot{\mathcal{A}}$  is supported on  $U_p$ , and  $\dot{\mathcal{A}}(p) = \mathcal{A}$ . Hence  $\partial_A A(p)(\dot{\mathcal{A}}) = \mathcal{A}$ , as we have claimed. It is similar when  $\text{per}(p) > 1$  where in this case

$$\partial_A A^{\text{per}(p)}(p)(\dot{A}) = A(f^{\text{per}(p)-1}(p)) \dots \dot{A}(p) + \dots + \dot{A}(f^{\text{per}(p)-1}(p)) \dots A(p)$$

which, for tangent vectors supported on  $U_p$ , reduces to

$$A(f^{\text{per}(p)-1}(p)) \dots \dot{A}(p).$$

The proof is now completed. □

## 2. Homoclinic orbits

Assume that  $p$  is a periodic point of  $f$  and  $q$  some homoclinic point of  $p$ . The derivative of  $\Psi_{B,p,q} = H_{f^m(q),p}^s.B^m(q).H_{p,q}^u$  at a vector  $\dot{B}$  is given by

$$(6) \quad \begin{aligned} & \partial_B H_{f^m(q),p}^s(\dot{B}).B^m(q).H_{p,q}^u + \\ & H_{f^m(q),p}^s \cdot \partial_B B^m(q)(\dot{B}).H_{p,q}^u + \\ & H_{f^m(q),p}^s.B^m(q).\partial_B H_{p,q}^u(\dot{B}) \end{aligned}$$

where

$$\partial_B B^m(q)(\dot{B}) = B(f^{m-1}(q)) \dots \dot{B}(q) + \dots + \dot{B}(f^{m-1}(q)) \dots B(q),$$

by definition.

**Proposition 5.2.** *The application*

$$\mathcal{U} \ni B \mapsto \Psi_{B,p,q}$$

*is a submersion, even restricted to tangent vectors  $\dot{B}$  supported on a neighborhood of  $q$ , for any periodic point  $p$  and each homoclinic point  $q$  of  $p$ .*

PROOF. Without loss of generality, we assume that  $p$  is a fixed point of  $f$ , and  $m = 1$ . Let  $U_q$  be any neighborhood of  $q$  which is disjoint from the orbit of  $p$  and  $\{f^j(q) : j \neq 0\}$ . So, the expression in (6) reduces to

$$H_{f(q),p}^s \cdot \partial_B B(q)(\dot{B}) \cdot H_{p,q}^u = H_{f(q),p}^s \cdot \dot{B}(q) \cdot H_{p,q}^u.$$

Thus,  $\partial_B \Psi_{B,p,q}$  is given by

$$\dot{B} \mapsto H_{f(q),p}^s \cdot \dot{B}(q) \cdot H_{p,q}^u,$$

for any vector  $\dot{B}$  supported on  $U_q$ . We claim that

$$\Phi(\dot{B}) = H_{f(q),p}^s \cdot \dot{B}(q) \cdot H_{p,q}^u$$

is surjective on  $T_B \mathcal{C}^{r,\rho}(M, d, \mathbb{C})$ .

Let  $\beta : \Sigma \rightarrow [0, 1]$  be a  $C^{r,\rho}$  function vanishing outside  $U_q$ , where  $\beta(q) = 1$ . For any  $\mathcal{B} \in \text{GL}(d, \mathbb{C})$ , define  $\dot{\mathcal{B}} \in T_B \mathcal{C}^{r,\rho}(M, d, \mathbb{C})$  as

$$\dot{\mathcal{B}}(w) = (H_{B,f(q),p}^s)^{-1} \cdot \mathcal{B} \cdot B(q)^{-1} \cdot \beta(w) B(w) \cdot (H_{B,p,q}^u)^{-1}.$$

Note that  $\dot{\mathcal{B}}(q) = H_{B,f(q),p}^s \cdot \mathcal{B} \cdot H_{B,p,q}^u$ , and so  $\Phi(\dot{\mathcal{B}}) = \mathcal{B}$ , as we have claimed. The proof is now completed.  $\square$

### 3. The main perturbation

Now, we consider the main perturbation including both periodic and homoclinic orbits.

**Proposition 5.3.** *If  $A \in \mathcal{C}^{r,\rho}(M, d, \mathbb{C})$  is fiber bunched then the application*

$$\Theta : \mathcal{U} \rightarrow \text{GL}(d, \mathbb{C})^2$$

$$\Theta(B) = (B(p), \Psi_{B,p,q}),$$

$B \in \mathcal{U}$ , is a submersion, even restricted to the subspace of tangent vectors  $\dot{B}$  supported on some neighborhoods of  $p, q$ .

PROOF. Take  $U_p$  such that  $U_p \cap \text{orb}(p) = \{p\}$ ,  $U_p \cap \text{orb}(q) = \emptyset$ , and similarly  $U_q$  so that  $U_q \cap \text{orb}(q) = \{q\}$ ,  $U_q \cap \text{orb}(p) = \emptyset$ .

First note that, if  $\dot{B}$  is a tangent vector supported on  $U_p \cup U_q$ , so, there exist two tangent vectors  $\dot{B}_1$  supported on  $U_p$ , and  $\dot{B}_2$  supported on  $U_q$  such that  $\dot{B} = \dot{B}_1 + \dot{B}_2$ . Indeed, we may assume that

$$\dot{B}_1(x) = \begin{cases} \dot{B}(x) & x \in U_p \\ \mathbf{0} & x \notin U_p \end{cases}$$

and

$$\dot{B}_2(x) = \begin{cases} \dot{B}(x) & x \in U_q \\ \mathbf{0} & x \notin U_q. \end{cases}$$

So

$$\partial_B \Theta(B)(\dot{B}) = \partial_B \Theta(B)(\dot{B}_1) + \partial_B \Theta(B)(\dot{B}_2)$$

which is equal to

$$\begin{aligned} & (\partial_B B(p)(\dot{B}_1), \partial_B \Psi_{B,p,q}(\dot{B}_1)) + (\partial_B B(\dot{B}_2), \partial_B \Psi_{B,p,q}(\dot{B}_2)) = \\ & (\dot{B}_1(p), \partial_B \Psi_{B,p,q}(\dot{B}_1)) + (\mathbf{0}, \partial_B \Psi_{B,p,q}(\dot{B}_2)) \end{aligned}$$

By Propositions 5.1 and 5.2, for any  $(\mathcal{B}_1, \mathcal{B}_2) \in \mathrm{GL}(d, \mathbb{C})^2$ , there exist tangent vectors  $\dot{\mathcal{B}}_1$  supported on  $U_p$ , and then  $\dot{\mathcal{B}}_2$  supported on  $U_q$  such that

$$\partial_B \Psi_{B,p,q}(\dot{\mathcal{B}}_2) = \mathcal{B}_2 - \partial_B \Psi_{B,p,q}(\dot{\mathcal{B}}_1),$$

and therefore

$$\partial_B \Theta(B)(\dot{\mathcal{B}}) = (\mathcal{B}_1, \mathcal{B}_2)$$

where  $\dot{\mathcal{B}} = \dot{\mathcal{B}}_1 + \dot{\mathcal{B}}_2$  is supported on  $U_p \cup U_q$ .  $\square$

## CHAPTER 6

# Proof of the Main Results

Assume that  $p$  is a periodic point of  $f$  and  $q$  a homoclinic point of  $p$ .

**Definition 6.1.** we say that a cocycle  $A$  is pinching at  $p$  if all eigenvalues of  $A^{\text{per}(p)}(p)$  have distinct absolute values.  $A$  is twisting at  $p, q$  if, for any pair of invariant subspaces  $E_1, E_2$  of  $A^{\text{per}(p)}(p)$  with  $\dim E_1 + \dim E_2 = d$ ,

$$\Psi_{A,p,q}(E_1) \cap E_2 = \{\mathbf{0}\}.$$

A cocycle  $A$  is simple if there exist some periodic point  $p$  and some homoclinic point  $q$  of  $p$  such that  $A$  is both pinching at  $p$  and twisting at  $p, q$ .

**Theorem 6.1.** [AV07] If  $A$  is simple then the Lyapunov spectrum of  $A$  is simple.

So, to prove Theorem 2, it is sufficient to prove that simple cocycles are typical in the space of all fiber bunched linear cocycles.

### 1. Proof of Theorem 2

It suffices to show that the complement of simple cocycles does have arbitrarily high codimension.

**1.1. Pinching.** Let  $Z$  be the subset of matrices  $A \in \text{GL}(d, \mathbb{C})$  whose eigenvalues are not all distinct in norm.  $Z$  is closed and contained in a finite union of closed submanifolds of  $\text{GL}(d, \mathbb{C})$  with codimension  $\geq 1$ .

It follows, by Proposition, that the subset of cocycles  $B \in \mathcal{U}$  for which  $B^{\text{per}(p)}(p) \in Z$  is closed and contained in a finite union of closed submanifolds with codimension  $\geq 1$ .

Now, for any  $l \geq 1$ , consider periodic points  $p_1, \dots, p_l$ . We imply that the subset of linear cocycles  $B \in \mathcal{U}$  where  $B^{\text{per}(p_i)}(p_i) \in Z$  is closed and contained in a finite union of closed submanifolds with codimension  $\geq l$ .

**1.2. Twisting.** The subset  $Y$  of all pairs of matrices  $(T_1, T_2)$  such that there exist  $T_2$ -invariant subspaces  $E_1, E_2$  with  $\dim E_1 + \dim E_2 = d$  where  $T_1(E_1) \cap E_2 \neq \{\mathbf{0}\}$ , is closed and contained in a finite union of closed submanifolds of positive codimension.

Indeed, Fixing  $E_1, E_2$ , the application

$$\text{GL}(d, \mathbb{C}) \ni T \mapsto T(E_1) \in \text{Grass}(\dim E_1, d)$$

is a submersion. In the other hand,

$$\{T : T(E_1) \text{ do not intersect transversally } E_2\}$$

is a submanifold with codimension  $\geq 1$ , since

$$\{E \in \text{Grass}(\dim E_1, d) : E \text{ do not intersect transversally } E_2\}$$

is a submanifold of positive codimension. Now, for any fixed matrix  $T_2$ , the set  $Y$  is contained in a finite number of submanifolds of positive codimension. So,  $Y$  is contained in a finite number of submanifolds of positive codimension in  $\mathrm{GL}(\mathbb{C}, d)^2$ .

Therefore, by Proposition 5.3, the subset of cocycles  $B \in \mathcal{U}$  so that

$$(B^{\mathrm{per}(p)}(p), \Psi_{B,p,q}) \in Y$$

is closed and contained in a finite union of closed submanifolds with positive codimension.

Given  $l \geq 1$ , if  $q_1, \dots, q_l$  are some homoclinic points of periodic points  $p_1, \dots, p_l$ , respectively, then the subset of cocycles  $B \in \mathcal{U}$  for which

$$(B^{\mathrm{per}(p_i)}(p_i), \Psi_{B,p_i,q_i}) \in Y$$

is closed and contained in a finite union of closed submanifolds with codimension  $\geq l$ .

**1.3. Real valued cocycles.** All results in [AV07] and Perturbation arguments of the last chapter are valid for cocycles with values in  $\mathrm{GL}(d, \mathbb{R})$ . But, in this case, there is the possibility of existence of pairs of complex conjugate eigen-values. Indeed, the subset of matrices whose eigen-values are not all distinct in norm has non-empty interior in  $\mathrm{GL}(d, \mathbb{R})$ .

The way to bypass this, is treated in [BGV03] and [BV04]:

Excluding a codimension 1 subset of cocycles, one may assume that

- (i) all the eigenvalues of  $B^{\mathrm{per}(p)}(p)$  are real and have distinct norms, except for  $c \geq 0$  pairs of complex conjugate eigenvalues,
- (ii)  $\Psi_{A,p,q}(E_1) \cap E_2 = \{\mathbf{0}\}$ , for any direct sums  $E_1$  and  $E_2$  of eigen-spaces of  $B^{\mathrm{per}(p)}(p)$  with  $\dim E_1 + \dim E_2 \leq d$ .

Avoiding another subset of positive codimention, we can choose a new priodic point  $\hat{p}$  so that all the eigenvalues of  $B^{\mathrm{per}(\hat{p})}(\hat{p})$  are real and distinct.

Now, in this way, for any  $l \geq 1$ , avoiding a codimension  $l$  subset of cocycles, one may suppose that periodic points  $\hat{p}_1, \dots, \hat{p}_l$  are defined.

The proof of Theorem 2 is now completed.

## 2. Proof of Theorem 1

Now, we carry out simplicity results in Theorem 2, for cocycles over Lorenz attractor to prove Theorem 1.

**2.1. Suspending cocycles by the flow.** Let  $A^t$  be a linear cocycle over  $\Lambda$ . We define a neighborhood  $\mathcal{V}$  of  $A^t$  as the subset of all cocycles  $B^t$  over  $\Lambda$  for which  $B_{\hat{P}} \in \mathcal{U}$ .

**Proposition 6.1.** *The application*

$$\mathcal{V} \ni B^t \mapsto B_{\hat{P}} \in \mathcal{U}$$

*is a submersion.*

PROOF. By definition,

$$\partial_{B^t} B_{\hat{P}}(\dot{B}_t) = \dot{B}_{\hat{P}}.$$

Let  $\mathcal{B} \in \mathcal{C}^{r,\rho}(\Lambda_{\hat{P}}, d, \mathbb{C})$ . The suspension  $\dot{\mathcal{B}}^t$  of  $\mathcal{B}$  is defined by

$$\dot{\mathcal{B}}^t(X^s(w)) = (\text{id}, t + s), \quad 0 < t + s \leq \hat{R}(w),$$

identifying  $(\text{id}, \hat{R}(w))$  and  $(\mathcal{B}(w), 0)$ , for any  $w \in \Lambda_{\hat{P}}$ , and set  $\dot{\mathcal{B}}^0 = \text{id}$ .  $\dot{\mathcal{B}}^t$  is a point of  $\mathcal{C}^{r,\rho}(\Lambda, d, \mathbb{C})$  for which  $\dot{\mathcal{B}}_{\hat{P}}(w) = (\mathcal{B}(w), 0)$ . This shows that the derivative is surjective. The proof is now completed.  $\square$

Now, Theorem 1 is a direct consequence of Theorem 2.



## CHAPTER 7

### Final Remarks

**Singular hyperbolicity.** The notion of *singular hyperbolic attractor* is motivated by geometric model of Lorenz attractor. It is proved in [MPP04] that such features present for any robustly transitive set of a flow: any attractor of a 3-dimensional flow containing in a robust fashion equilibria together with regular orbits must admit an invariant splitting

$$E^s \oplus E^{cu}$$

of the tangent bundle into a 1-dimensional uniformly contracting sub-bundle and a 2-dimensional volume-expanding sub-bundle.

It is shown that singular hyperbolicity implies the main geometric and ergodic features of the classical models [APPV09]. Namely, one may construct convenient cross-sections and invariant contracting foliations for corresponding Poincaré transformation that allows us to reduce the flow dynamics to a 1-dimensional map which is uniformly expanding. As an extension of Theorem 1,

**Problem 1.** *Typical fiber bunched linear cocycles over any singular hyperbolic attractor are simple?*

**Fiber bunching.** Fiber bunching behaviour is a condition derived from hyperbolicity of the base dynamics. Indeed, it guarantees existence and differentiability of holonomy maps which is essential to define the transition maps and then verify perturbation arguments.

Bonatti and Viana in [BGV03] proved that generic fiber bunched linear cocycles over any hyperbolic transformation have some non-zero Lyapunov exponent. But, Viana in [V08] removing fiber bunching proved existence of non-zero exponents for typical linear cocycles.

In [ASV] it is shown that typical fiber bunched linear cocycles over some ergodic class of partially hyperbolic diffeomorphisms have non-zero exponents.

**Problem 2.** *The results of [ASV] are valid removing the condition of fiber bunching?*



## Bibliography

- [AP07] V. Araujo and M. Pacifico, Three dimensional flows, Brazillian Mathematics Colloquium IMPA (2007).
- [APPV09] V. Araujo, M. Pacifico, E. Pujals and M. Viana, Singular hyperbolic attractors, Transactions of the American Mathematical Society 361 (2009) 2431-2485.
- [ASV] A. Avila, J. Santamaria and M. Viana, Cocycles over partially hyperbolic maps, preprint IMPA.
- [AV07] A. Avila and M. Viana, Simplicity of Lyapunov spectra: a sufficient condition, Potugaliae Matematica 64 (2007) 311-376.
- [AV10] A. Avila and M. Viana, External Lyapunov exponents: an invariance principles and applications, preprint IMPA.
- [BC91] M. Benedicks and L. Carleson, The dynamics of the  $H_{\lambda}^{\frac{1}{2}}$  non map, Annals of Mathematics 133 (1991) 73-169.
- [BDV04] C. Bonatti, L. Diaz and M. Viana, Dynamics Beyond Uniform Hyperbolicity: A Global Geometric and Probabilistic Perspective, Encyclopaedia of Mathematical Sciences 102 Springer-Verlag (2004).
- [BGV03] C. Bonatti, X. Gomez-Mont and M. Viana, Géricité d'exposants de Lyapunov non-nuls pour des produits déterministes de matrices, Annales de l'Institute Henri Poincaré 20 (2003) 579-624.
- [BV04] C. Bonatti and M. Viana, Lyapunov exponents with multiplicity 1 for deterministic products of matrices, Ergodic Theory and Dynamical Systems 24 (2004) 1295-1330.
- [D06] K. Díaz-Ordaz, Decay of correlation for non-Hölder observables for one-dimensional expanding Lorenz-like maps, Discrete and Continuous Dynamical Systems 15 (2006) 159-176.
- [F63] H. Furstenberg, Non-commuting random products, Transactions of the American Mathematical Society 108 (1963) 377-428.
- [GH83] J. Guckenheimer, P. Holmes, Nonlinear oscillations, dynamical systems, and bifurcations of vector fields, Applied Mathematical Sciences 42 Springer-Verlag (1983).
- [GW79] J. Guckenheimer and R. Williams, Structural stability of Lorenz attractors, Publications Mathématiques de l'IHÉS 50 (1979) 59-72.
- [H83] M. Herman. Une méthode nouvelle pour minorer les exposants de Lyapunov et quelques exemples montrant le caractère local d'un théorème d'Arnold et de Moser sur le tore de dimension 2, Commentarii Mathematici Helvetici 58 (1983) 453-502.
- [H76] M. Hirsch, Differential topology, Springer-Verlag (1976).
- [HS] M. Hirsch and S. Smale, Differential equations, dynamical systems, and linear algebra, Academic Press (1974).
- [J81] M. Jacobson, Absolutely continuous invariant measures for one parameter families of one-dimensional maps, Communications in Mathematical Physics 81 (1981) 39-88.
- [L63] E. Lorenz, Deterministic nonperiodic flow, Journal of the Atmospheric Sciences, 20 (1963) 130-141.
- [M80] R. Mane, Ergodic theory and differentiable dynamics, Springer-Verlag (1980).
- [MP82] W. de Melo and J. Palis, Geometric theory of dynamical systems, Springer-Verlag (1982).
- [MPP04] C. Morales, M. Pacifico and E. Pujals, robust transitive singular sets for 3-flows are partially hyperbolic attractors or repellers, Annals of Mathematics 160 (2004) 357-432.
- [O68] V. Oseledets, A multiplicative ergodic theorem, Transactions of the Moscow Mathematical Society 19 (1968) 197-231.
- [S78] M. Shub, Global stability of dynamical systems, Springer-Verlag (1978).
- [S68] S. Smale, Differentiable dynamical systems, Bulletin of the American Mathematical Society 73 (1967) 747-817.

- [T98] W. Tuker, The Lorenz attractor exists, PhD thesis, University of Uppsala (1998).
- [V97] M. Viana, Stochastic dynamics of deterministic systems, Brazillian Mathematics Colloquium IMPA (1997).
- [V00] M. Viana, What's new on Lorenz strange attractors?, *The Mathematical Intelligencer* 22 (2000), 6-19
- [V08] M. Viana, Almost all cocycles over any hyperbolic system have non-vanishing Lyapunov exponents, *Annals of Mathematics* 167 (2008) 643-680.
- [W79] R. Williams, The structure of the Lorenz attractor, *Publications Mathématiques de l'IHÈS* 50 (1979) 73-99.

# Livros Grátis

( <http://www.livrosgratis.com.br> )

Milhares de Livros para Download:

[Baixar livros de Administração](#)

[Baixar livros de Agronomia](#)

[Baixar livros de Arquitetura](#)

[Baixar livros de Artes](#)

[Baixar livros de Astronomia](#)

[Baixar livros de Biologia Geral](#)

[Baixar livros de Ciência da Computação](#)

[Baixar livros de Ciência da Informação](#)

[Baixar livros de Ciência Política](#)

[Baixar livros de Ciências da Saúde](#)

[Baixar livros de Comunicação](#)

[Baixar livros do Conselho Nacional de Educação - CNE](#)

[Baixar livros de Defesa civil](#)

[Baixar livros de Direito](#)

[Baixar livros de Direitos humanos](#)

[Baixar livros de Economia](#)

[Baixar livros de Economia Doméstica](#)

[Baixar livros de Educação](#)

[Baixar livros de Educação - Trânsito](#)

[Baixar livros de Educação Física](#)

[Baixar livros de Engenharia Aeroespacial](#)

[Baixar livros de Farmácia](#)

[Baixar livros de Filosofia](#)

[Baixar livros de Física](#)

[Baixar livros de Geociências](#)

[Baixar livros de Geografia](#)

[Baixar livros de História](#)

[Baixar livros de Línguas](#)

[Baixar livros de Literatura](#)

[Baixar livros de Literatura de Cordel](#)

[Baixar livros de Literatura Infantil](#)

[Baixar livros de Matemática](#)

[Baixar livros de Medicina](#)

[Baixar livros de Medicina Veterinária](#)

[Baixar livros de Meio Ambiente](#)

[Baixar livros de Meteorologia](#)

[Baixar Monografias e TCC](#)

[Baixar livros Multidisciplinar](#)

[Baixar livros de Música](#)

[Baixar livros de Psicologia](#)

[Baixar livros de Química](#)

[Baixar livros de Saúde Coletiva](#)

[Baixar livros de Serviço Social](#)

[Baixar livros de Sociologia](#)

[Baixar livros de Teologia](#)

[Baixar livros de Trabalho](#)

[Baixar livros de Turismo](#)