



Instituto de Matemática Pura e Aplicada

Doctoral Thesis

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**Rio de Janeiro
2010**

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Instituto de Matemática Pura e Aplicada

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**SIMONS TYPE EQUATION IN $S^2 \times \mathbb{R}$ AND $\mathbb{H}^2 \times \mathbb{R}$ AND
APPLICATIONS**

Thesis presented to the Post-graduate Program in Mathematics at Instituto de Matemática Pura e Aplicada as partial fulfillment of the requirements for the degree of Doctor in Philosophy in Mathematics.

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**Rio de Janeiro
2010**

To my parents, João Faustino and Maria Madalena and
my wife Sidiene Batista.

Acknowledgments

Escreverei os agradecimentos em português já que é o idioma vernáculo a todas as pessoas a quem dirijo estes.

- Agradeço, em primeiro lugar, a **Deus** por ter me conduzido em suas veredas de maneira tão suave e delicada que só agora é possível perceber tão grande cuidado e amor para comigo.
- Sobremaneira aos meus pais, João Faustino e Maria Madalena, por todo cuidado e carinho a me dedicados. A minha queridíssima esposa Sidiane Ferreira Batista que, afetosamente, tem me oferecido todo apoio humano necessário.
- Ao Professor Manfredo sobremaneira por sua orientação, sua amizade e seus conselhos que tanto foram úteis academicamente e em minha vida pessoal.
- Ao Professor Hilário Alencar por toda sua dedicação e amizade a mim ofertadas de forma tão espontânea.
- De maneira especial a todos os meus amigos do Centro Cultural das Laranjeira, atual CCUB, em particular a André Medrado, Benedito Montenegro, João Malheiro, João Carlos Nara e Rafael Teixeira, por todo auxílio no meu desenvolvimento humano.

- Aos meus amigos de todas as horas: Almir Santos, Claudemir Leandro e Ivaldo Nunes. Também aos colegas da Geometria Diferencial do IMPA, em especial a Cristina Levina e Maria Costa.
- A Senhora Hilda junto a sua família que tão bem me acolheram em sua casa por mais de dois anos.
- Aos Professores do IM-UFAL que colaboraram, com sua amizade, para a finalização de toda essa etapa.
- A todos os funcionários do IMPA pelo excelente e incomparável suporte técnico.
- Enfim, ao CNPq pelo apoio financeiro.

Qual é o segredo da perseverança?
O Amor. - Enamora-te, e não O deixarás.
(Caminho, 999)

Abstract

Equations of Simons type are presented. They are satisfied by a pair of special operators associated to the immersion $\Sigma^2 \hookrightarrow M^2(c) \times \mathbb{R}$ with constant mean curvature. Some immersions are characterized.

Keywords: Simons Type Equation, Constant Mean Curvature, Immersed Surface.

Resumo

Apresentamos equações tipo Simons. Estas são satisfeitas por um par de operadores especiais associados a imersões $\Sigma^2 \looparrowright M^2(c) \times \mathbb{R}$ com curvatura média constante. Utilizando tais equações, caracterizamos algumas imersões.

Palavras-chave: Equações Tipo Simons, Curvatura Média Constante, Superfícies Imersas.

Contents

Acknowledgments	iv
Abstract	vii
Resumo	viii
Introduction	1
1 Preliminaries	6
2 Simons type equation for ϕ and S in $M^2(c) \times \mathbb{R}$	9
3 Applications	23
Bibliography	32

Introduction

In 1994, using the traceless second fundamental form $\phi = A - HI$ associated to an immersed hypersurface $M^n \looparrowright \mathbb{S}^{n+1}$, H. Alencar and M. do Carmo, see [AdC], proved that

Theorem. *Let $M^n \looparrowright \mathbb{S}^{n+1}$ be an immersed hypersurface. If M^n is compact and orientable with constant mean curvature H and*

$$|\phi|^2 \leq B_H,$$

where B_H is the square of the positive root of

$$P_H(x) = x^2 + \frac{n(n-2)}{\sqrt{n(n-1)}}Hx - n(H^2 + 1).$$

Then:

- (a) Either $|\phi|^2 = 0$ (and M^n is totally umbilic) or $|\phi|^2 = B_H$.
- (b) The $H(r)$ -tori $\mathbb{S}^{n-1}(r) \times \mathbb{S}^1(\sqrt{1-r^2})$ with $r^2 < \frac{n-1}{n}$ are the only hypersurfaces with constant mean curvature H and $|\phi|^2 = B_H$.

Motivated by this result we decided to study this problem for surfaces in $M^2(c) \times \mathbb{R}$ with $c = \pm 1$, where $M^2(-1) = \mathbb{H}^2$ and $M^2(1) = \mathbb{S}^2$.

We begin by using the traceless second fundamental form ϕ associated to an immersed surface $\Sigma^2 \looparrowright M^2(c) \times \mathbb{R}$ to characterize some immersed surfaces.

Then we study, in chapter 2, a special tensor S defined by

$$SX = 2HAX - c\langle X, T \rangle T + \frac{c}{2}(1 - \nu^2)X - 2H^2X, \quad (1)$$

where $X \in T_p\Sigma$, A is the Weingarten operator associated to the second fundamental form, H is the mean curvature, T is the tangential component of the parallel field ∂_t , tangent to \mathbb{R} in $M^2(c) \times \mathbb{R}$, and $\nu = \langle N, \partial_t \rangle$.

This operator satisfies Codazzi's equation, provided H is constant, with vanishing trace, see Proposition 2.3. We remark that any surface with $|S| = 0$ and constant mean curvature is interesting, because the $(2,0)$ -part of the quadratic differential Q ,

$$Q(X, Y) = 2H\langle AX, Y \rangle - c\langle X, \partial_t \rangle \langle Y, \partial_t \rangle,$$

of these surfaces vanishes. In [AR], section 2, Abresch and Rosenberg described four distinct classes of complete, possibly immersed, constant mean curvature surfaces $\Sigma^2 \looparrowright M^2(c) \times \mathbb{R}$ with vanishing $(2,0)$ -part of quadratic differential Q .

More precisely, the four classes are

- (i) Σ^2 is an embedded rotationally invariant constant mean curvature sphere S_H^2 ;
- (ii) Σ^2 is a convex rotationally invariant constant mean curvature graph D_H^2 over the horizontal leaf $M^2(c) \times \{t_0\}$;
- (iii) Σ^2 is an embedded annulus, rotationally invariant constant mean curvature surface C_H^2 with two asymptotically conical ends;
- (iv) Σ^2 is embedded constant mean curvature surface P_H^2 ; it is an orbit under some two dimensional solvable subgroup of ambient isometries.

The surface in (i) was known to W.Y. Hsiang, in [Hs], and to R. Pedrosa and M. Ritoré, in [PR]. We shall refer to S_H^2 as the embedded rotationally invariant constant mean curvature spheres. In this paper we will call these surfaces described in [AR] by Abresch-Rosenberg surfaces.

Remark.1: In $\mathbb{S}^2 \times \mathbb{R}$ only the spheres S_H^2 occur.

Next, we obtain an equation of Simons type for S and apply it in some particular cases:

Theorem 0.1 (Simons Type Equation for S). *Let $\Sigma^2 \looparrowright M^2(c) \times \mathbb{R}$ be an immersed surface with non zero constant mean curvature H and S as defined in (1). Then,*

$$\begin{aligned} \langle (\nabla^2 S)x, y \rangle &= 2cv^2 \langle Sx, y \rangle + 2H \langle Ax, Sy \rangle - \langle A^2 x, Sy \rangle + \\ &+ \langle Ay, SAx \rangle - \langle Ax, y \rangle \text{tr}(AS) \end{aligned}$$

and

$$\begin{aligned} \frac{1}{2} \Delta |S|^2 &= |\nabla S|^2 - |S|^4 + |S|^2 \left(\frac{5cv^2}{2} - \frac{c}{2} + 2H^2 - \frac{c}{H} \langle ST, T \rangle \right) + \\ &+ c|ST|^2 - \frac{1}{4H^2} \langle ST, T \rangle^2. \end{aligned}$$

Let us consider the polynomial $p_H(t) = -t^2 - \frac{1}{H}t + \left(\frac{4H^2 - 1}{2} \right)$. When H is greater than one half, p_H has a positive root denoted by L_H ; In fact,

$$L_H = \frac{4H^2 - 1}{\sqrt{8H^4 - 2H^2 + 1} - 1}.$$

One has:

Theorem 0.2. *Let $\Sigma^2 \looparrowright \mathbb{S}^2 \times \mathbb{R}$ be an immersed surface with constant mean curvature H greater than one half. If*

$$\Sigma^2 \text{ is complete and } \sup_{\Sigma} |S| < L_H$$

or

$$\Sigma^2 \text{ is closed and } |S| \leq L_H,$$

then $\Sigma^2 = S_H^2$, i.e, Σ^2 is an embedded rotationally invariant constant mean curvature sphere.

Let us consider the polynomial $q_H(t) = -t^2 - \frac{1}{H}t + \left(\frac{8H^4 - 12H^2 - 1}{4H^2}\right)$.

When H is greater than $\sqrt{\frac{12 + \sqrt{176}}{16}}$, q_H has a positive root denoted by M_H ; In fact,

$$M_H = \frac{8H^4 - 12H^2 - 1}{2H(\sqrt{8H^4 - 12H^2 - 1} + 1)}.$$

One has:

Theorem 0.3. *Let $\Sigma^2 \looparrowright \mathbb{H}^2 \times \mathbb{R}$ be an immersed surface with constant mean curvature H greater than $\sqrt{\frac{12 + \sqrt{176}}{16}} \approx 1.25664$. If*

$$\Sigma^2 \text{ is complete and } \sup_{\Sigma} |S| < M_H$$

or

$$\Sigma^2 \text{ is closed and } |S| \leq M_H,$$

then $\Sigma^2 = S_{H}^2$, i.e, Σ^2 is an embedded rotationally invariant constant mean curvature sphere.

Remark.2: Besides Theorems 0.2 and 0.3, we obtain in chapter 3 further applications of Simons equation of Theorem 0.1.

Remark.3: Besides Simons type equation for S , we obtain the Simons type equation for ϕ .

The organization of this thesis is as follows.

In chapter 1 we comment some notation about covariant derivatives of tensors fields, the curvature tensor and operators associated to the Weingarten operator A that will be used throughout the thesis.

We will recall results on the second covariant derivative of the Weingarten operator, see Theorem 2 in [B], and the result known as the Omori-Yau Maximum Principle to complete manifolds, see Theorem 1 in [Y]. Finally we recall the Gauss' equation for the product space $M^2(c) \times \mathbb{R}$.

In chapter 2, we will obtain an equation of Simons type for the traceless second fundamental form ϕ and for S defined in (1). This chapter involves many curvature computations and it is quite technical.

Finally, in chapter 3, we gives several geometric applications based on the results found in chapter 2 together with the Omori-Yau's Theorem.

CHAPTER 1

Preliminaries

Let $\Sigma^2 \hookrightarrow M^3$ be an immersed surface. Let $\bar{\nabla}$ denote the Levi-Civita connection on M^3 and let ∇ denote the Levi-Civita connection on Σ for the induced metric.

Generally speaking, objects defined on M^3 will be denoted by the same symbols as the corresponding objects defined on Σ plus a bar over the symbol.

The Riemannian metric extends to natural inner products on space of tensors and the above connections induce natural covariant derivatives of tensor fields. For example, if $\{e_1, e_2\}$ is a geodesic frame in $p \in \Sigma^2$ and ψ is a tensor on Σ^2 , we have

$$\nabla^2 \psi(p) = \sum_{i=1}^2 (\nabla_{e_i} \nabla_{e_i} \psi)(p).$$

For more details about covariant derivatives of tensor fields see [S], sections 1 and 2.

We adopt the following convention for the curvature tensor: if $x, y, z \in T_p \Sigma$, we define $R_{x,y,z}$ by

$$R_{x,y,z} = R(X, Y)Z(p) = (\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z)(p),$$

for any local vector fields which extend the given vectors x, y, z .

The second fundamental form is defined by $\alpha(X, Y) = (\bar{\nabla}_X Y)^\perp$ and the associated Weingarten operator is given by $Av = -(\bar{\nabla}_v N)^T$, where N is a unit normal field on Σ^2 . We use the Weingarten operator to define the following operators

$$\begin{aligned} \langle \bar{R}(A)x, y \rangle := & \sum_{i=1}^2 (-\langle Ax, \bar{R}_{e_i, y} e_i \rangle - \langle Ay, \bar{R}_{e_i, x} e_i \rangle \\ & + \langle Ay, x \rangle \langle N, \bar{R}_{e_i, N} e_i \rangle - 2\langle Ae_i, \bar{R}_{e_i, x} y \rangle) \end{aligned} \quad (1.1)$$

and

$$\langle \bar{R}'x, y \rangle := \sum_{i=1}^2 \{ \langle (\bar{\nabla}_x \bar{R})_{e_i, y} e_i, N \rangle + \langle (\bar{\nabla}_{e_i} \bar{R})_{e_i, x} y, N \rangle \},$$

where $\{e_1, e_2\}$ is a orthonormal basis of $T_p \Sigma$.

With this notation we have the following result:

Theorem 1.1. *Let $\Sigma^2 \looparrowright M^3$ be an immersed surface with constant mean curvature H . For any $x, y \in T_p \Sigma$ we have*

$$\begin{aligned} \langle (\nabla^2 A)x, y \rangle = & -|A|^2 \langle Ax, y \rangle + \langle \bar{R}(A)x, y \rangle \\ & + \langle \bar{R}'x, y \rangle + 2H \langle \bar{R}_{N, x} y, N \rangle + 2H \langle Ax, Ay \rangle. \end{aligned} \quad (1.2)$$

Proof. See Theorem 2 in [B] and observe that the codimension here is one. ■

We will also use the result known as the Omori-Yau Maximum Principle whose proof can be found in [Y], Theorem 1.

Theorem 1.2 (Omori-Yau Maximum Principle). *Let M be a complete Riemannian manifold with Ricci curvature bounded from below. If $u \in C^\infty(M)$ is bounded from above, then there exist a sequence of points $\{p_j\} \in M$ such that*

$$\lim_{j \rightarrow \infty} u(p_j) = \sup_M u, \quad |\nabla u|(p_j) < \frac{1}{j}, \quad \text{and} \quad \Delta u(p_j) < \frac{1}{j}.$$

Let us recall Gauss' equation for Σ^2 in $M^2(c) \times \mathbb{R}$:

$$\begin{aligned}
 R(Y, X)Z = & \langle AX, Z \rangle AY - \langle AY, Z \rangle AX + c(\langle X, Z \rangle Y - \langle Y, Z \rangle X + \\
 & -\langle Y, T \rangle \langle X, Z \rangle T - \langle X, T \rangle \langle Z, T \rangle Y + \\
 & + \langle X, T \rangle \langle Y, Z \rangle T + \langle Y, T \rangle \langle Z, T \rangle X), \tag{1.3}
 \end{aligned}$$

where X, Y, Z in $T_p\Sigma$, N is a unitary normal field on Σ^2 and T is the tangential component of the parallel field ∂_t . For more details see [D].

CHAPTER 2

Simons type equation for ϕ and S in $M^2(c) \times \mathbb{R}$

In this chapter we will obtain an equation of Simons type for the traceless second fundamental form ϕ and for S defined in (1).

Let $M^2(c) \times \mathbb{R}$, where $M^2(-1) = \mathbb{H}^2$ and $M^2(1) = \mathbb{S}^2$. In this case we have that $\bar{R}' = 0$, because $M^2(c) \times \mathbb{R}$ is locally symmetric.

In Lemmas 2.1 and 2.2 we will consider an immersed surface $\Sigma^2 \looparrowright M^2(c) \times \mathbb{R}$ with constant mean curvature H where A is the Weingarten operator associated to the second fundamental form on Σ^2 .

Lemma 2.1. *Denoting the identity by I , we have that*

$$\bar{R}(A) = c(5v^2 - 1)A - 4cHv^2I.$$

Proof. Consider an orthonormal basis $\{e_1, e_2\}$ in $T_p\Sigma^2$ such that $Ae_i = k_i e_i$, $i = 1, 2$. Consider $x, y \in T_p\Sigma$. We have

$$x = x_1 e_1 + x_2 e_2 \text{ e } y = y_1 e_1 + y_2 e_2.$$

Computing the first sum in (1.1)

$$\sum_{i=1}^2 \langle \bar{R}_{e_i, y} e_i, Ax \rangle = \langle \bar{R}_{e_1, y} e_1, k_1 x_1 e_1 + k_2 x_2 e_2 \rangle + \langle \bar{R}_{e_2, y} e_2, k_1 x_1 e_1 + k_2 x_2 e_2 \rangle$$

$$\begin{aligned}
&= k_2 x_2 \langle \bar{R}_{e_1, y} e_1, e_2 \rangle + k_1 x_1 \langle \bar{R}_{e_2, y} e_2, e_1 \rangle \\
&= k_2 x_2 \langle \bar{R}_{e_1, y_1 e_1 + y_2 e_2} e_1, e_2 \rangle + k_1 x_1 \langle \bar{R}_{e_2, y_1 e_1 + y_2 e_2} e_2, e_1 \rangle \\
&= k_2 x_2 y_2 \langle \bar{R}_{e_1, e_2} e_1, e_2 \rangle + k_1 x_1 y_1 \langle \bar{R}_{e_2, e_1} e_2, e_1 \rangle \\
&= -\bar{K}_\Sigma (k_2 x_2 y_2 + k_1 x_1 y_1) = -\bar{K}_\Sigma \langle Ax, y \rangle,
\end{aligned}$$

where $\bar{K}_\Sigma = \langle \bar{R}_{e_1, e_2} e_2, e_1 \rangle$.

Hence,

$$\sum_{i=1}^2 \langle \bar{R}_{e_i, y} e_i, Ax \rangle = -\bar{K}_\Sigma \langle Ax, y \rangle. \quad (2.1)$$

It's simple see that

$$\sum_{i=1}^2 \langle \bar{R}_{e_i, x} e_i, Ay \rangle = -\bar{K}_\Sigma \langle Ax, y \rangle. \quad (2.2)$$

In the third sum in (1.1) we have

$$\begin{aligned}
\langle \bar{R}_{e_i, N} e_i, N \rangle &= -c \{ (1 - \langle e_i, \partial_t \rangle)^2 (1 - v^2) - v^2 \langle e_i, \partial_t \rangle^2 \} \\
&= -c \{ 1 - \langle e_i, \partial_t \rangle^2 - v^2 + \langle e_i, \partial_t \rangle^2 v^2 - \langle e_i, \partial_t \rangle^2 v^2 \} \\
&= -c \{ 1 - v^2 - \langle e_i, \partial_t \rangle^2 \}.
\end{aligned}$$

Therefore,

$$\sum_{i=1}^2 \langle \bar{R}_{e_i, N} e_i, N \rangle = -c(1 - v^2). \quad (2.3)$$

To finish, we computing the fourth sum.

$$\begin{aligned}
\sum_{i=1}^2 \langle \bar{R}_{e_i, x} y, A e_i \rangle &= \langle \bar{R}_{e_1, x} y, k_1 e_1 \rangle + \langle \bar{R}_{e_2, x} y, k_2 e_2 \rangle \\
&= \langle \bar{R}_{e_1, x_1 e_1 + x_2 e_2} y, k_1 e_1 \rangle + \langle \bar{R}_{e_2, x_1 e_1 + x_2 e_2} y, k_2 e_2 \rangle \\
&= k_1 x_2 \langle \bar{R}_{e_1, e_2} y, e_1 \rangle + k_2 x_1 \langle \bar{R}_{e_2, e_1} y, e_2 \rangle \\
&= k_1 x_2 \langle \bar{R}_{e_1, e_2} (y_1 e_1 + y_2 e_2), e_1 \rangle + k_2 x_1 \langle \bar{R}_{e_2, e_1} (y_1 e_1 + y_2 e_2), e_2 \rangle
\end{aligned}$$

$$\begin{aligned}
&= k_1 x_2 y_2 \langle \bar{R}_{e_1, e_2} e_2, e_1 \rangle + k_2 x_1 y_1 \langle \bar{R}_{e_2, e_1} e_1, e_2 \rangle \\
&= \bar{K}_\Sigma (k_1 x_2 y_2 + k_2 x_1 y_1) \\
&= \bar{K}_\Sigma ([2H - k_2] x_2 y_2 + [2H - k_1] x_1 y_1) \\
&= \bar{K}_\Sigma (2H x_2 y_2 - k_2 x_2 y_2 + 2H x_1 y_1 - k_1 x_1 y_1) \\
&= \bar{K}_\Sigma (2H[x_1 y_1 + x_2 y_2] - [k_1 x_1 y_1 + k_2 x_2 y_2]) \\
&= \bar{K}_\Sigma (2H \langle x, y \rangle - \langle Ax, y \rangle),
\end{aligned}$$

where we used that $2H = k_1 + k_2$.

Thus,

$$\sum_{i=1}^2 \langle \bar{R}_{e_i, x} y, A e_i \rangle = \bar{K}_\Sigma (2H \langle x, y \rangle - \langle Ax, y \rangle). \quad (2.4)$$

Now, we need computing \bar{K}_Σ . Using the tensor of curvature in $M^2(c) \times \mathbb{R}$ we have:

$$\bar{K}_\Sigma = \langle \bar{R}_{e_1, e_2} e_2, e_1 \rangle = c(1 - \langle e_1, T \rangle^2 - \langle e_2, T \rangle^2) = c(1 - |T|^2)$$

Therefore,

$$\bar{K}_\Sigma = cv^2. \quad (2.5)$$

Substituting (2.1), (2.2), (2.3) and (2.4) into (1.1), obtain

$$\langle \bar{R}(A)x, y \rangle = 2\bar{K}_\Sigma \langle Ax, y \rangle - c(1 - v^2) \langle Ax, y \rangle - 2\bar{K}_\Sigma (2H \langle x, y \rangle - \langle Ax, y \rangle).$$

Using (2.5) we obtain

$$\langle \bar{R}(A)x, y \rangle = 5cv^2 \langle Ax, y \rangle - c \langle Ax, y \rangle - 4cv^2 H \langle x, y \rangle.$$

Thus,

$$\bar{R}(A) = c(5v^2 - 1)A - 4cHv^2I.$$

■

Lemma 2.2. $\langle \bar{R}_{N,x}y, N \rangle = -c\{\langle x, T \rangle \langle y, T \rangle - \langle x, y \rangle \langle T, T \rangle\}$.

Proof. We observe that

$$\langle x^*, y^* \rangle = \langle x, y \rangle - \langle x, T \rangle \langle y, T \rangle,$$

$$\langle x^*, N^* \rangle = v \langle x, T \rangle$$

and

$$\langle N^*, N^* \rangle = 1 - v^2,$$

where we have used $v^* = v - \langle v, \partial_t \rangle \partial_t$ for any $v \in T_p(M^2(c) \times \mathbb{R})$.

It follows that

$$\begin{aligned} \langle \bar{R}_{N,x}y, N \rangle &= -c\{\langle N^*, x^* \rangle \langle N^*, y^* \rangle - \langle N^*, N^* \rangle \langle x^*, y^* \rangle\} \\ &= -c\{(v \langle x, T \rangle)(v \langle y, T \rangle) - (\langle x, y \rangle - \langle x, T \rangle \langle y, T \rangle) \langle T, T \rangle\} \\ &= -c\{v^2 \langle x, T \rangle \langle y, T \rangle - \langle x, y \rangle \langle T, T \rangle + \langle x, T \rangle \langle y, T \rangle \\ &\quad - v^2 \langle x, T \rangle \langle y, T \rangle\} = -c\{\langle x, T \rangle \langle y, T \rangle - \langle x, y \rangle \langle T, T \rangle\}. \end{aligned}$$

This concludes the proof. ■

Proposition 2.1. *Let $\Sigma^2 \looparrowright M^2(c) \times \mathbb{R}$ be an immersed surface with constant mean curvature H and let A be the Weingarten operator associated to the second fundamental form on Σ^2 . Then,*

$$\begin{aligned} \langle (\nabla^2 A)x, y \rangle &= -|A|^2 \langle Ax, y \rangle + c(5v^2 - 1) \langle Ax, y \rangle - 4cHv^2 \langle x, y \rangle \\ &\quad - 2cH\{\langle x, T \rangle \langle y, T \rangle - \langle x, y \rangle \langle T, T \rangle\} + 2H \langle Ax, Ay \rangle. \end{aligned}$$

Proof. Consider equation (1.2)

$$\begin{aligned} \langle (\nabla^2 A)x, y \rangle &= -|A|^2 \langle Ax, y \rangle + \langle \bar{R}(A)x, y \rangle \\ &\quad + \langle \bar{R}'x, y \rangle + 2H \langle \bar{R}_{N,x}y, N \rangle + 2H \langle Ax, Ay \rangle. \end{aligned}$$

Now, we use Lemmas 2.1 and 2.2 and the fact $\bar{R}' = 0$ to obtain

$$\begin{aligned} \langle (\nabla^2 A)x, y \rangle &= -|A|^2 \langle Ax, y \rangle + c(5v^2 - 1) \langle Ax, y \rangle - 4cHv^2 \langle x, y \rangle \\ &\quad - 2Hc\{\langle x, T \rangle \langle y, T \rangle - \langle x, y \rangle \langle T, T \rangle\} + 2H \langle Ax, Ay \rangle. \end{aligned}$$

This concludes the proof. ■

Consider two tensors Z, W on Σ^2 . We define the inner product $\langle Z, W \rangle$ in $p \in \Sigma^2$ as

$$\langle Z, W \rangle = \sum_{i=1}^2 \langle Ze_i, We_i \rangle,$$

where $\{e_1, e_2\}$ is an orthonormal basis for $T_p\Sigma$.

Lemma 2.3. *Let $\Sigma^2 \looparrowright M^2(c) \times \mathbb{R}$ be an immersed surface with constant mean curvature and let A be the Weingarten operator associated to the second fundamental form on Σ^2 . Then,*

$$(a) \quad \langle \nabla^2 A, I \rangle = 0.$$

$$(b) \quad \langle \nabla^2 A, A \rangle = -|A|^4 + c(5v^2 - 1)|A|^2 - 8cH^2v^2 - 2cH\langle AT, T \rangle + 4cH^2|T|^2 + 2H\text{tr}(A^3).$$

Proof. Consider $\{e_1, e_2\}$ an orthonormal basis of $T_p\Sigma$. We use the definition of the inner product between tensors and the expression in Proposition 2.1 to obtain

$$\begin{aligned} \langle \nabla^2 A, A \rangle &= \sum_{i=1}^2 \langle (\nabla^2 A)e_i, Ae_i \rangle = -|A|^2 \sum_{i=1}^2 \langle Ae_i, Ae_i \rangle + \\ &c(5v^2 - 1) \sum_{i=1}^2 \langle Ae_i, Ae_i \rangle - 4cHv^2 \sum_{i=1}^2 \langle Ae_i, e_i \rangle - 2cH \left\{ \sum_{i=1}^2 \langle AT, e_i \rangle \langle e_i, T \rangle \right. \\ &\left. - \langle T, T \rangle \sum_{i=1}^2 \langle Ae_i, e_i \rangle \right\} + 2H \sum_{i=1}^2 \langle A^2 e_i, Ae_i \rangle. \end{aligned}$$

Therefore,

$$\langle \nabla^2 A, A \rangle = -|A|^4 + c(5v^2 - 1)|A|^2 - 8cH^2v^2 - 2cH\langle AT, T \rangle + 4cH^2|T|^2 + 2H\text{tr}(A^3).$$

Using the definition of the inner product and Proposition 2.1 we obtain

$$\langle \nabla^2 A, I \rangle = \sum_{i=1}^2 \langle (\nabla^2 A)e_i, e_i \rangle = -|A|^2 \sum_{i=1}^2 \langle Ae_i, e_i \rangle + \dots$$

$$\begin{aligned} & \dots + c(5v^2 - 1) \sum_{i=1}^2 \langle Ae_i, e_i \rangle - 8cHv^2 - 2cH \left\{ \sum_{i=1}^2 \langle T, e_i \rangle \langle e_i, T \rangle \right. \\ & \quad \left. - 2\langle T, T \rangle \right\} + 2H \sum_{i=1}^2 \langle A^2 e_i, e_i \rangle. \end{aligned}$$

Therefore,

$$\begin{aligned} \langle \nabla^2 A, I \rangle &= -2H|A|^2 + c(5v^2 - 1)2H - 8cHv^2 + 2cH\langle T, T \rangle + 2H|A|^2 = 0 \\ &= -2H|A|^2 + c(5v^2 - 1)2H - 8cHv^2 + 2cH(1 - v^2) + 2H|A|^2 \\ &= 10cHv^2 - 2cH - 8cHv^2 + 2cH - 2cHv^2 = 0, \end{aligned}$$

where we have used that $v^2 + |T|^2 = 1$. ■

Proposition 2.2. *Let $\Sigma \looparrowright M^2(c) \times \mathbb{R}$ be an immersed surface with constant mean curvature H and ϕ the traceless second fundamental form, then*

- (a) $|\phi|^2 = |A|^2 - 2H^2$.
- (b) $\nabla\phi = \nabla A$.
- (c) $\text{tr}A^3 = 3H|\phi|^2 + 2H^3$.

Proof. The proof of item (a) is:

$$\begin{aligned} |\phi|^2 &= \langle \phi, \phi \rangle = \langle A - HI, A - HI \rangle = \langle A, A \rangle - 2H\langle A, I \rangle + H^2\langle I, I \rangle \\ &= |A|^2 - 4H^2 + 2H^2 = |A|^2 - 2H^2, \end{aligned}$$

where $\langle A, I \rangle = 2H$ and $\langle I, I \rangle = 2$.

To prove item (b), we consider tangent fields X, Y . Then,

$$\begin{aligned} (\nabla_X \phi)Y &= (\nabla_X A)Y - (\nabla_X(HI))Y = (\nabla_X A)Y - \nabla_X HI(Y) + H\nabla_X Y \\ &= (\nabla_X A)Y - H\nabla_X Y - X(H)Y + H\nabla_X Y = (\nabla_X A)Y, \end{aligned}$$

because H is constant.

Finally, the proof of item(c) is:

$$\text{tr}(A^3) = \sum_{i=1}^2 \langle A^3 e_i, e_i \rangle = \sum_{i=1}^2 \langle (\phi + HI)^3 e_i, e_i \rangle$$

$$= \sum_{i=1}^2 \langle (\phi^3 + 3H\phi^2 + 3H^2\phi + H^3I)e_i, e_i \rangle = 3H|\phi|^2 + 2H^3,$$

because $\text{tr}\phi = \text{tr}\phi^3 = 0$.

■

Next we shall derive an equation of Simons type for the traceless second fundamental form ϕ :

Theorem 2.1. *Let $\Sigma \looparrowright M^2(c) \times \mathbb{R}$ be an immersed surface with constant mean curvature H and ϕ the traceless second fundamental form. Then*

$$\langle \nabla^2\phi, \phi \rangle = -|\phi|^4 + (2H^2 + 5c\nu^2 - c)|\phi|^2 - 2cH\langle \phi T, T \rangle$$

and

$$\frac{1}{2}\Delta|\phi|^2 = |\nabla\phi|^2 - |\phi|^4 + (2H^2 + 5c\nu^2 - c)|\phi|^2 - 2cH\langle \phi T, T \rangle.$$

Proof. We use Proposition 2.2 to show that

$$\langle \nabla^2\phi, \phi \rangle = \langle \nabla^2A, A - HI \rangle = \langle \nabla^2A, A \rangle - H\langle \nabla^2A, I \rangle.$$

Now, we use Lemma 2.3 to obtain

$$\begin{aligned} \langle \nabla^2\phi, \phi \rangle &= -|A|^4 + c(5\nu^2 - 1)|A|^2 - 8cH^2\nu^2 + 2cH\langle AT, T \rangle \\ &\quad + 4cH^2|T|^2 + 2H\text{tr}(A^3). \end{aligned}$$

Therefore,

$$\begin{aligned} \langle \nabla^2\phi, \phi \rangle &= -(|\phi|^2 + 2H^2)^2 + c(5\nu^2 - 1)(|\phi|^2 + 2H^2) - 8cH^2\nu^2 + \\ &\quad - 2cH\langle (\phi + HI)T, T \rangle + 4cH^2|T|^2 + 2H(3H|\phi|^2 + 2H^3), \end{aligned}$$

which brings us to

$$\begin{aligned} \langle \nabla^2\phi, \phi \rangle &= -|\phi|^4 - 4H^2|\phi|^2 - 4H^4 + c(5\nu^2 - 1)|\phi|^2 + 2c(5\nu^2 - 1)H^2 - 8cH^2\nu^2 \\ &\quad - 2cH\langle \phi T, T \rangle - 2cH^2|T|^2 + 4cH^2|T|^2 + 6H^2|\phi|^2 + 4H^4. \end{aligned}$$

Hence,

$$\langle \nabla^2\phi, \phi \rangle = -|\phi|^4 + 2H^2|\phi|^2 + c(5\nu^2 - 1)|\phi|^2 - 2cH\langle \phi T, T \rangle.$$

To finish, we use that $\frac{1}{2}\Delta|\phi|^2 = |\nabla\phi|^2 + \langle \nabla^2\phi, \phi \rangle$.

■

Now we evaluate the Laplacian of $|S|^2$ where S is defined by (1), i.e,

$$S = 2HA - c\langle T, \cdot \rangle T + \frac{c}{2}(1 - v^2)I - 2H^2I.$$

Proposition 2.3 (Codazzi's Equation). *Let $\Sigma^2 \looparrowright M^2(c) \times \mathbb{R}$ be an immersed surface with constant mean curvature and S defined in (1). Then*

$$(\nabla_X S)Y = (\nabla_Y S)X,$$

for all tangent fields X, Y on Σ^2 and $\text{tr}(S) = 0$.

Proof. We consider (u, v) an isothermals parameters of the surface Σ^2 . Now, we consider the complex parametric, $z = u + iv$. Let us set

$$T_S(X, Y) := (\nabla_X S)Y - (\nabla_Y S)X = \nabla_X(SY) - \nabla_Y(SX) - S[X, Y].$$

We will prove that T_S is null. For this, consider the derivative

$$\partial_z = \frac{1}{2}(\partial_u - i\partial_v) \text{ and } \partial_{\bar{z}} = \frac{1}{2}(\partial_u + i\partial_v).$$

We will compute T_S in the basis $\{\partial_z, \partial_{\bar{z}}\}$. First note that,

$$\begin{aligned} \langle T_S(\partial_z, \partial_{\bar{z}}), \partial_z \rangle &= \partial_z \langle S\partial_{\bar{z}}, \partial_z \rangle - \langle S\partial_{\bar{z}}, \nabla_{\partial_z} \partial_z \rangle + \\ &\quad - \partial_{\bar{z}} \langle S\partial_z, \partial_z \rangle + \langle S\partial_z, \nabla_{\partial_{\bar{z}}} \partial_z \rangle \\ &= -Q_{\bar{z}}^{(2,0)} = 0, \end{aligned}$$

because $Q^{(2,0)}$ is holomorphic and using the fact that $\nabla_{\partial_z} \partial_{\bar{z}} = 0$, $\nabla_{\partial_z} \partial_z = \frac{\lambda_{\bar{z}}}{\lambda} \partial_z$, $\langle S\partial_z, \partial_z \rangle = Q^{(2,0)}$ and $\langle S\partial_{\bar{z}}, \partial_{\bar{z}} \rangle = 0$, where $\lambda = \langle \partial_z, \partial_{\bar{z}} \rangle$.

Next,

$$\begin{aligned} \langle T_S(\partial_z, \partial_{\bar{z}}), \partial_{\bar{z}} \rangle &= \partial_{\bar{z}} \langle \partial_z, S\partial_z \rangle - \langle S\partial_z, \nabla_{\partial_{\bar{z}}} \partial_{\bar{z}} \rangle + \\ &\quad - \partial_z \langle S\partial_{\bar{z}}, \partial_{\bar{z}} \rangle + \langle S\partial_{\bar{z}}, \nabla_{\partial_z} \partial_{\bar{z}} \rangle \\ &= -\overline{Q_{\bar{z}}^{(2,0)}} = 0, \end{aligned}$$

where we have used that $\nabla_{\partial_z} \partial_{\bar{z}} = \frac{\lambda_{\bar{z}}}{\lambda} \partial_{\bar{z}}$ and $\overline{Q^{(2,0)}}_z = \overline{Q^{(2,0)}_{\bar{z}}}$. It follows that $T_S = 0$.

To finish,

$$\text{tr}(S) = 2H\text{tr}(A) - c|T|^2 + c(1 - v^2) - 4H^2 = 0,$$

where we used that $|T|^2 + v^2 = 1$ and $\text{tr}(A) = 2H$.

■

The result below is known, see [S], p. 81, adapted for codimension 1, but, for completeness we will give its proof.

Lemma 2.4. *Let Z be a symmetric operator satisfying Codazzi's equation and $\text{tr}(Z) = 0$, then*

$$\langle (\nabla^2 Z)x, y \rangle = \sum_{i=1}^2 \{-\langle Zy, R_{e_i, x} e_i \rangle - \langle Z e_i, R_{e_i, x} y \rangle\}, \quad (2.6)$$

where $\{e_1, e_2\}$ is an orthonormal basis of $T_p \Sigma$.

Proof. Let us consider a geodesic frame $\{E_1, E_2\}$ in $p \in \Sigma^2$ which extends the basis $\{e_1, e_2\}$ and X, Y local vector parallel fields which extend the given vectors x, y . Compute

$$\begin{aligned} (\nabla^2 Z)X &= \sum_{i=1}^2 (\nabla_{E_i} \nabla_{E_i} Z)X = \sum_{i=1}^2 \nabla_{E_i} ((\nabla_{E_i} Z)X) \\ &= \sum_{i=1}^2 \nabla_{E_i} ((\nabla_X Z)E_i) = \sum_{i=1}^2 (\nabla_{E_i} \nabla_X Z)E_i \\ &= \sum_{i=1}^2 (\nabla_X \nabla_{E_i} Z)E_i + \sum_{i=1}^2 (R(E_i, X)Z)E_i \\ &= \sum_{i=1}^2 \nabla_X ((\nabla_{E_i} Z)E_i) + \sum_{i=1}^2 (R(E_i, X)Z)E_i \\ &= \nabla_X \left(\sum_{i=1}^2 (\nabla_{E_i} Z)E_i \right) + \sum_{i=1}^2 (R(E_i, X)Z)E_i, \end{aligned}$$

taking into account that Z satisfies Codazzi's equation.

Then,

$$\begin{aligned}
 \langle (\nabla^2 Z)x, y \rangle &= \langle \nabla_X \left(\sum_{i=1}^2 (\nabla_{E_i} Z) E_i \right), Y \rangle + \sum_{i=1}^2 \langle (R(E_i, X)Z) E_i, Y \rangle \\
 &= X \left(\sum_{i=1}^2 \langle (\nabla_{E_i} Z) E_i, Y \rangle \right) + \sum_{i=1}^2 \langle (R(E_i, X)Z) E_i, Y \rangle \\
 &= X(\text{tr}(\nabla_Y Z)) + \sum_{i=1}^2 \langle (R(E_i, X)Z) E_i, Y \rangle \\
 &= XY(\text{tr}(Z)) + \sum_{i=1}^2 (\langle (R(E_i, X)(Z E_i), Y \rangle - \langle Z(R(E_i, X)E_i), Y \rangle).
 \end{aligned}$$

By noting that $\text{tr}Z = 0$, and computing the above expression at the point p , we obtain

$$\langle (\nabla^2 Z)x, y \rangle = \sum_{i=1}^2 \{ -\langle (R(e_i, x)y, Ze_i) \rangle - \langle R(e_i, x)e_i, Zy \rangle \}.$$

■

Let us evaluate each summand in expression (2.6).

Lemma 2.5. *Let Z be an operator as in Lemma 2.4. Then,*

$$i) \sum_{i=1}^2 \langle Zy, R_{e_i, x} e_i \rangle = -cv^2 \langle Zx, y \rangle - 2H \langle Ax, Zy \rangle + \langle A^2 x, Zy \rangle.$$

and

$$ii) \sum_{i=1}^2 \langle Ze_i, R_{e_i, x} y \rangle = -cv^2 \langle Zx, y \rangle - \langle Ay, ZAx \rangle + \langle Ax, y \rangle \text{tr}(AZ).$$

Proof. Consider $\{e_1, e_2\}$ an orthonormal basis of $T_p \Sigma$. Using Gauss' equation (1.3) we find

$$\langle Zy, R_{e_i, x} e_i \rangle = -c \{ \langle x, Zy \rangle - \langle x, e_i \rangle \langle Zy, e_i \rangle - \langle x, T \rangle \langle Zy, T \rangle + \dots$$

$$\begin{aligned} & \dots - \langle e_i, T \rangle^2 \langle x, Zy \rangle + \langle e_i, T \rangle \langle x, e_i \rangle \langle Zy, T \rangle + \\ & + \langle x, T \rangle \langle e_i, T \rangle \langle e_i, Zy \rangle - \langle Ae_i, e_i \rangle \langle Ax, Zy \rangle + \\ & + \langle Ax, e_i \rangle \langle Ae_i, Zy \rangle. \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{i=1}^2 \langle Zy, R_{e_i, x} e_i \rangle &= -c \{ 2 \langle x, Zy \rangle - \sum_{i=1}^2 \langle x, e_i \rangle \langle Zy, e_i \rangle + \\ & - 2 \langle x, T \rangle \langle Zy, T \rangle - \langle x, Zy \rangle \sum_{i=1}^2 \langle e_i, T \rangle^2 + \\ & + \langle Zy, T \rangle \sum_{i=1}^2 \langle e_i, T \rangle \langle x, e_i \rangle + \langle x, T \rangle \sum_{i=1}^2 \langle e_i, T \rangle \langle e_i, Zy \rangle \} + \\ & - \langle Ax, Zy \rangle \sum_{i=1}^2 \langle Ae_i, e_i \rangle + \sum_{i=1}^2 \langle Ax, e_i \rangle \langle Ae_i, Zy \rangle, \end{aligned}$$

which implies that

$$\begin{aligned} \sum_{i=1}^2 \langle Zy, R_{e_i, x} e_i \rangle &= -c \{ 2 \langle x, Zy \rangle - \langle Zx, y \rangle - 2 \langle x, T \rangle \langle Zy, T \rangle + \\ & - \langle x, Zy \rangle |T|^2 + \langle Zy, T \rangle \langle x, T \rangle + \langle x, T \rangle \langle T, Zy \rangle \} + \\ & - \langle Ax, Zy \rangle 2H + \langle Ax, AZy \rangle. \end{aligned}$$

Hence,

$$\sum_{i=1}^2 \langle Zy, R_{e_i, x} e_i \rangle = -c(1 - |T|^2) \langle Zx, y \rangle - 2H \langle Ax, Zy \rangle + \langle A^2 x, Zy \rangle,$$

which shows the validity of (i). Now, one may verify that

$$\begin{aligned} \langle Ze_i, R_{e_i, x} y \rangle &= -c \{ \langle e_i, y \rangle \langle Ze_i, x \rangle - \langle x, y \rangle \langle Ze_i, e_i \rangle + \\ & - \langle x, T \rangle \langle Ze_i, T \rangle \langle e_i, y \rangle - \langle e_i, T \rangle \langle y, T \rangle \langle x, Ze_i \rangle + \\ & + \langle e_i, T \rangle \langle x, y \rangle \langle Ze_i, T \rangle + \langle x, T \rangle \langle y, T \rangle \langle e_i, Ze_i \rangle \} + \dots \end{aligned}$$

$$\dots - \langle Ae_i, y \rangle \langle Ax, Ze_i \rangle + \langle Ax, y \rangle \langle Ae_i, Ze_i \rangle.$$

Therefore

$$\begin{aligned} \sum_{i=1}^2 \langle Ze_i, R_{e_i, x} y \rangle &= -c \left\{ \sum_{i=1}^2 \langle e_i, y \rangle \langle Ze_i, x \rangle - \langle x, y \rangle \sum_{i=1}^2 \langle Ze_i, e_i \rangle + \right. \\ &\quad - \langle x, T \rangle \sum_{i=1}^2 \langle Ze_i, T \rangle \langle e_i, y \rangle - \langle y, T \rangle \sum_{i=1}^2 \langle e_i, T \rangle \langle x, Ze_i \rangle + \\ &\quad + \langle x, y \rangle \sum_{i=1}^2 \langle e_i, T \rangle \langle Ze_i, T \rangle + \langle x, T \rangle \langle y, T \rangle \sum_{i=1}^2 \langle e_i, Ze_i \rangle + \\ &\quad \left. - \sum_{i=1}^2 \langle Ae_i, y \rangle \langle Ax, Ze_i \rangle + \langle Ax, y \rangle \sum_{i=1}^2 \langle Ae_i, Ze_i \rangle. \right\} \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{i=1}^2 \langle Ze_i, R_{e_i, x} y \rangle &= -c \{ \langle Zx, y \rangle - \langle x, T \rangle \langle Zy, T \rangle - \langle y, T \rangle \langle Zx, T \rangle + \\ &\quad + \langle ZT, T \rangle \langle x, y \rangle \} - \langle Ay, ZAx \rangle + \langle Ax, y \rangle \text{tr}(AZ), \end{aligned}$$

noting that $\text{tr}Z = 0$.

Considering that

$$\begin{aligned} &-\langle x, T \rangle \langle Zy, T \rangle - \langle y, T \rangle \langle Zx, T \rangle + \langle ZT, T \rangle \langle x, y \rangle \\ &= -(1 - v^2) \langle Zx, y \rangle, \end{aligned}$$

we find

$$\sum_{i=1}^2 \langle Ze_i, R_{e_i, x} y \rangle = -cv^2 \langle Zx, y \rangle - \langle Ay, ZAx \rangle + \langle Ax, y \rangle \text{tr}(AZ),$$

which demonstrates (ii). ■

Theorem 2.2. *Let $\Sigma^2 \looparrowright M^2(c) \times \mathbb{R}$ be an immersed surface with non zero constant mean curvature H and let Z be an operator on Σ^2 satisfying Codazzi's equation with $tr(Z) = 0$. Then,*

$$\begin{aligned} \langle (\nabla^2 Z)x, y \rangle &= 2cv^2 \langle Zx, y \rangle + 2H \langle Ax, Zy \rangle - \langle A^2x, Zy \rangle + \\ &+ \langle Ay, ZAx \rangle - \langle Ax, y \rangle tr(AZ). \end{aligned}$$

Proof. We use the expressions of Lemma 2.5 in equation (2.6) obtained in Lemma 2.4. ■

Next we derive an equation of Simons type for the operator S as defined in (1).

Theorem 2.3 (Thm 0.1 in Introduction). *Let $\Sigma^2 \looparrowright M^2(c) \times \mathbb{R}$ be an immersed surface with non zero constant mean curvature H and S as defined in (1). Then,*

$$\begin{aligned} \langle (\nabla^2 S)x, y \rangle &= 2cv^2 \langle Sx, y \rangle + 2H \langle Ax, Sy \rangle - \langle A^2x, Sy \rangle + \\ &+ \langle Ay, SAx \rangle - \langle Ax, y \rangle tr(AS) \end{aligned}$$

and

$$\begin{aligned} \frac{1}{2} \Delta |S|^2 &= |\nabla S|^2 - |S|^4 + |S|^2 \left(\frac{5cv^2}{2} - \frac{c}{2} + 2H^2 - \frac{c}{H} \langle ST, T \rangle \right) + \\ &+ c|ST|^2 - \frac{1}{4H^2} \langle ST, T \rangle^2. \end{aligned}$$

Proof. First, since S satisfies Proposition 2.3, we can use the Theorem 2.2 with $Z = S$.

Now, we know that $\frac{1}{2} \Delta |S|^2 = |\nabla S|^2 + \langle \nabla^2 S, S \rangle$. Furthermore, we finds that

$$\langle \nabla^2 S, S \rangle = 2cv^2 |S|^2 + 2H tr(AS^2) - [tr(AS)]^2.$$

Now, we need to compute $tr(AS^2)$ and $tr(AS)$, as follows:

$$\begin{aligned} tr(AS^2) &= tr \left\{ S^2 \left(S + \frac{c}{2H} \langle T, \cdot \rangle T - \frac{c}{4H} (1 - v^2) I + HI \right) \right\} \\ &= tr S^3 + \frac{c}{2H} tr \langle T, S^2 \cdot \rangle T - \left(\frac{c}{4H} (1 - v^2) - H \right) tr S^2 \end{aligned}$$

$$= 0 + \frac{c}{2H}|ST|^2 - \left(\frac{c}{4H}(1 - v^2) - H\right)|S|^2$$

and

$$\begin{aligned} tr(AS) &= tr\left\{S\left(S + \frac{c}{2H}\langle T, \cdot \rangle T - \frac{c}{4H}(1 - v^2)I - HI\right)\right\} \\ &= trS^2 + \frac{c}{2H}tr(\langle T, S \cdot \rangle T) - \left(\frac{c}{4H}(1 - v^2) - H\right)trS \\ &= |S|^2 + \frac{c}{2H}\langle ST, T \rangle - 0, \end{aligned}$$

noting that $tr S = tr S^3 = 0$, also that

$$tr(\langle T, S \cdot \rangle T) = \sum_{i=1}^2 \langle T, S e_i \rangle \langle T, e_i \rangle = \langle ST, T \rangle$$

and that

$$tr(\langle T, S^2 \cdot \rangle T) = \sum_{i=1}^2 \langle T, S^2 e_i \rangle \langle T, e_i \rangle = \langle S^2 T, T \rangle.$$

Therefore,

$$\begin{aligned} \frac{1}{2}\Delta|S|^2 &= |\nabla S|^2 + 2cv^2|S|^2 + 2H\left(\frac{c}{2H}|ST|^2 - \left(\frac{c}{4H}(1 - v^2) - H\right)|S|^2\right) \\ &\quad - \left(|S|^2 + \frac{c}{2H}\langle ST, T \rangle\right)^2, \end{aligned}$$

in this way,

$$\begin{aligned} \frac{1}{2}\Delta|S|^2 &= |\nabla S|^2 + 2cv^2|S|^2 + c|ST|^2 - \left(\frac{c}{2}(1 - v^2) - 2H^2\right)|S|^2 + \\ &\quad - |S|^4 - \frac{c}{H}\langle ST, T \rangle|S|^2 - \frac{1}{4H^2}\langle ST, T \rangle^2. \end{aligned}$$

Rearranging terms, we obtain finally

$$\begin{aligned} \frac{1}{2}\Delta|S|^2 &= |\nabla S|^2 - |S|^4 + |S|^2\left(\frac{5cv^2}{2} - \frac{c}{2} + 2H^2 - \frac{c}{H}\langle ST, T \rangle\right) + \\ &\quad + c|ST|^2 - \frac{1}{4H^2}\langle ST, T \rangle^2. \end{aligned}$$

■

CHAPTER 3

Applications

In this chapter, we will apply the results found in chapter 2 together with the Omori-Yau's Theorem to classify some surfaces in $M^2(c) \times \mathbb{R}$.

Theorem 3.1. *Let $\Sigma^2 \looparrowright \mathbb{H}^2 \times \mathbb{R}$ be an oriented complete immersed minimal surface. Assume that*

$$\sup_{\Sigma} (|A|^2 + 5v^2) < 1.$$

Then Σ^2 is a vertical plane $\gamma \times \mathbb{R}$ for some geodesic γ in \mathbb{H}^2 .

Proof. Using Theorem 2.1 with $H = 0$ and $c = -1$, one finds

$$\frac{1}{2} \Delta |A|^2 = |\nabla A|^2 - |A|^4 + (1 - 5v^2)|A|^2 \geq |A|^2 (-|A|^2 + 1 - 5v^2).$$

Let $\frac{d}{2} := -\sup_{\Sigma} (|A|^2 + 5v^2) + 1 > 0$. Therefore,

$$\Delta |A|^2 \geq d \cdot |A|^2. \tag{3.1}$$

Using Gauss' equation (1.3) in $\mathbb{H}^2 \times \mathbb{R}$ we have

$$K_{\Sigma} = K_{\text{ext}} - v^2 = -\frac{|A|^2 + 5v^2}{2} + \frac{3v^2}{2} \geq -\frac{1}{2}.$$

Now we can use Theorem 1.2 with $u = |A|^2$, i.e, there exist $\{p_j\}$ in Σ^2 such that

$$\lim_{j \rightarrow \infty} |A|^2(p_j) = \sup_{\Sigma} |A|^2 \text{ and } \lim_{j \rightarrow \infty} \Delta |A|^2(p_j) \leq 0.$$

Next, we use inequality (3.1) to conclude that $\sup_{\Sigma} |A|^2 = 0$, i.e, Σ^2 is totally geodesic with $|v| < \sqrt{0.2}$.

Since Σ^2 is totally geodesic and $|v| < \sqrt{0.2}$ it cannot be a slice, it must be a vertical plane $\gamma \times \mathbb{R}$ for some geodesic γ in \mathbb{H}^2 .

■

Theorem 3.2. *Let $\Sigma^2 \looparrowright \mathbb{H}^2 \times \mathbb{R}$ be a complete immersed surface with constant mean curvature H . Assume that*

$$\sup_{\Sigma} (|\phi|^2 + 5v^2) < 2H^2 + 1 \text{ and } \langle \phi T, T \rangle \geq 0.$$

Then Σ^2 is a vertical plane $\gamma \times \mathbb{R}$ for some geodesic γ in \mathbb{H}^2 .

Proof. We consider the expression in Theorem 2.1 for the particular case $c = -1$:

$$\frac{1}{2} \Delta |\phi|^2 = |\nabla \phi|^2 - |\phi|^4 + (2H^2 + 1 - 5v^2) |\phi|^2 + 2H \langle \phi T, T \rangle.$$

As $\langle \phi T, T \rangle \geq 0$, we find

$$\frac{1}{2} \Delta |\phi|^2 \geq -|\phi|^4 + (2H^2 + 1 - 5v^2) |\phi|^2.$$

Consider $\frac{d}{2} := 2H^2 + 1 - \sup_{\Sigma} (|\phi|^2 + 5v^2) > 0$. Then

$$\Delta |\phi|^2 \geq 2|\phi|^2 (2H^2 + 1 - 5v^2 - |\phi|^2) \geq d|\phi|^2,$$

which implies,

$$\Delta |\phi|^2 \geq d|\phi|^2. \tag{3.2}$$

Using Gauss' equation (1.3) in $\mathbb{H}^2 \times \mathbb{R}$ we have

$$K_{\Sigma} = K_{ext} - v^2 = -\frac{|\phi|^2 + 5v^2 - 2H^2}{2} + \frac{3v^2}{2} \geq -\frac{1}{2}.$$

Now we can use Theorem 1.2 with $u = |\phi|^2$, i.e, there exist $\{p_j\}$ in Σ^2 such that

$$\lim_{j \rightarrow \infty} |\phi|^2(p_j) = \sup_{\Sigma} |\phi|^2 \text{ and } \lim_{j \rightarrow \infty} \Delta |\phi|^2(p_j) \leq 0.$$

Furthermore, we use inequality (3.2) to conclude that $\sup_{\Sigma} |\phi|^2 = 0$, i.e, Σ^2 is totally umbilical. Next, we use that if Σ^2 is totally umbilical with constant mean curvature in $\mathbb{H}^2 \times \mathbb{R}$ then Σ^2 is totally geodesic, which follows from [ST] section 4. Since Σ^2 is totally geodesic and $|v| < \sqrt{0.2}$ it must be a vertical plane $\gamma \times \mathbb{R}$ for some geodesic γ in \mathbb{H}^2 . This concludes the proof. ■

We need the following result:

Lemma 3.1. *Let $\Sigma^2 \looparrowright M^2(c) \times \mathbb{R}$ be a complete immersed surface with non zero constant mean curvature H . Then $|S| = 0$ if and only if Σ^2 is an Abresch-Rosenberg surface.*

Proof. We consider (u, v) an isothermals parameters to a surface Σ^2 . Now, we consider the complex parameter, $z = u + iv$ and $(2,0)$ -part of the Abresch-Rosenberg differential

$$Q(x, y) = 2H\langle Ax, y \rangle - c\langle x, T \rangle \langle y, T \rangle.$$

We can rewrite Q as

$$Q(x, y) = \langle Sx, y \rangle - \frac{c}{2}(1 - v^2)\langle x, y \rangle + 2H^2\langle x, y \rangle.$$

Next we evaluate $Q(\partial_z, \partial_z)$ noting that $\langle \partial_z, \partial_z \rangle = 0$:

$$Q(\partial_z, \partial_z) = \langle S\partial_z, \partial_z \rangle = \left(\frac{\tilde{e} - \tilde{g}}{4} \right) - i\frac{\tilde{f}}{2},$$

where $\tilde{e} = \langle S\partial_u, \partial_u \rangle = -\langle S\partial_v, \partial_v \rangle = -\tilde{g}$ and $\tilde{f} = \langle S\partial_u, \partial_v \rangle$. Therefore

$$|Q^{(2,0)}| = \sqrt{\left(\frac{\tilde{e} - \tilde{g}}{4} \right)^2 + \frac{\tilde{f}^2}{4}} = \sqrt{\frac{\tilde{e}^2}{4} + \frac{\tilde{f}^2}{4}} = \frac{\lambda^2}{2\sqrt{2}}|S|,$$

where $\lambda = |\partial_u| > 0$. This concludes the proof. ■

Let us consider the polynomial $p_H(t) = -t^2 - \frac{1}{H}t + \left(\frac{4H^2 - 1}{2}\right)$. When H is greater than one half there is a positive root for p_H . Let L_H be the positive root. One has:

Theorem 3.3 (Thm 0.2 in Introduction). *Let $\Sigma^2 \looparrowright S^2 \times \mathbb{R}$ be an immersed surface with constant mean curvature H greater than one half. If*

$$\Sigma^2 \text{ is complete and } \sup_{\Sigma} |S| < L_H;$$

or

$$\Sigma^2 \text{ is closed and } |S| \leq L_H.$$

Then $\Sigma^2 = S_{H}^2$, i.e, Σ^2 is an embedded rotationally invariant constant mean curvature sphere.

Proof. Let consider two cases. First, Σ is complete and second, Σ is closed.

First Case. Consider the expression in Theorem 2.3 with $c = 1$:

$$\begin{aligned} \frac{1}{2}\Delta|S|^2 &= |\nabla S|^2 - |S|^4 + |S|^2 \left(\frac{5\nu^2}{2} - \frac{1}{2} + 2H^2 - \frac{1}{H}\langle ST, T \rangle \right) \\ &\quad + |ST|^2 - \frac{1}{4H^2}\langle ST, T \rangle^2. \end{aligned}$$

As $|\langle ST, T \rangle| \leq |ST| \leq |S|$, we have

$$\frac{1}{2}\Delta|S|^2 \geq -|S|^4 + |S|^2 \left(\frac{5\nu^2}{2} + \frac{4H^2 - 1}{2} - \frac{1}{H}|S| \right) + \left(\frac{4H^2 - 1}{4H^2} \right) \langle ST, T \rangle^2,$$

hence,

$$\frac{1}{2}\Delta|S|^2 \geq |S|^2 \left(\frac{4H^2 - 1}{2} - \frac{1}{H}|S| - |S|^2 \right) + \frac{5}{2}\nu^2|S|^2, \quad (3.3)$$

because $H > \frac{1}{2}$.

Observe that

$$\frac{4H^2 - 1}{2} - \frac{1}{H}|S| - |S|^2 \geq p_H(\sup_{\Sigma} |S|) =: \frac{d}{2} > 0$$

and $v^2|S|^2 \geq 0$. Therefore

$$\Delta|S|^2 \geq d|S|^2. \quad (3.4)$$

Now we estimate $|S|$.

$$|S| \geq 2H|A| - \langle T, \cdot \rangle T - (1 - v^2) - 4H^2 \geq 2H|A| - 2(1 - v^2) - 4H^2,$$

that is,

$$L_H \geq |S| \geq 2H|A| - 2 - 4H^2.$$

Using Gauss' equation (1.3) in $\mathbb{S}^2 \times \mathbb{R}$ we find

$$K_\Sigma = K_{ext} + v^2 = -\frac{|A|^2}{2} + 2H^2 + v^2 \geq -\frac{1}{2} \left(\frac{L_H + 2 + 4H^2}{2H} \right)^2.$$

Now we can use Theorem 1.2 with $u = |S|^2$, i.e, there exists a $\{p_j\}$ in Σ^2 such that

$$\lim_{j \rightarrow \infty} |S|^2(p_j) = \sup_{\Sigma} |S|^2 \text{ and } \lim_{j \rightarrow \infty} \Delta|S|^2(p_j) \leq 0.$$

By means of inequality (3.4) we conclude that $\sup_{\Sigma} |S|^2 = 0$, i.e, $|S| = 0$ in Σ^2 . Using Lemma 3.1 and Remark 1 of the Introduction we conclude the proof.

Second case. Let us consider expression (3.3)

$$\frac{1}{2}\Delta|S|^2 \geq |S|^2 \left(\frac{4H^2 - 1}{2} - \frac{1}{H}|S| - |S|^2 \right) + \frac{5}{2}v^2|S|^2.$$

As $|S| \leq L_H$, we have $\frac{4H^2 - 1}{2} - \frac{1}{H}|S| - |S|^2 \geq 0$. Hence,

$$\frac{1}{2}\Delta|S|^2 \geq \frac{5}{2}v^2|S|^2.$$

Integrating and using Stokes' Theorem we find

$$0 \geq \frac{5}{2} \int_{\Sigma} v^2|S|^2 d\Sigma \geq 0.$$

It follows that

$$|S| \cdot v = 0. \quad (3.5)$$

Let $\Theta = \{p \in \Sigma^2 : v(p) = 0\} = v^{-1}(0)$ be the nodal lines of v . We know that

$$\Delta v + (|A|^2 + Ric(N, N))v = 0.$$

Hence, we could apply Theorem 2.5 in [C], p. 49, to conclude that Θ has empty interior. Thus, using (3.5), $|S|$ vanishes in an open and dense set. By continuity, $|S| = 0$ in Σ .

Using Lemma 3.1 and Remark.1 of the Introduction we conclude the proof. ■

Theorem 3.4. *There exists no $\Sigma^2 \looparrowright \mathbb{S}^2 \times \mathbb{R}$ complete immersed surface with constant mean curvature greater than one half such that $|S| = L_H$.*

Proof. Suppose that there exist $\Sigma^2 \looparrowright \mathbb{S}^2 \times \mathbb{R}$ satisfying the condition of the theorem. Using expression (3.3)

$$\frac{1}{2}\Delta|S|^2 \geq |S|^2 \left(\frac{4H^2 - 1}{2} - \frac{1}{H}|S| - |S|^2 \right) + \frac{5}{2}v^2|S|^2,$$

with $|S| = L_H$ one finds that

$$0 \geq 0 + \frac{5}{2}v^2L_H^2 \geq 0.$$

Hence $v = 0$, i.e., $\Sigma^2 \looparrowright \mathbb{S}^2 \times \mathbb{R}$ is a cylinder $\gamma \times \mathbb{R}$ for some $\gamma \in \mathbb{S}^2$ with constant curvature $2H$.

On the other hand, for a cylinder $\gamma \times \mathbb{R}$, where $\gamma \in \mathbb{S}^2$ with constant curvature $2H$, we may write

$$S = \begin{pmatrix} 2H^2 + \frac{1}{2} & 0 \\ 0 & -2H^2 - \frac{1}{2} \end{pmatrix}.$$

As $|S| = \frac{\sqrt{2}}{2}(4H^2 + 1) > L_H$ we have a contradiction. ■

In next theorem we need the following result:

Lemma 3.2. *Any Abresch-Rosenberg surface $\Sigma^2 \hookrightarrow \mathbb{H}^2 \times \mathbb{R}$ with $H > \frac{1}{2}$ is an embedded rotationally invariant constant mean curvature sphere.*

Proof. See Proposition 4.3 in [AR], p. 19. ■

Let us consider the polynomial $q_H(t) = -t^2 - \frac{1}{H}t + \left(\frac{8H^4 - 12H^2 - 1}{4H^2}\right)$. When H is greater than a positive root of the polynomial $r(x) = 8x^4 - 12x^2 - 1$, i.e, H is greater than $\sqrt{\frac{12 + \sqrt{176}}{16}}$, there is a positive root for q_H . Let M_H be the positive root.

Theorem 3.5 (Thm 0.3 in Introduction). *Let $\Sigma^2 \hookrightarrow \mathbb{H}^2 \times \mathbb{R}$ be an immersed surface with constant mean curvature H greater than $\sqrt{\frac{12 + \sqrt{176}}{16}} \approx 1.25664$. If*

$$\Sigma^2 \text{ is complete and } \sup_{\Sigma} |S| < M_H$$

or

$$\Sigma^2 \text{ is closed and } |S| \leq M_H,$$

then $\Sigma^2 = S_{\frac{1}{H}}^2$, i.e, Σ^2 is an embedded rotationally invariant constant mean curvature sphere.

Proof. Let us consider two cases. First, Σ is complete and second, Σ is closed.

First case. Consider the expression in Theorem 2.3 with $c = -1$

$$\begin{aligned} \frac{1}{2}\Delta|S|^2 &= |\nabla S|^2 - |S|^4 + |S|^2 \left(-\frac{5\nu^2}{2} + \frac{1}{2} + 2H^2 + \frac{1}{H}\langle ST, T \rangle \right) \\ &\quad - |ST|^2 - \frac{1}{4H^2}\langle ST, T \rangle^2. \end{aligned}$$

As $|\langle ST, T \rangle| \leq |ST| \leq |S|$, we may write

$$\frac{1}{2}\Delta|S|^2 \geq -|S|^4 + |S|^2 \left(\frac{4H^2 + 1 - 5\nu^2}{2} - \frac{1}{H}|S| \right) - \left(\frac{4H^2 + 1}{4H^2} \right) |S|^2,$$

i.e,

$$\frac{1}{2}\Delta|S|^2 \geq |S|^2 \left(\frac{4H^2 - 4 + 5 - 5\nu^2}{2} - \frac{1}{H}|S| - \frac{4H^2 + 1}{4H^2} - |S|^2 \right).$$

This may be rewritten as,

$$\frac{1}{2}\Delta|S|^2 \geq |S|^2 \left(\frac{8H^4 - 12H^2 - 1}{4H^2} - \frac{1}{H}|S| - |S|^2 \right) + \frac{5}{2}(1 - \nu^2)|S|^2. \quad (3.6)$$

Observe that

$$\frac{8H^4 - 12H^2 - 1}{4H^2} - \frac{1}{H}|S| - |S|^2 \geq q_H(\sup_{\Sigma} |S|) =: \frac{d}{2} > 0$$

and $(1 - \nu^2)|S|^2 \geq 0$. Therefore,

$$\Delta|S|^2 \geq d|S|^2. \quad (3.7)$$

Next we estimate $|S|$.

$$|S| \geq 2H|A| - |\langle T, \cdot \rangle T| - (1 - \nu^2) - 4H^2 \geq 2H|A| - 2(1 - \nu^2) - 4H^2,$$

i.e,

$$M_H \geq |S| \geq 2H|A| - 2 - 4H^2.$$

Using Gauss' equation (1.3) in $\mathbb{H}^2 \times \mathbb{R}$ we find

$$K_{\Sigma} = K_{ext} - \nu^2 = -\frac{|A|^2}{2} + 2H^2 - \nu^2 \geq -\frac{1}{2} \left(\frac{M_H + 2 + 4H^2}{2H} \right)^2.$$

Now we can use Theorem 1.2 with $u = |S|^2$, i.e, there exists a $\{p_j\}$ in Σ^2 such that

$$\lim_{j \rightarrow \infty} |S|^2(p_j) = \sup_{\Sigma} |S|^2 \text{ and } \lim_{j \rightarrow \infty} \Delta|S|^2(p_j) \leq 0.$$

Inequality (3.7) allows us conclude that $\sup_{\Sigma} |S|^2 = 0$, i.e, $|S| = 0$ in Σ^2 . Then, by using Lemmas 3.1 and 3.2, we conclude the proof.

Second case. Let us consider expression (3.6)

$$\frac{1}{2}\Delta|S|^2 \geq |S|^2 \left(\frac{8H^4 - 12H^2 - 1}{4H^2} - \frac{1}{H}|S| - |S|^2 \right) + \frac{5}{2}(1 - \nu^2)|S|^2.$$

As $|S| \leq M_H$, we have that $\frac{8H^4 - 12H^2 - 1}{4H^2} - \frac{1}{H}|S| - |S|^2 \geq 0$. Hence,

$$\frac{1}{2}\Delta|S|^2 \geq \frac{5}{2}(1 - \nu^2)|S|^2.$$

Integrating and using Stokes' Theorem we write

$$0 \geq \frac{5}{2} \int_{\Sigma} (1 - \nu^2)|S|^2 d\Sigma \geq 0.$$

Moreover

$$(1 - \nu^2) \cdot |S|^2 = 0. \quad (3.8)$$

Consider $\Theta = \{p \in \Sigma^2; \nu^2(p) = 1\} \subset \mathbb{H}^2 \times \{t_0\}$, for any t_0 . Since H is positive we have that Θ has empty interior. Thus, using (3.8), we conclude that $|S|$ vanishes in an open and dense set. By continuity, $|S| = 0$ in Σ . Using Lemma 3.1 and the fact that the only closed surface is S_H^2 we conclude the proof. ■

Theorem 3.6. *There exists no $\Sigma^2 \looparrowright \mathbb{H}^2 \times \mathbb{R}$ a complete immersed surface with constant mean curvature greater than $\sqrt{\frac{12 + \sqrt{176}}{16}} \approx 1.25664$ such that $|S| = M_H$.*

Proof. Suppose that there exists $\Sigma^2 \looparrowright \mathbb{H}^2 \times \mathbb{R}$ satisfying the condition of the theorem. Using expression (3.6)

$$\frac{1}{2}\Delta|S|^2 \geq |S|^2 \left(\frac{8H^4 - 12H^2 - 1}{4H^2} - \frac{1}{H}|S| - |S|^2 \right) + \frac{5}{2}(1 - \nu^2)|S|^2$$

with $|S| = M_H$ we obtain:

$$0 \geq 0 + \frac{5}{2}(1 - \nu^2)M_H^2 \geq 0.$$

Hence $\nu^2 = 1$, i.e, $\Sigma^2 \looparrowright \mathbb{H}^2 \times \mathbb{R}$ is a slice $\mathbb{H}^2 \times \{t_0\}$. But $\mathbb{H}^2 \times \{t_0\}$ has zero mean curvature, and this is impossible because H is positive. ■

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