Escola de Pós-Graduação em Economia - EPGE
Fundação Getulio Vargas

# Endogenous Growth with Deferred Technological Change

Dissertação submetida à Escola de Pós-Graduação em Economia da Fundação Getulio Vargas como requisito de obtenção do título de Mestre em Economia

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Rio de Janeiro
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**Abstract**

Lucas (2009) has proposed a two-sector model to account for patterns in growth data. However, Lucas's analysis does not involve any inter-temporal decision by the consumers. The behavior of the variables is determined a priori by the technology that is chosen. Rodriguez (2006) proposed a model with the two-sector technology presented by Lucas adding an inter-temporal decision process for the consumer. In addition to the results obtained by Rodriguez, we characterize sufficiency and provide elucidating examples of particular cases of the model. Moreover, we make an effort to derive new insights from the model and clarify some technical points. Finally, we obtain conditions under which the economy invests in human capital even though benefits are deferred.

**Keywords:** Endogenous growth, development, human capital, income distribution.

**JEL Classification codes:** O11, O15.

**Resumo**

Lucas (2009) propôs um modelo com dois setores para explicar padrões observados em dados de crescimento. Entretanto, a análise de Lucas não envolve uma decisão intertemporal para o consumidor. O comportamento das variáveis é determinado à priori pela tecnologia escolhida. Rodriguez (2006) propôs um modelo com a tecnologia com dois setores apresentada por Lucas adicionando um processo de decisão intertemporal para o consumidor. Adicionalmente aos resultados obtidos por Rodriguez, nós caracterizamos suficiência e apresentamos exemplos esclarecedores de casos particulares do modelo. Ademais, nós fazemos um esforço para derivar novos *insights* e esclarecer alguns pontos técnicos. Finalmente, nós obtemos condições sob as quais a economia investe em capital humano mesmo com benefícios diferidos.

**Palavras-chave:** Crescimento endógeno, desenvolvimento, capital humano, distribuição de renda.

* I am especially grateful to Prof. Rubens Penha Cysne for guiding me throughout this process. I would also like to thank Prof. Jair Koiller, Prof. Moacyr Alvim and my colegues Marinho Bertanha, Jefferson Bertolai and Vitor Farinha for their patience and useful insights. All errors are mine.

# Contents

# 1 Introduction

In 1798, Thomas Malthus proposed his Principle of Population:

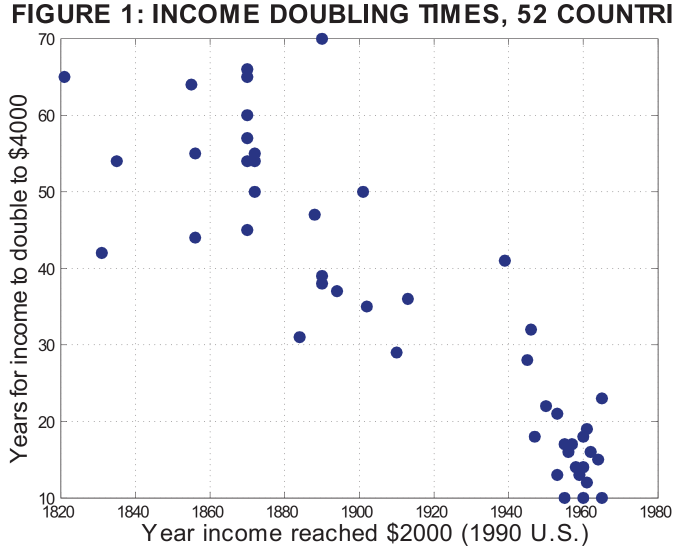
> "The power of population is indefinitely greater than the power in the earth to produce subsistence for man. Population, when unchecked, increases in a geometrical ratio. Subsistence increases only in an arithmetical ratio. A slight acquaintance with numbers will show the immensity of the first power in comparison with the second."

However, soon after that came the industrial revolution and this logic was not fitting anymore, since production increases had also become of geometrical proportions. Lucas (2009) postulates that much of the evolution of cross-country GDP differences from that point on can be explained by how much each country has taken advantage of this revolution:

> "(...) a study of economic growth in the world as a whole must be a study of the diffusion of the industrial revolution across economies."

Parente and Prescott (2002) present compelling evidence on this. Figure 1 uses data from Maddison (2003) and it plots the number of years it took for each country's income to grow from \$2000 (1990 U.S.) to \$4000 against the date that country reached \$2000.

**FIGURE 1: INCOME DOUBLING TIMES, 52 COUNTRIES**



It is clear in Figure 1 that as the years go by it takes less time for a country to double its income (after it reaches the level of $2000). It took the first countries to industrialize around 50 years to double their income while the average number of years that took the countries that doubled their income in the 1950's and 1960's is around 15.

Much work has been done in an endeavor to reconcile these facts. Lucas (2009) provides two deliberately mechanical models that assess these attempts.

The first model is a simpler version of the one used in Tamura (1991) and Lucas (1993) that introduced the idea of modifying the usual human capital accumulation technology so that any one country's rate of human capital growth is influenced by the world's level of human capital. It can be summarized by these bare equations:

$$\eta(t) = \eta_0 e^{\mu t}, \qquad \dot{h}\,(t) = \mu h\,(t)^{1-\theta}\,\eta\,(t)^{\theta}$$

where $\eta\,(t)$ is the human capital of a leading economy that grows at the constant rate $\mu$, $h\,(t)$ is the human capital of other countries and $\theta$ is an externality parameter.

The idea is that the world's human capital (represented by the leading economy's human capital) generates an externality on the human capital accumulation technology of the other countries. This externality is added in order to account for the fact that once a country develops some knowledge; the other countries need only to learn it and not reinvent it. Hence, the higher the world's level of knowledge is the easier it should be for a country to develop human capital.

With this simple model at hand, choosing $\mu$ and $\theta$ adequately, Lucas (2009) is able to generate the pattern in Figure 1. Notice, however, that the countries do not make any decisions in this model; the environment determines exogenously the evolution of the variables[1].

Nevertheless, this model generates the groundless prediction that the poorest countries have the highest growth rates. This is the usual convergence result predicted by the neoclassical growth theory. In his seminal paper Lucas (1988) establishes that a theory of development needs to be able to reconcile the known fact that:

[1] Building on these equations, Rodriguez (2004) introduces an inter-temporal decision by adding the problem of how to allocate an endowment of time each period between capital accumulation and the production of a consumption good.

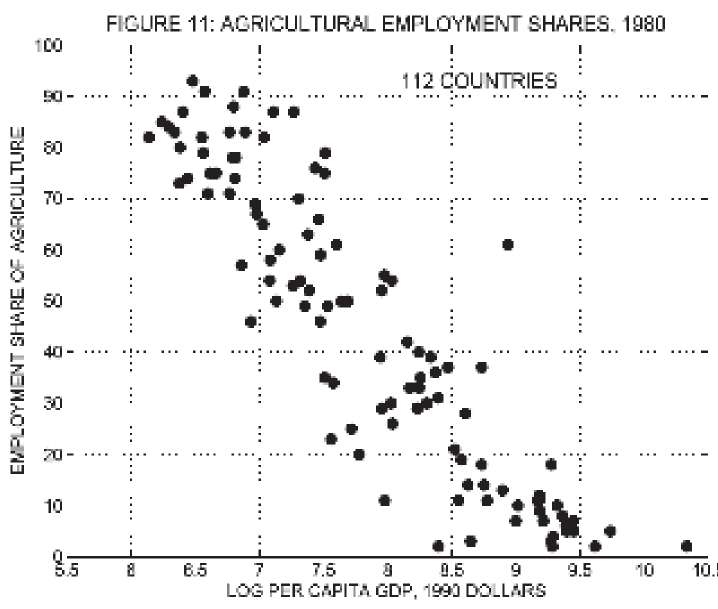
"(...) the poorest countries tend to have the lowest growth; the wealthiest next; the 'middle-income' countries highest."

Indeed, the sample in Figure 1 excludes the countries that did not reach the arbitrary level of \$4000 by 1990. The choice of this level was not fully explained by Parente and Prescott. Lucas (2009) notices that the set of countries excluded is compatible with the ones that have been classified by Sachs and Warner (1995) as closed. To put it briefly, this is the connection made by Parente and Prescott when they classify the impediments imposed on international trade as "barriers to riches".

The second model in Lucas (2009) addresses this issue. In contrast with the focus of the first model on the technology of capital accumulation the second model focus on the production of a consumption good.

Production takes place in two sectors. The first, identified as modern, is human capital intensive. The second is denominated traditional since it uses less human capital. The idea is that countries that use relatively more the traditional sector are less benefited by the world's accumulation of capital.

Interpreting the employment share of agriculture as the level of use of the traditional sector, Lucas motivates the following graph[2]:




[2]This is a reproduction of the graph presented by Lucas (2009). For employment it uses data from the 1984 World Development Report and for real income Maddison (2003).

It is clear that there is a negative correlation between a country's employment share of agriculture and its per capita GDP. Perhaps, and a third model in Lucas (2009) goes in the direction of this account of the facts, the more industrialized a country is the more it gains from the world's human capital accumulation which, throughout time, explains the differences in GPD we observe now.

Hence, there is a great deal that can be learned from the simple models in Lucas (2009). Though, those models do not involve any decision making on behalf of the consumers in a country. The behavior of the variables is determined a priori by the technology that is chosen.

Rodriguez (2006)[3] proposed a model that introduces a consumer decision process to this setting. His model comprises both of the features that issue from Lucas (2009) analysis. That is, it incorporates a spillover effect in the human capital accumulation technology and includes a two-sector production technology. Furthermore, the model introduces a dynamic-optimization problem for the consumer. At each moment in time the consumer will need to choose how much it is willing to work on each sector and how much effort it is willing to devote to human capital accumulation.

Rodriguez (2006) solves the model with a somewhat non-canonical approach. We propose to solve it through a more standard technique. The solution we present uses only usual optimal control theory and enables us to prove sufficiency which is not attempted by Rodriguez (2006). We also provide elucidating examples without externality and with finite horizon. Moreover, we make an effort to derive new insights from the model and clarify some technical points.

At some stage in the process of solving the model we will need to deal with the question of whether it is desirable for an economy to transition from one technology to the other. The country would need to forfeit consumption during a transitional period in order to increase its productivity in the future. We establish sufficient conditions under which a transition takes place.

Section 2 introduces the model proposed by Rodriguez (2006). Section 3 starts with the solution of the general case. Then, we tackle the particular cases without externality and with finite horizon. Section 4 concludes.

[3] This paper was downloaded from http://home.uchicago.edu/~amr/RODRIGUEZ_ALEJANDRO_PAPER.pdf. The author explicitly asks not to be quoted, however, since our work greatly bennefited from his we thought it would not be fair not to give him credit. Besides, we were not able to get in touch with him.

## 2 The Model

We will start by presenting the model put forward by Rodriguez (2006). We intend to examine the behavior of a country that faces the problem of deciding how much effort to put into producing a good (and thus consuming it) and how much to allocate to human capital accumulation (which will enhance the production possibilities in the future).

We will suppress any issue related to trade assuming the country is closed. We will, also, be abstracting altogether from the economics of demography, normalizing population to a constant.

Furthermore, we will assume the decision process of the country can be embodied by a representative agent and that the time horizon is infinite. The representative agent derives utility only from a non storable consumption good, $c$, and the instantaneous utility of consumption will be given by the standard constant elasticity of substitution function:

$$v\left(c\left(t\right)\right)=\frac{c\left(t\right)^{1-\sigma}}{1-\sigma}$$

The country has an endowment of one unity of time per period. This endowment can be allocated between human capital accumulation (or studying), $u\left(t\right)$, and working, $l\left(t\right)$. Accordingly, we set:

$$\begin{aligned} u\left(t\right),l\left(t\right) &\geq 0 \\ u\left(t\right)+l\left(t\right) &= 1 \end{aligned}$$

The production technology is divided into two sectors. The work devoted to sector $i$ at each period will be designated $l_i\left(t\right)$ and we will obviously need to set: $l_1\left(t\right)+l_2\left(t\right)=l\left(t\right)$. Sector 1 will be referred to as traditional, seeing as it uses only work as an input to produce the consumption good:

$$y_1\left(t\right)=al_1\left(t\right)$$

where $a$ is a positive constant that represents the productivity of labor in this sector. Sector 2, on the other hand, uses human capital as an input also; hence, we call it modern. Its technology is given by:

$$y_2\left(t\right)=h\left(t\right)l_2\left(t\right)$$

Observe that the productivity of labor in sector 2 is given by the level of human capital. The aggregate technology function will be given by[4]:

$$y(t) = al_1(t) + h(t)\, l_2(t)$$

Moreover, we assume that labor is mobile, thus, the optimal allocation is equivalent to the equilibrium allocation which we will solve for. This linear technology, proposed by Rodriguez (2006), is a very convenient simplification of the two-sector technology presented by Lucas (2009). The linearity assumption makes to model a lot tractable and makes it possible to introduce the inter-temporal decision for the consumer. All credit to this insight is due to Rodriguez (2006).

We will use the concept of human capital that is typical in growth models. This concept is broader than the labor market type of human capital, as noted by Romer (1990), incorporating not only person specific abilities but also a notion of knowledge, or technology, in a sense that trespasses individuals. In this sense human capital is unbounded, since it is not tied to individuals.

Following our motivation to generate the Parente and Prescott result in Figure 1, we will introduce a technology spillover to the law of motion for human capital, so that we will have:

$$\dot{h}(t) = \delta h(t)^{1-\theta}\, \eta(t)^{\theta}\, u(t)$$

where $\eta(t)$ is an exogenous variable that varies according to:

$$\dot{\eta}(t) = \gamma \eta(t)$$

Thus, the evolution of $\eta(t)$ has an external impact on the capital accumulation process of the country. We could just assume that $\eta(t)$ is some measure of the rest of the world's human capital in order to generate the technology spillover intuition. Though, following Rodriguez (2004) we will assume that $\eta(t)$ is the human capital of the most advanced country (where most advanced means with the highest human capital level)[5]. This allows us

[4]Since $l_1(t)$ and $l_2(t)$ are measured in unities of the time endowment and the output is going to be the consumption good, we need to adjust the unities of $a$ and $h(t)$ to be unities of consumption good over unities of the time endowment good.

[5]Lucas (1993) and Tamura (1996) use the world's average human capital instead of the leader's. Though in order to extend the model in one direction (here extending to a two sector technology) we need to make simplifying assumptions on other.

to determine $\gamma$ endogenously by solving the problem of the leader assuming that he behaves as any other country.

Notice that this human capital accumulation technology is not affected by movements in labor from one sector to the other. That is, with regard to the effects on human capital accumulation all that matters is the time that is not spent working, $1-(l_1+l_2)$, and not in which sector the work is done. This fact together with the linearity in labor on both sectors entitles a simplification to the production technology. Both sectors will not be used together in any case but if $h(t)=a$. In other words, $h(t)>a$ implies $l_1(t)=0$ and $h(t)<a$ implies $l_2(t)=0$. Though this is a maximization argument and it does not quite belong to the definition of the environment, we will work from now on with[6]:

$$y(t)=\max(a,h(t))(1-u(t))$$

We can, now, determine the problem faced by the representative agent:

**Problem 1**

$$\max_{\{u(t)\in[0,1]\}_{t=0}^{\infty}}\int_0^{\infty}e^{-\rho t}\frac{(\max(a,h(t))(1-u(t)))^{1-\sigma}}{1-\sigma}dt$$

*subject to*

$$\dot{h}(t)=\delta h(t)^{1-\theta}\eta(t)^{\theta}u(t), \qquad h(0)=h_0$$
$$\dot{\eta}(t)=\gamma\eta(t), \qquad \eta(0)=\eta_0$$

Whenever we refer to Problem 1, we will presume that the following assumptions are being satisfied:

**Assumption 1.1** $(\delta,\rho,\sigma,a,h_0,\gamma)\in\mathbb{R}_{++}^6$ *and* $\theta\in(0,1)$.

**Assumption 1.2** $\rho>\gamma(1-\sigma)$.

**Assumption 1.3** $\rho<\delta$.

**Assumption 1.4** $\eta_0>a$.

**Assumption 1.5** $\frac{\theta}{1+\theta}<\sigma<\frac{1+\theta}{\theta}$.

[6] If you prefer just take this as the definition of the technology function.

Assumption 1.1 is set in order for the parameters to have reasonable values. For a solution to exist we need the objective function of the problem to be bounded under all feasible paths of $u(t)$, accordingly, we need Assumption 1.2. To assure that the solution is interior[7] we need Assumption 1.3. Assumption 1.4 is imposed in order for the leader to start with enough human capital to use the modern sector. Finally, we will need Assumption 1.5 to establish sufficiency[8].

After the solution to this problem is established, we will analyze the particular case in which there is no externality in the human capital accumulation technology, that is, $\theta = 0$. We will work, then, with the very standard law of motion that was used by Uzawa (1965) and latter on by Lucas (1988):

$$\dot{h}(t) = \delta h(t) u(t)$$

This means that there are constant returns to human capital accumulation and there is a constant $\delta$ that affects its productivity. If the agent decides to devote no effort to human capital accumulation, $u(t) = 0$, then none accumulates. If all effort is devoted to this purpose, $u(t) = 1$, $h(t)$ grows at its maximal rate $\delta$.

Observe that the simplification in the production function applies once again. Therefore, the problem faced by the country is:

**Problem 2**

$$\max_{\{u(t)\in[0,1]\}_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} \frac{(\max(a, h(t))(1-u(t)))^{1-\sigma}}{1-\sigma} dt$$

*subject to*

$$\dot{h}(t) = \delta h(t) u(t), \quad h(0) = h_0$$

When we refer to Problem 2 we will always assume that these assumptions are valid:

**Assumption 2.1** $(\delta, \rho, \sigma, a, h_0) \in \mathbb{R}^5_{++}$.

[7] Actually this assumption assures only that the solution of the Modern Economy Problem as defined below is interior.

[8] Note that since $\theta \in (0,1)$, the restriction $\theta/(1+\theta) < \sigma < (1+\theta)/\theta$ is not very tight. For instance, if $\theta = 1/2$ then the restriction is $1/3 < \sigma < 3$ and if $1/2 < \sigma < 2$ the restriction is satisfied for any choice of $\theta$.

**Assumption 2.2** $\rho > \delta (1 - \sigma)$.

**Assumption 2.3** $\rho < \delta$.

Observe that these assumptions are analogous to those of the first problem. We choose to restate them to easy referencing.

# 3 Main Results

Here we solve both of the problems that have been described. We start with Problem 1; the general case. We, then, turn to Problem 2 following the same steps.

## 3.1 The Problem With Externality

We will start by solving Problem 1, the case in which there is externality. The reason we do not have a trivial solution to the problem lies on the fact that the integrand in the objective function is not differentiable on the state variable. In particular it is not differentiable at $h(t) = a$ since the function $\max(a, h(t))$ is not differentiable at that point. Therefore we need to separate the solution on the sets of $h(t)$ for which the function is differentiable.

First, we will solve the much simpler case in which the country starts with enough human capital to work on the modern sector, that is $h_0 \geq a$, we call this the Modern Economy Problem. We will, then, analyze the Traditional Economy Problem for $h_0 < a$. In order to do that, we first try to answer the question: will this economy accumulate enough capital to transition to the modern technology? We prove that the answer is always yes. Then, we characterize the optimal solution and provide a numerical example.

### 3.1.1 Modern Economy Problem

Here we will just follow Rodriguez (2004). The solution for $h_0 \geq a$ is easy to determine. Since $h(t)$ is non-decreasing (see the law of motion for human capital), $h_0 \geq a$ implies $h(t) \geq a$ for all $t$. Therefore the traditional sector will never be used, that is $\max(a, h(t)) = h(t)$ for all $t$. Then, the problem can be restated in the following way:

$$\max_{\{u(t)\in[0,1]\}_{t=0}^{\infty}}\left\{\begin{array}{cc} \int_0^\infty e^{-\rho t}\frac{(h(1-u))^{1-\sigma}}{1-\sigma}dt & \\ & \dot{h}=\delta\eta^\theta h^{1-\theta}u \\ s.t. & h(0)=h_0 \\ & \dot{\eta}=\gamma\eta \\ & \eta(0)=\eta_0 \end{array}\right\}$$

Since the path of $\eta$ is independent from the other variables, the ones that are under the control of the agent, it can be interpreted as a continuum of parameters. Having this in mind we define a new state variable: $x := h/\eta$. This way we incorporate the fact that what is relevant to the decision making of the agent is the relation between its human capital and the leader's, since the leader's follows an exogenous process. Therefore, we can rewrite the problem in the following way:

$$\max_{\{u(t)\in[0,1]\}_{t=0}^{\infty}}\int_0^\infty \eta_0^{1-\sigma}e^{-(\rho-(1-\sigma)\gamma)t}\frac{(x(1-u))^{1-\sigma}}{1-\sigma}dt$$

subject to

$$\dot{x}=\delta x^{1-\theta}u-\gamma x, \quad x(0)=\frac{h_0}{\eta_0}$$

Current value Hamiltonian (assuming the solution is interior):

$$H(u,x,\phi,t)=\eta_0^{1-\sigma}\frac{(x(1-u))^{1-\sigma}}{1-\sigma}+\phi\left(\delta x^{1-\theta}u-\gamma x\right)$$

Necessary conditions for maximization:

$$-\eta_0^{1-\sigma}(x(1-u))^{-\sigma}x+\phi\delta x^{1-\theta}=0 \tag{1}$$

$$\dot{\phi}=(\rho-(1-\sigma)\gamma)\phi-\eta_0^{1-\sigma}(x(1-u))^{-\sigma}(1-u)-\phi\left((1-\theta)\delta x^{-\theta}u-\gamma\right) \tag{2}$$

$$\dot{x}=\delta x^{1-\theta}u-\gamma x,\ \ x(0)=\frac{h_0}{\eta_0} \tag{3}$$

$$\lim_{t\to\infty}e^{-(\rho-(1-\sigma)\gamma)t}\phi(t)x(t)=0 \tag{4}$$

Substituting (1) in (2):

$$\frac{\dot{\phi}}{\phi}=\rho+\sigma\gamma-\delta x^{-\theta}(1-\theta u) \tag{5}$$

And from (1):

$$\begin{aligned}
\phi &= \frac{\eta_0^{1-\sigma}}{\delta}(1-u)^{-\sigma}x^{\theta-\sigma} \\
\frac{\dot{\phi}}{\phi} &= \sigma\frac{\dot{u}}{1-u}+(\theta-\sigma)\frac{\dot{x}}{x} \qquad (6)
\end{aligned}$$

Then, (5) and (6) imply, using (3):

$$\dot{u}=\frac{\left[\rho+\theta\gamma-\delta x^{-\theta}(1-\sigma u)\right](1-u)}{\sigma} \qquad (7)$$

Hence, (3) and (7) form a system of differential equations which combined with the initial condition in (3) and with the transversality condition, determine the optimal[9] paths $u^*(t)$ and $x^*(t)$.

On the steady state we have:

$$\begin{aligned}
\dot{u} &= 0 \Longrightarrow \rho+\theta\gamma-\delta x^{-\theta}(1-\sigma u)=0 \\
\dot{x} &= 0 \Longrightarrow \delta x^{1-\theta}u-\gamma x=0
\end{aligned}$$

Solving we find the steady state of the system:

$$\begin{aligned}
x^{SS} &= \left(\frac{(\sigma+\theta)\gamma+\rho}{\delta}\right)^{-\frac{1}{\theta}} \qquad (8a) \\
u^{SS} &= \frac{\gamma}{(\sigma+\theta)\gamma+\rho} \qquad (8b)
\end{aligned}$$

We can, now, calculate the rate of growth of the product in this economy:

$$\frac{\dot{y}}{y}=\frac{\delta}{\sigma}\left(\frac{\eta}{h}\right)^{\theta}-\frac{\theta\gamma+\rho}{\sigma}$$

Observe that a higher level of human capital, $h$, implies a lower rate of growth. Thus, for the economies that make use of the modern sector the model predicts the usual convergence result.

Also, the growth rate is increasing with $\eta$, this is consistent with the evidence from Parente and Prescott (2002) of Figure 1. As time goes by the world's (or leading economy's) human capital increases. Hence, countries

[9] Sufficiency is established in Appendix D using the Arrow Theorem and Assumption 2.5.

that start their growth period later on will have a higher productivity in their human capital accumulation technology which essentially induces faster growth.

Furthermore, if the only technology available was the modern one, we would have the ungrounded prediction that the poorer a country the higher is its rate of growth. That is the reason we introduced the traditional sector in the first place.

To facilitate a procedure in the solution for the Traditional Economy we define the value function of this solution, $W$, if the initial time is $t_0$:

$$W(t_0, h(t_0)) := \int_{t_0}^{\infty} e^{-\rho(t-t_0)} \eta(t)^{1-\sigma} \frac{(x^*(t)(1-u^*(t)))^{1-\sigma}}{1-\sigma} dt$$

Given $t_0$ and $h(t_0)$ we can calculate the value function. The path for $\eta(t)$ is determined exogenously and the solution $(u^*(t), x^*(t))$ is determined by the system of differential equations (3) and (7) with the initial condition $x(t_0) = h(t_0)/\eta(t_0)$ and the terminal condition in (4). Although we cannot solve this explicitly we will present a numerical example below.

#### 3.1.2 Transition

Before we tackle the Traditional Economy Problem we will provide a helpful and strong result regarding the question of when it will be desirable for the country to transition to the modern sector.

**Proposition 1** *If Assumptions 1.1→1.4 are satisfied then the solution $(u^*(t), h^*(t))$ to Problem 1 with $h_0 < a$ will* ***always*** *involve capital accumulation and there exists a finite $t^*$ such that $h^*(t^*) = a$.*

**Proof.** In Appendix B. ■

The rationality behind this proposition is based on the law of motion for the human capital, $\dot{h} = \delta h^{1-\theta}\eta^{\theta}u$. Observe that for the set of parameters we are taking under consideration, the return (in unities of human capital) of allocating time to human capital accumulation, $\delta h^{1-\theta}\eta^{\theta}$, goes to infinity as $\eta \to \infty$. Therefore, there will be a time in which the level of the leader's human capital is so large that if the country devotes a small portion of his time endowment for a infinitesimal period it can multiply its human capital by a very large factor.

In other words, the cost of making a transition tends to zero and the benefits, collected as improvements on the labor productivity, are significant. Consequently, at some point it must become optimal to make a transition.

### 3.1.3 Traditional Economy Problem

The result in Proposition 1 entitles the following simplification to Problem 1. There is always a finite time, $t^*$, such that $h(t^*) = a$, i.e., there is a transition to the modern sector. Therefore, using the Bellman Principle of Optimality[10], we can rewrite the problem as

$$\max_{\{u(t)\in[0,1]\}_{t=0}^{t^*}} \left\{ \begin{array}{cc} & \int_0^{t^*} e^{-\rho t} \frac{(a(1-u))^{1-\sigma}}{1-\sigma} dt \\ & \dot{h} = \delta \eta^\theta h^{1-\theta} u \\ s.t. & h(0) = h_0,\, h(t^*) = a \\ & \dot{\eta} = \gamma \eta \\ & \eta(0) = \eta_0,\ \ \eta(t^*) = \eta_0 e^{\gamma t^*} \end{array} \right\} + \max_{\{u(t)\in[0,1]\}_{t=t^*}^{\infty}} \left\{ \begin{array}{cc} & \int_{t^*}^{\infty} e^{-\rho t} \frac{(h(1-u))^{1-\sigma}}{1-\sigma} dt \\ & \dot{h} = \delta \eta^\theta h^{1-\theta} u \\ s.t. & h(t^*) = a \\ & \dot{\eta} = \gamma \eta \\ & \eta(t^*) = \eta_0 e^{\gamma t^*} \end{array} \right\}$$

Observe that the second maximization has been solved above, except for a multiplying constant $e^{-\rho t^*}$, it is precisely the Modern Economy Problem with $h_0 = a$ and $\eta_0 = \eta_0 e^{\gamma t^*}$. Thus

$$e^{-\rho t^*} W(t^*, a) = \max_{\{u(t)\in[0,1]\}_{t=t^*}^{\infty}} \left\{ \begin{array}{cc} & \int_{t^*}^{\infty} e^{-\rho t} \frac{(h(1-u))^{1-\sigma}}{1-\sigma} dt \\ & \dot{h} = \delta \eta^\theta h^{1-\theta} u \\ s.t. & h(t^*) = a \\ & \dot{\eta} = \gamma \eta \\ & \eta(t^*) = \eta_0 e^{\gamma t^*} \end{array} \right\}$$

where $W(\cdot,\cdot)$ represents the value function of the Modern Economy Problem. Therefore, the Traditional Economy Problem can be restated in the following way:

$$\max_{\{u(t)\in[0,1]\}_{t=t^*}^{\infty}} \left\{ \begin{array}{cc} & \int_0^{t^*} e^{-\rho t} \frac{(a(1-u))^{1-\sigma}}{1-\sigma} dt \\ & \dot{h} = \delta \eta^\theta h^{1-\theta} u \\ s.t. & h(0) = h_0,\, h(t^*) = a \\ & \dot{\eta} = \gamma \eta \\ & \eta(0) = \eta_0,\, \eta(t^*) = \eta_0 e^{\gamma t^*} \end{array} \right\} + e^{-\rho t^*} W(t^*, a)$$

[10] "Principle of Optimality: An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision" Bellman (1957).

To conclude the simplification, note that the path of $\eta$ is exogenous, hence, the agent will solve:

$$\max_{\{u(t)\in[0,1]\}_{t=0}^{t^*}} \int_0^{t^*} e^{-\rho t}\frac{(a(1-u))^{1-\sigma}}{1-\sigma}dt + e^{-\rho t^*}W(t^*,a)$$

subject to

$$\dot{h} = \delta\eta_0^{\theta}e^{\gamma\theta t}h^{1-\theta}u, \quad h(0)=h_0, \quad h(t^*)=a, \quad t^* \text{ free} \tag{9}$$

Where $e^{-\rho t^*}W(t^*,a)$ will be treated as a salvage term and $t^*$ as a free endpoint to be determined optimally. Also, in addition to the initial condition on the state variable we also have a terminal condition. We can, now, solve it using the standard optimal control theory. We first define the Hamiltonian

$$H(u,h,\lambda,t) = \lambda_0 e^{-\rho t}\frac{(a(1-u))^{1-\sigma}}{1-\sigma} + \lambda\left(\delta\eta_0^{\theta}e^{\gamma\theta t}h^{1-\theta}u\right)$$

Assume that $(u^*(t), h^*(t))$ is a solution to our problem, then $\lambda(t)$ satisfies

$$\dot{\lambda} = -\frac{\partial H(u^*,h^*,\lambda,t)}{\partial h} = -(1-\theta)\lambda\delta\eta_0^{\theta}e^{\gamma\theta t}(h^*)^{-\theta}u^* \tag{10}$$

Since we are dealing with a free endpoint problem, the following transversality condition must be satisfied

$$H(u^*(t^*), h^*(t^*)\lambda(t^*), t^*) + \frac{d}{dt^*}\left(e^{-\rho t^*}W(t^*,a)\right) = 0 \tag{11}$$

Also, for each $t\in[0,t^*]$, $u^*(t)$ is a value of $u\in[0,1]$ which maximizes

$$g(u) := H(h^*(t), u, \lambda(t), t) = \lambda_0\frac{(a(1-u))^{1-\sigma}}{1-\sigma} + \lambda(t)\delta\eta_0^{\theta}e^{\gamma\theta t}(h^*(t))^{1-\theta}u$$

Differentiating:

$$\begin{aligned} g'(u) &= -\lambda_0 e^{-\rho t}a^{1-\sigma}(1-u)^{-\sigma} + \lambda(t)\delta\eta_0^{\theta}e^{\gamma\theta t}(h^*(t))^{1-\theta} \\ g''(u) &= -\sigma\lambda_0 e^{-\rho t}a^{1-\sigma}(1-u)^{-\sigma-1} \end{aligned}$$

Observe that $g'(u) \to -\infty$ as $u \to 1$, thus $u = 1$ cannot maximize $g(u)$. Furthermore, $g(u)$ is strictly concave on $[0, 1)$. Then the following must hold:

$$u^*(t) = 0 \text{ maximizes } g(u) \Leftrightarrow g'(0) \leq 0 \Leftrightarrow$$
$$\Leftrightarrow \lambda(t) \leq \lambda_0 \delta^{-1} \eta_0^{-\theta} \left(h^*(t)\right)^{\theta-1} e^{-(\rho+\gamma\theta)t} a^{1-\sigma} \tag{13a}$$
$$u^*(t) \in (0,1) \text{ maximizes } g(u) \Leftrightarrow g'(u^*(t)) = 0 \Leftrightarrow$$
$$\Leftrightarrow \lambda(t) = \lambda_0 \delta^{-1} \eta_0^{-\theta} \left(h^*(t)\right)^{\theta-1} e^{-(\rho+\gamma\theta)t} a^{1-\sigma} \left(1 - u^*(t)\right)^{-\sigma} \tag{13b}$$

First note that if $\lambda_0 = 0$, then the fact that $(\lambda_0, \lambda(t)) \neq 0$ for all $t$ combined with (10), ($13a$) and ($13b$) imply that $\lambda(t) = \bar{\lambda} < 0$ and we must have $u^*(t) = 0$ for all $t$. Consequently, we have a contradiction with the result in Proposition 1. Thus $\lambda_0 = 1$.

We know, from Proposition 1, that the optimal path involves a transition and this can only occur if the country is accumulating human capital in a final period before the transition. Thus, we guess that either $u^*(t) \in (0, 1)$ for all $t \in [0, t^*]$ or there exist a $\hat{t} \in (0, t^*)$ such that $u^*(t) = 0$ for $t \in \left[0, \hat{t}\right]$ and $u^*(t) \in (0, 1)$ for all $t \in (\hat{t}, t^*]$.

If $u^*(t) \in (0, 1)$ we know that ($13b$) must apply, then differentiating it we get

$$\frac{\dot{\lambda}(t)}{\lambda(t)} = (\theta - 1) \frac{\dot{h}^*(t)}{h^*(t)} - (\rho + \gamma\theta) + \sigma \frac{\dot{u}^*(t)}{1 - u^*(t)}$$

Observe that (9) and (10) imply that

$$\frac{\dot{\lambda}(t)}{\lambda(t)} = (\theta - 1) \frac{\dot{h}^*(t)}{h^*(t)}$$

Then, we obtain that if $u^*(t) \in (0, 1)$ it must follow this law of motion

$$\frac{\dot{u}^*(t)}{1 - u^*(t)} = \frac{\rho + \gamma\theta}{\sigma}$$

which implies the one parameter solution

$$u^*(t) = 1 - C_1 e^{-\left(\frac{\gamma\theta+\rho}{\sigma}\right)t} \tag{14}$$

This means that, if $u^*(t) \in (0, 1)$, the path for $u^*(t)$ is increasing implying that once the country starts accumulating human capital it will accumulate more and more until it gets to the point of the transition. This is in agreement with our guess for the possible paths of $u^*(t)$.

Furthermore, notice that equation (14) accounts for the possibility that $u^*(t) = 0$ for an initial period, seeing as $C_1 > 1$ implies $u^*(0) < 0$. In that case we would have $u^*(t) = 0$ for $t \in \left[0, \hat{t}\right]$ with $\hat{t}$ being defined by the equation $\lambda\left(\hat{t}\right) = \delta^{-1}\eta_0^{-\theta}\left(h^*\left(\hat{t}\right)\right)^{\theta-1} e^{-(\rho+\gamma\theta)\hat{t}} a^{1-\sigma}$, considering (13$a$), which is equivalent to $u^*\left(\hat{t}\right) = 0$.

Then, substituting (14) into the differential equation in (9) we get for $\hat{t} \geq 0$:

$$h^*(t) = \left(C_2 + \frac{\delta\eta_0^\theta}{\gamma}e^{\gamma\theta t} - \frac{\theta\sigma\delta\eta_0^\theta}{\gamma\theta(\sigma-1)-\rho}e^{\frac{(\gamma\theta(\sigma-1)-\rho)}{\sigma}t}C_1\right)^{\frac{1}{\theta}}$$

and using the initial and final conditions in (9) we determine

$$\begin{aligned} C_1 &= \frac{\gamma\theta(\sigma-1)-\rho}{\theta\sigma\delta\eta_0^\theta}\frac{\left(\frac{\delta\eta_0^\theta}{\gamma}\left(e^{\gamma\theta t^*} - e^{\gamma\theta\hat{t}}\right) - \left(a^\theta - h_0^\theta\right)\right)}{\left(e^{\frac{(\gamma\theta(\sigma-1)-\rho)}{\sigma}t^*} - e^{\frac{(\gamma\theta(\sigma-1)-\rho)}{\sigma}\hat{t}}\right)} \\ C_2 &= h_0^\theta - \frac{\delta\eta_0^\theta}{\gamma}e^{\gamma\theta\hat{t}} + \frac{\theta\sigma\delta\eta_0^\theta}{\gamma\theta(\sigma-1)-\rho}e^{\frac{(\gamma\theta(\sigma-1)-\rho)}{\sigma}\hat{t}}C_1 \end{aligned}$$

Moreover, with the optimal[11] path already determined for each $t^*$ we can use equation (11) to determine the optimal $t^*$; substituting for $\lambda(t^*)$ using (13$b$) we obtain

$$\frac{a^{1-\sigma}}{1-\sigma}\left(\frac{1-\sigma u^*(t^*)}{(1-u^*(t^*))^\sigma}\right) = \rho e^{-\rho t^*}W(t^*, a) - e^{-\rho t^*}\frac{\partial}{\partial t}W(t^*, a)$$

The growth rate of a country that starts with $h_0 < a$ will have three stages:

$$\frac{\dot{y}}{y} = \begin{cases} 0 & t \in \left[0, \hat{t}\right] \\ -\frac{\rho+\gamma\theta}{\sigma} & t \in \left(\hat{t}, t^*\right] \\ \frac{\delta}{\sigma}\left(\frac{\eta}{h}\right)^\theta - \frac{\rho+\theta\gamma}{\sigma} & t \in (t^*, \infty) \end{cases}$$

Thus, as in the case with no externality, the country will have to forfeit consumption for an intermediate period during which it will have a negative growth rate. After that, it will enjoy ever increasing growth rates until the convergence is over with and it enters the balanced growth path.

[11] Sufficiency is established in Appendix D.

### 3.1.4 A Numerical Example

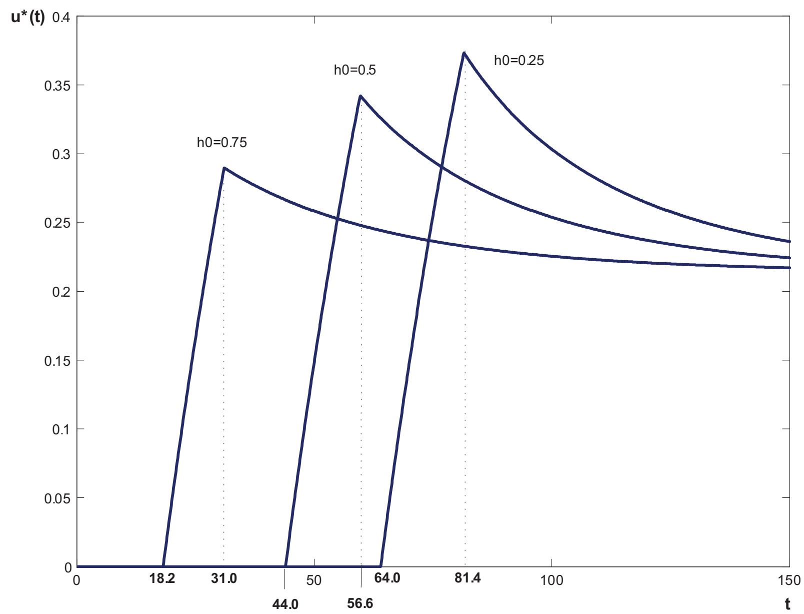
We will, now, present a numerical example of the solution we have just arrived at. Following the calibrations in Lucas (2009) we set the leader's growth rate, $\gamma$, to 0.02, and take his calibrated level for the externality parameter, $\theta$, 0.67. For the elasticity of substitution and discount parameters we use the standard numbers: $\sigma = 2$ and $\rho = 0.04$. In order to have $x^{SS}$ in ($8a$) equal to 1, so that the leader always remains the leader and that there is convergence in the long run, we set $\delta = 0.0934$. Also, we normalize the parameters $a$ and $\eta_0$ to 1.

Then, we calculate the optimal paths for three initial levels of human capital, $h_0$: 0.25, 0.5, 0.75. We first present the times in which the country starts accumulating capital, $\hat{t}$, and the time of the transition to the modern sector, $t^*$, for each initial level of human capital:

| $h_0$ | 0.25 | 0.5 | 0.75 |
|---|---|---|---|
| $\hat{t}$ | 64.0 | 44.0 | 18.2 |
| $t^*$ | 81.4 | 56.6 | 31.0 |

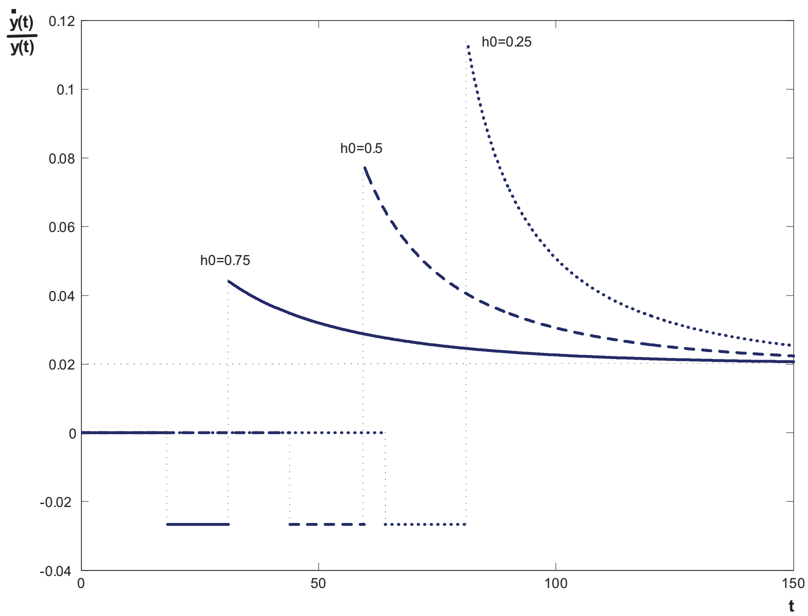
With a higher initial level of human capital a country will start accumulating earlier and, thus transition earlier. This is what we would expect, since the cost of the transition is lower when $h_0$ is higher. Also, the transition period, $t^* - \hat{t}$, is reduced. A difference in the initial level of human capital from 0.75 to 0.5 practically doubles the time in which the transition will take place.

Now, we look at the optimal paths for the control variable $u$:




Notice that with a higher initial level of human capital the country needs to invest less time to human capital accumulation in order to transition. After the transition, the country will invest decreasing amounts. What happens is that increasing levels of human capital will enhance both the productivity of the human capital technology, as it did before the transition, and the productivity of the consumption good production technology. The second effect dominates.

Next, we present the plot for the growth rates:




In the beginning the country does not invest in human capital and uses only the traditional sector technology, thus, the paths are constant and the growth rate is zero. Then, as the country starts to invest, human capital starts to increase. Though, since it is still using the traditional sector the output decreases and at this transition period the growth rates are negative. After the transition to the modern sector the growth rate will always be positive.

The country is willing to face the negative growth rate period because after the transition it will have positive growth rates forever. It will start with really high growth rates (a period that can be interpreted as a "miracle") and in the long run its growth rate tends to the leaders growth rate level, $\gamma = 0.02$.

## 3.2 The Problem Without Externality

In this subsection we will follow closely the steps taken in the previous subsection since the absence of externality is a particular case, with $\theta = 0$, of the problem solved above. However the solution is not completely analogous.

For instance, the result in Proposition 1 is not valid if there is no externality. We arrive at a similar result that gives us sufficient conditions under which a transition takes place, however our condition is not necessary. We start, again, by solving the Modern Economy Problem.

### 3.2.1 Modern Economy Problem

We search for a solution to Problem 2 with $h_0 \geq a$. Since $h(t)$ is non-decreasing (see its law of movement), $h_0 \geq a$ implies $h(t) \geq a$ for all $t$. Therefore, the traditional technology will never be used. In this case, the problem can be restated as:

$$\max_{\{u(t)\in[0,1]\}_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} \frac{(h(t)(1-u(t)))^{1-\sigma}}{1-\sigma} dt$$

subject to

$$\dot{h}(t) = \delta h(t) u(t), \quad h(0) = h_0 > a, \quad \lim_{t\to\infty} h(t) \text{ free}$$

The solution to this problem is fairly simple, though the familiar transversality condition $\lim_{t\to\infty} p(t) x(t) = 0$, $x(t)$ being the state variable and $p(t)$ the costate variable, does not apply. In view of the fact that the state variable $h(t)$ might be unbounded for some paths of the control $u(t)$. This does not allow the determination of a solution through the usual steps. Therefore we need a different approach to arrive at the solution.

In Appendix A we solve the same problem with a finite horizon $T$. Having that solution at hand we now use its limit as $T \to \infty$, say $(u^*(t), h^*(t))$, as a candidate for the optimal solution to the infinite horizon problem. Seeing as it seems sensible that the limit of the finite horizon solution as $T \to \infty$ would be optimal for the infinite horizon problem. Then we show that this path satisfies the necessary condition in the Maximum Principle. We conclude proving that it is indeed the optimal solution to the infinite horizon problem by checking the sufficiency conditions of an infinite horizon version of the Arrow Theorem[12].

Let us see what happens to the solutions in $(52a)$ and $(52b)$ when $T \to \infty$. First notice that $\hat{t} \to \infty$ as $T \to \infty$. Thus, using L'Hôpital's rule we can

[12] Here we use Theorem 3.14 in Seierstad and Sydsaeter (1999).

easily calculate the limits for the equations in (52$a$), (52$b$) and (52$c$) as $T \to \infty$ that provide the following candidates for $(u^*(t), h^*(t))$ and $\phi(t)$:

$$
\begin{aligned}
u^*(t) &= \frac{\delta - \rho}{\delta\sigma} && t \geq 0 && \text{(16a)}\\
h^*(t) &= h_0 e^{\left(\frac{\delta-\rho}{\sigma}\right)t} && t \geq 0 && \text{(16b)}\\
\phi(t) &= \left(\frac{\rho - \delta(1-\sigma)}{\delta\sigma}\right)^{-\sigma} \frac{h_0^{-\sigma}}{\delta} e^{(\rho-\delta)t} && t \geq 0 && \text{(16c)}
\end{aligned}
$$

Notice that this means that an economy that starts with $h_0 \geq a$ will be on a balanced growth path at all times. We use the Arrow Theorem to prove that this solution is effectively optimal. To begin we need a convex set $A(t)$ for each $t \geq 0$ such that for all admissible pair $(u(t), h(t))$ we have $h(t) \in A(t)$. So, we take $A(t) = [0, \infty)$ for all $t \geq 0$. We need $\phi_0 = 1$ as well. Then, we need to check if our candidate satisfies the conditions in the Maximum Principle. Moreover, we need to check if

$$\hat{H}(h, \phi, t) = \max_{u \in [0,1]} H(h, u, \phi(t), t) \text{ exists and is concave in } h \text{ for } h \in A(t) \text{ and each } t \geq 0 \tag{17}$$

and if for all admissible $h(t)$

$$\lim_{t \to \infty} \inf e^{-\rho t} \phi(t)(h(t) - h^*(t)) \geq 0 \tag{18}$$

First observe that $(u^*(t), h^*(t))$ is admissible. Also, the function $\phi(t)$ is easily seen to satisfy the differential equation in (43). Furthermore, the Hamiltonian $g(u) = H(h^*(t), u, \phi(t), t)$ has been shown to be strictly concave on $[0, 1)$. In addition, we know that $\lim_{u \to 1} g'(u) = -\infty$ then $u = 1$ can never be optimal and it is easy to see that $\phi(t) > \delta^{-1}(h^*(t))^{-\sigma} \Leftrightarrow \delta > \rho$ which has been assumed. Then for all $t \geq 0$, $g(u)$ is maximized by $u^* = \frac{\delta-\rho}{\delta\sigma}$ since $g'(u^*) = 0$.

The concavity condition (17) has been verified above, so it remains to verify the limit condition (18). For any admissible path $(u(t), h(t))$ the law of motion $\dot{h}(t) = \delta h(t) u(t)$ with initial condition $h(0) = h_0$ must apply. Hence, integrating we obtain

$$h(t) = h_0 e^{\delta \int_0^t u(\tau) d\tau}$$

Then using (16$c$) we can calculate the limit for any admissible path of $h(t)$

$$\lim_{t\to\infty}\inf e^{-\rho t}\phi(t)\,h(t) = \left(\frac{\rho-\delta(1-\sigma)}{\delta\sigma}\right)^{-\sigma}\frac{h_0^{1-\sigma}}{\delta}\lim_{t\to\infty}\inf e^{\delta\left(\int_0^t u(\tau)d\tau - t\right)} \geq 0$$

where the inequality follows from the assumption that $\rho > \delta(1-\sigma)$. Moreover

$$\lim_{t\to\infty} e^{-\rho t}\phi(t)\,h^*(t) = \left(\frac{\rho-\delta(1-\sigma)}{\delta\sigma}\right)^{-\sigma}\frac{h_0^{1-\sigma}}{\delta}\lim_{t\to\infty} e^{-\frac{(\rho-\delta(1-\sigma))}{\sigma}t} = 0$$

where the last equality follows, again, from the assumption that $\rho > \delta(1-\sigma)$. This implies that for all admissible $h(t)$

$$\lim_{t\to\infty}\inf e^{-\rho t}\phi(t)\,(h(t)-h^*(t)) \geq 0$$

which verifies condition (18). Thus the optimality of our candidate path $(u^*(t), h^*(t))$ is assured.

At last, we calculate the value function of this optimal path, $V$:

$$\begin{aligned} V(h_0) &:= \int_0^\infty e^{-\rho t}\frac{(h^*(t)(1-u^*(t)))^{1-\sigma}}{1-\sigma}dt \\ V(h_0) &= \frac{\sigma}{1-\sigma}(\delta\sigma)^{\sigma-1}(\rho-\delta(1-\sigma))^{-\sigma}h_0^{1-\sigma} \end{aligned}$$

The growth rate generated by this path can be easily calculated:

$$\frac{\dot{y}}{y} = \frac{\delta-\rho}{\sigma}$$

Since the economy is in a balanced growth path the rate of growth is evidently constant.

### 3.2.2 Transition

We will, now, provide a condition on the exogenous parameters of the Traditional Economy Problem, that is the problem for $h_0 < a$, under which we can guarantee that there will be capital accumulation and a transition to the modern sector technology on the optimal path. Proposition 1, for the reasons stated at the end of the proof in Appendix B, is not valid if $\theta \to 0$. Thus, for the case at hand, with $\theta = 0$, if have a weaker result.

**Proposition 2** *If Assumptions 2.1→2.3 are satisfied then there exists an exogenous and unique* $h^{\#} < a$ *such that if* $h_0 \in \left(h^{\#}, a\right)$ *the solution* $\left(u^*(t), h^*(t)\right)$ *to Problem 2 with* $h_0 < a$ *involves capital accumulation and there exists a finite* $t^*$ *such that* $h^*(t^*) = a$.

**Proof.** In Appendix C. ■

In other words, we show that it will be optimal to accumulate human capital until the country's human capital reaches the level $a$ and the sector 2 is used as long as we have $h_0 \in \left(h^{\#}, a\right)$ for some $h^{\#} < a$. The proof follows the steps we describe now.

First notice that if only the traditional sector is ever used, that is, if $h(t) < a$ for all $t$, it cannot be optimal to accumulate any human capital since it is only used as an input to the technology in the modern sector. This means that the best path amongst those that do not involve a transition to the modern sector is the path with no human capital accumulation whatsoever. Then, in order to show that it is optimal to transition to the modern sector, it is sufficient to show that $u(t) = 0$ for all $t \geq 0$ is not an optimal path.

After establishing this, we show that the path with no human capital accumulation is dominated by the path $u(t) = \frac{\delta-\rho}{\delta\sigma}$ for all $t$[13], as long as $h_0 \in \left(h^{\#}, a\right)$ with $h^{\#}$ being defined by:

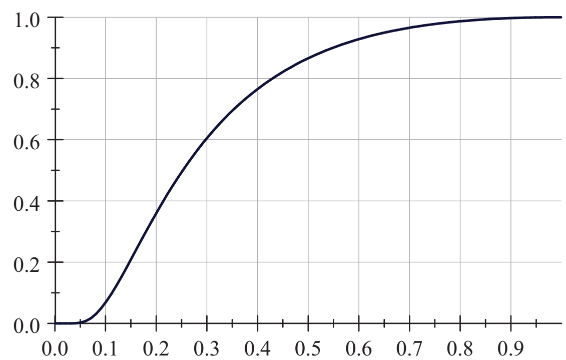
$$h^{\#} = a\left(\left(\frac{\rho-(1-\sigma)\delta}{(1-\sigma)(\delta-\rho)}\right)\left(\left(\frac{\rho-(1-\sigma)\delta}{\delta\sigma}\right)^{\sigma-1} - 1\right)\right)^{\frac{\delta-\rho}{\sigma\rho}}$$

Finally, we show that $h^{\#} < a$ for all set of parameters and the result is established.

To illustrate the relationship between $h^{\#}$ and the model's parameters we present in following graphs. First we take $a = 1$, $\sigma = 2$ and $\delta = 1$ and we

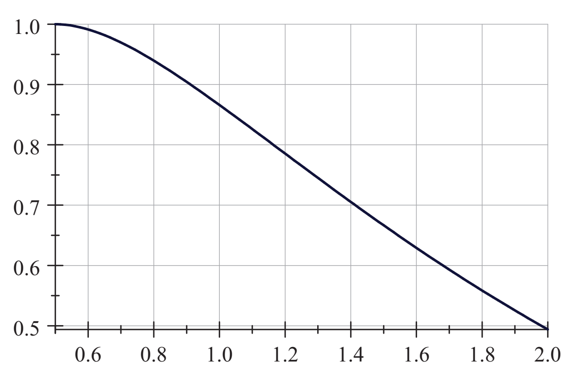
[13] This is the solution to the Modern Economy Problem applied to the Traditional Economy Model.

graph $h^{\#}$ on the y-axis and $\rho$ on the x-axis:



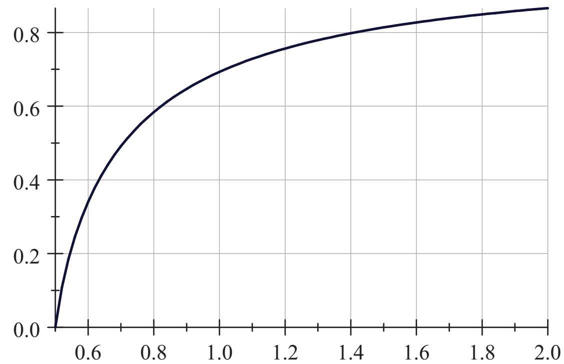

Notice that there is a positive relation between $h^{\#}$ and $\rho$. The higher is the agent's discount factor the less she will care about the future. Then, we need a higher initial level of human capital, $h_0$, to assure that it is desirable for the agent to incur on the cost of accumulating capital knowing that the benefits are deferred. Notice that for all values of $\rho$ plotted we have $\delta(1-\sigma) < \rho < \delta$.

To analyze the impact of $\delta$ we again trace the graph of $h^{\#}$ now with $\delta$ in the x-axis, taking $a = 1$, $\sigma = 2$ and $\rho = 0.5$:




The correlation is negative seeing as a higher level of $\delta$ implies a higher productivity of human capital accumulation. Thus, increasing $\delta$ makes it easier for the agent to transition to the modern sector; with less effort she can accumulate more capital. Therefore, a lower level of $h_0$ is required for the transition to be desirable. Notice that for all values of $\delta$ plotted we have $\delta > \rho$.

Lastly, we look at the relationship between $h^{\#}$ and $\sigma$, taking $a = 1$, $\delta = 1$ and $\rho = 0.5$:




The intuition is again very simple. The instantaneous elasticity of substitution[14] is given by $1/\sigma$, as a result we have that a higher level of $\sigma$ implies a lower level of the instantaneous elasticity of substitution which entails a lower tolerance to delay consumption. Hence, higher levels of $\sigma$ lead to a tighter restriction on $h_0$ in order to have the transition being optimal.

### 3.2.3 Traditional Economy Problem

We, now, search for the solution to the slightly more complicated problem of an economy that starts with $h_0 < a$. Evidently, it is either optimal to accumulate human capital and transition to the modern sector or not, and the result in Proposition 2 implies that there are cases in which it is optimal to transition. If the solution does not involve a transition it is straightforward to see that the solution is to never accumulate human capital at all, that is, choose $u(t) = 0$ for all $t$. If the a transition will never take place, human capital will never be used to enhance utility, thus, there is no reason to accumulate it[15].

---

[14] Remember that, when $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$, the instantaneous elasticity of substitution is given by:

$$-\frac{u'(c_s)/u'(c_t)}{c_s/c_t}\frac{d(c_s/c_t)}{d(u'(c_s)/u'(c_t))} = -\frac{c_s^{-\sigma}/c_t^{-\sigma}}{c_s/c_t}\frac{d(c_s/c_t)}{d\left(c_s^{-\sigma}/c_t^{-\sigma}\right)} = \frac{1}{\sigma}.$$

[15] This point is formally made in the proof of Proposition 1.

Therefore, in order to characterize the solution for every case all that is left to do is to establish the solution for the cases in which the transition is optimal. This is what we do now.

In order to get to the solution we need first to restate the problem in a more tractable manner. We search for the best path amongst those that have a transition to the sector that is human capital intensive. So that we will have $h(t^*) = a$ for some $t^*$. Thus, f this is the case, we can rewrite the problem in the following way:

$$\max_{\{u(t)\in[0,1]\}_{t=0}^{t^*}} \left\{ \begin{array}{c} \int_0^{t^*} e^{-\rho t} \frac{(a(1-u))^{1-\sigma}}{1-\sigma} dt \\ \quad \dot{h} = \delta h u \\ s.t. \quad h(0) = h_0 \\ \quad h(t^*) = a \end{array} \right\} + \max_{\{u(t)\in[0,1]\}_{t=t^*}^{\infty}} \left\{ \begin{array}{c} \int_{t^*}^{\infty} e^{-\rho t} \frac{(h(1-u))^{1-\sigma}}{1-\sigma} dt \\ \quad \dot{h} = \delta h u \\ s.t. \quad h(t^*) = a \end{array} \right\}$$

Notice that

$$e^{-\rho t^*} V(a) = \max_{\{u(t)\in[0,1]\}_{t=t^*}^{\infty}} \left\{ \begin{array}{c} \int_{t^*}^{\infty} e^{-\rho t} \frac{(h(1-u))^{1-\sigma}}{1-\sigma} dt \\ \quad \dot{h} = \delta h u \\ s.t. \quad h(t^*) = a \end{array} \right\}$$

Then the problem simplifies to the following:

$$\max_{\{u(t)\in[0,1]\}_{t=0}^{t^*}} \int_0^{t^*} e^{-\rho t} \frac{(a(1-u))^{1-\sigma}}{1-\sigma} dt + e^{-\rho t^*} V(a)$$

subject to

$$\dot{h} = \delta h u, \quad h(0) = h_0, \quad h(t^*) = a, \quad t^* \text{ free} \tag{19}$$

We define the Hamiltonian:

$$H(u, h, \lambda, t) = \lambda_0 e^{-\rho t} \frac{(a(1-u))^{1-\sigma}}{1-\sigma} + \lambda \delta h u$$

Denoting $(u^*(t), h^*(t))$ as a solution to our problem, then $\lambda(t)$ satisfies

$$\dot{\lambda} = -\frac{\partial H(u^*, h^*, \lambda, t)}{\partial h} = -\lambda \delta u^*, \tag{20}$$

The free endpoint implies the following transversality condition

$$H(u^*(t^*), h^*(t^*) \lambda(t^*), t^*) + \frac{d}{dt^*}\left(e^{-\rho t^*} V(a)\right) = 0. \tag{21}$$

Since for this case we do not have the usual transversality condition that implies $\lambda_0 = 1$[16], we need to contemplate the possibility that we might have $\lambda_0 = 0$.

Also, the Maximum Principle requires that for each $t \in [0, t^*]$, $u^*(t)$ is a value of $u \in [0,1]$ which maximizes

$$g(u) := H(h^*(t), u, \lambda(t), t) = \lambda_0 \frac{(a(1-u))^{1-\sigma}}{1-\sigma} + \lambda(t)\,\delta h^*(t)\,u$$

Differentiating:

$$\begin{aligned} g'(u) &= -\lambda_0 e^{-\rho t} a^{1-\sigma} (1-u)^{-\sigma} + \lambda(t)\,\delta h^*(t) \\ g''(u) &= -\sigma \lambda_0 e^{-\rho t} a^{1-\sigma} (1-u)^{-\sigma-1} \end{aligned}$$

Observe that $g'(u) \to -\infty$ as $u \to 1$, thus $u = 1$ cannot maximize $g(u)$. Besides, $g(u)$ is strictly concave on $[0,1)$. Then the following must hold:

$$u^*(t) = 0 \text{ maximizes } g(u) \Leftrightarrow g'(0) \leq 0 \Leftrightarrow$$
$$\Leftrightarrow \lambda(t) \leq \lambda_0 \delta^{-1} (h^*(t))^{-1} e^{-\rho t} a^{1-\sigma} \tag{23a}$$
$$u^*(t) \in (0,1) \text{ maximizes } g(u) \Leftrightarrow g'(u^*(t)) = 0 \Leftrightarrow$$
$$\Leftrightarrow \lambda(t) = \lambda_0 \delta^{-1} (h^*(t))^{-1} e^{-\rho t} a^{1-\sigma} (1-u)^{-\sigma} \tag{23b}$$

First note that if $\lambda_0 = 0$, then the fact that $(\lambda_0, \lambda(t)) \neq 0$ for all $t$ combined with (20), ($23a$) and ($23b$) imply that $\lambda(t) = \bar{\lambda} < 0$ and we must have $u^*(t) = 0$ for all $t$. Consequently, for $h_0 \geq h^{\#}$ this contradicts the result in Proposition 1. Thus, $\lambda_0 = 1$ if $h_0 \geq h^{\#}$.

Furthermore, notice that $h_0 \geq h^{\#}$ is a sufficient condition for Proposition 1 to apply, not necessary. Hence, there could be $h_0 < h^{\#}$ such that it is still desirable to make a transition. If $\lambda_0 = 0$ we know that there will not be a transition and the value function of the solution, $V_1$, is:

$$\begin{aligned} V_1 &= \int_0^{\infty} e^{-\rho t} \frac{a^{1-\sigma}}{1-\sigma} dt \\ V_1 &= \frac{1}{\rho} \frac{a^{1-\sigma}}{1-\sigma} \end{aligned} \tag{24}$$

[16] For example, on the simplest optimal control problem with finite fixed endpoint, $T$, the transversality condition implies $\lambda(T) = 0$. Then, since the Pontryagin Maximum Principle requires $(\lambda_0, \lambda(t)) \neq (0,0)$ for all $t$ and $\lambda_0$ to be either 0 or 1, we can establish that $\lambda_0$ must be 1.

We will, now, solve for the case with $\lambda_0 = 1$. In the usual case in which we can assure that $\lambda_0 = 1$, the solution that we arrive after solving the necessary conditions is our obvious candidate for the optimal solution. Since in this case we might have $\lambda_0 = 0$, we need to check if the solution with $\lambda_0 = 1$ is better than the solution with $\lambda_0 = 0$. When we arrive at the solution, we will be able to calculate its value function. We will, then, compare it to (24) in order to check which one is better and, finally, choose our candidate for the optimal solution.

In order for the solution with $\lambda_0 = 1$ to be better then the one with $\lambda_0 = 0$ it must involve a transition. Thus, there must be human capital accumulation in a final period before the transition. Hence, we guess that either $u^*(t) \in (0,1)$ for all $t \in [0, t^*]$ or there exist a $\hat{t} \in (0, t^*)$ such that $u^*(t) = 0$ for $t \in [0, \hat{t}]$ and $u^*(t) \in (0,1)$ for all $t \in (\hat{t}, t^*]$.

If $u^*(t) \in (0,1)$ we know that $(23b)$ must apply, then differentiating it we get

$$\frac{\dot{\lambda}(t)}{\lambda(t)} = -\rho - \frac{\dot{h}^*(t)}{h^*(t)} + \sigma \frac{\dot{u}^*(t)}{1 - u^*(t)}$$

Notice that (19) and (20) imply that

$$\frac{\dot{\lambda}(t)}{\lambda(t)} = -\frac{\dot{h}^*(t)}{h^*(t)}$$

Then, if $u^*(t) \in (0,1)$ it must follow this law of motion

$$\frac{\dot{u}^*(t)}{1 - u^*(t)} = \frac{\rho}{\sigma}$$

Solving this differential equation we get

$$u^*(t) = 1 - C_3 e^{-\frac{\rho}{\sigma} t} \tag{25}$$

This means that, if $u^*(t) \in (0,1)$, the path for $u^*(t)$ is increasing. Then, once the country starts accumulating human capital it will accumulate more and more until it gets to the point of the transition. This is in agreement with our guess for the possible paths of $u^*(t)$.

Furthermore, notice that equation (25) takes account of the possibility that $u^*(t) = 0$ for an initial period, seeing as if $C_3 > 1$ it implies $u^*(0) < 0$. In that case we would have $u^*(t) = 0$ for $t \in [0, \hat{t}]$ with $\hat{t}$ being defined by the equation $\lambda\left(\hat{t}\right) = \delta^{-1}\left(h^*\left(\hat{t}\right)\right)^{-1} e^{-\rho\hat{t}} a^{1-\sigma}$, considering $(23a)$, which is equivalent to $u^*\left(\hat{t}\right) = 0$.

Then, substituting (25) into the differential equation in (19) we get for $\hat{t} \geq 0$:

$$h^*(t) = C_4 e^{\delta t + C_3 \frac{\delta\sigma}{\rho} e^{-\frac{\rho}{\sigma}t}}$$

and using the initial and final conditions in (19) we determine

$$\begin{aligned} C_3 &= \frac{\rho}{\delta\sigma} \frac{\left(\ln\left(\frac{a}{h_0}\right) - \delta\left(t^* - \hat{t}\right)\right)}{\left(e^{-\frac{\rho}{\sigma}t^*} - e^{-\frac{\rho}{\sigma}\hat{t}}\right)} \\ C_4 &= h_0 e^{-\delta\hat{t} - \left(\ln\left(\frac{a}{h_0}\right) - \delta\left(t^* - \hat{t}\right)\right)\left(e^{-\frac{\rho}{\sigma}t^*} - e^{-\frac{\rho}{\sigma}\hat{t}}\right)^{-1} e^{-\frac{\rho}{\sigma}\hat{t}}} \end{aligned}$$

The solution path $(u^*(t), h^*(t))$ defined by the last four equations is precisely the limit of the solution of Traditional Economy Problem with externality as $\theta$ tends to 0. This conclusion can be arrived at through tedious calculations, using L'Hôpital's rule, which we choose not to present here.

Then, with the optimal[17] path already determined for each $t^*$ we can use equation (21) to determine the optimal $t^*$:

$$e^{-\rho t^*} \frac{\left(a\left(1 - u^*(t^*)\right)\right)^{1-\sigma}}{1-\sigma} + \lambda(t^*)\,\delta h^*(t^*)\,u^*(t^*) + \frac{d}{dt^*}\left(e^{-\rho t^*} V(a)\right) = 0$$

Using $(23b)$ to substitute for $\lambda(t^*)$ we have, with some algebra:

$$\frac{1 - \sigma u^*(t^*)}{\left(1 - u^*(t^*)\right)^{\sigma}} = \frac{V(a)}{V_1} \tag{26}$$

The value function of this solution will be given by:

$$V_2 = \int_0^{\hat{t}} e^{-\rho t} \frac{a^{1-\sigma}}{1-\sigma} dt + \int_{\hat{t}}^{t^*} e^{-\rho t} \frac{\left(a\left(1 - u^*(t)\right)\right)^{1-\sigma}}{1-\sigma} dt + e^{-\rho t^*} V(a)$$

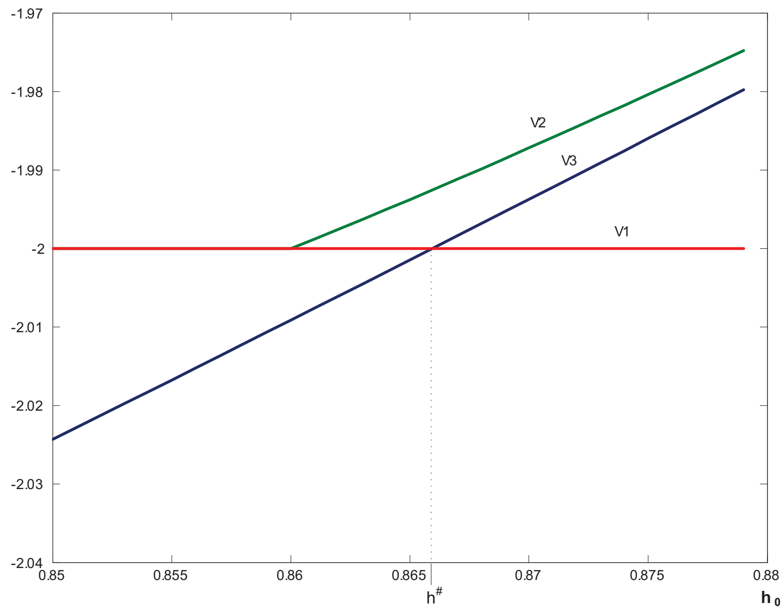
Notice that the growth of a country that never transitions to the modern sector is always equal to zero. For an economy that does transition we have:

$$\frac{\dot{y}}{y} = \begin{cases} 0 & t \in \left[0, \hat{t}\right] \\ -\frac{\rho}{\sigma} & t \in \left(\hat{t}, t^*\right] \\ \frac{\delta - \rho}{\sigma} & t \in \left(t^*, \infty\right) \end{cases}$$

[17] For $\lambda_0 = 0$ and for $\lambda_0 = 1$ and $u^*(t) = 0$ sufficiency is easily established through the Arrow Theorem. However, for $\lambda_0 = 1$ and $u^*(t) \in (0, 1)$ we were not able to establish sufficiency.

Therefore, the country goes through a period during which it has a negative growth rate in order to have a positive growth rate after the transition.

We can solve all of the equations above numerically. Taking $a = 1$, $\delta = 1$, $\rho = 0.5$ and $\sigma = 2$, we get the following graph of $V_1$ (red), $V_2$ (green) and of the value function of the path used in Proposition 1, $V_3$ (blue) with $h_0$ varying on the x-axis[18]:




For the set of parameters chosen, the threshold level given by Proposition 1, $h^{\#}$, is around 0.866 which is precisely the value for $h_0$ that gives $V_1 = V_3$ (this was, by the way the equation used to determine $h^{\#}$). Obviously we have $V_3 < V_1$ for $h_0 < h^{\#}$ and $V_3 > V_1$ for $h_0 > h^{\#}$.

Since the condition given by Proposition 1 is sufficient and not necessary for the transition to be desirable, we have an interval ranging from around 0.86 to $h^{\#}$ in which we have $V_3 < V_1$ but $V_2 > V_1$. For these values of $h_0$ it is desirable to transition even though the condition given by Proposition 1 is not being satisfied.

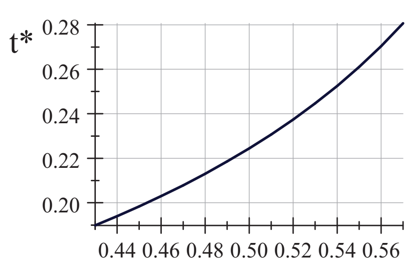
Also notice that, since $V_2$ is the value function of the best path with capital accumulation we never have $V_2 < V_1$ or $V_2 < V_3$. Now, for any value

[18] The restriction $C_1 < 1$ was checked for each computation and was never binding.

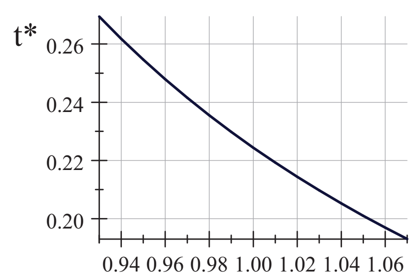
of $h_0$ lower than 0.86 equation (26) has no solution. This means that the optimal path does not involve a transition to the modern sector. Then it is best for the country never to accumulate human capital and keep using the traditional technology perpetually.

Finally, observe that the value function of the optimal solution, $V_1$ for low values of $h_0$ and $V_2$ for high enough values of $h_0$, has a kink at the point in which the country is indifferent from making a transition or not. We would not expect this since, usually, the value function is differentiable with respect to the initial level of the state variable. This could be explained by the fact that we are not dealing with a usual case of optimal control; our objective function is not differentiable at $h = a$. Also, if the country will never transition we would expect a zero derivative, because human capital will never be used and changes in its initial level should not change the value function. On the other hand, if the a transition takes place, changes in human capital will have a direct positive impact on the value function. However, we have not reached any definitive conclusion in this respect.

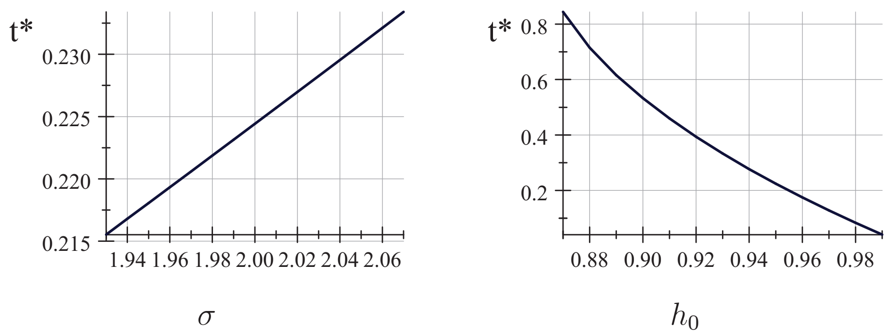
We also present the following graphs that will only corroborate the obvious intuition for the impact of the parameters on the time that the transition takes place, $t^*$ (evidently we choose parameters for which a transition takes place). We again take $a = 1$, $\delta = 1$, $\rho = 0.5$, $\sigma = 2$ and $h_0 = 0.95$. For the graph of each parameter we vary its value around those set up:




$\rho$




$\delta$




The higher the value of the discount the longer it takes to transition, since the agent has a lower tolerance for delaying consumption. For the other parameters the reasoning is the same.

### 3.2.4 Example with Finite Time Horizon

The idea of this example is to show that infinite time horizon is not necessary for the transition to modern sector to be desirable. We present an example in which the agent has a finite time horizon, still with no externality, and demonstrate that a transition might be desirable.

The agent's problem is the same we have been working with, adding the simplifying assumptions that $\sigma = 0$ (then the instantaneous utility function is linear), $\delta = 1$ and $\rho \in (0,1)$. Hence, the agent solves

$$\max_{\{u(t)\in[0,1]\}_{t=0}^{T}} \int_0^T e^{-\rho t} \max(a,h)(1-u)\,dt$$

subject to

$$\dot{h} = hu, \quad h(0) = h_0$$

We first look at the problem faced by a country that starts with $h_0 > a$. Since $h(t)$ is non-decreasing we have $\max(a, h(t)) = h(t)$ for all $t$. Thus, the problem can be restated in the following way[19]:

$$\max_{\{u(t)\in[0,1]\}_{t=t_0}^{T}} \int_{t_0}^T e^{-\rho t} h(1-u)\,dt$$

subject to

$$\dot{h} = hu, \quad h(t_0) = h_0$$

[19] We will solve with an initial time $t_0$ because that we facilitate a procedure done bellow. However, the solution applies, in particular, to the case where $t_0 = 0$.

The current value Hamiltonian is

$$H\left(u,h,\phi,t\right)=h\left(1-u\right)+\phi hu$$

These are the necessary conditions that follow from the Maximum Principle:

$$\phi=\rho\phi-H_h \implies \phi=\rho\phi-1+\left(1-\phi\right)u \tag{27}$$

$$u\in\arg\max_u H \tag{28}$$

$$e^{-\rho(T-t_0)}\phi\left(T\right)=0 \tag{29}$$

$$\dot{h}=hu,\quad h\left(t_0\right)=h_0 \tag{30}$$

Observe that the linearity of the Hamiltonian on the control variable will generate a bang-bang type of solution. From (27) and (28) we have that:

$$\begin{aligned}\text{If }\phi &< 1 \implies u^*\left(t\right)=0\text{ and }\phi=\rho\phi-1\\ \text{If }\phi &> 1 \implies u^*\left(t\right)=1\text{ and }\phi=\left(\rho-1\right)\phi\end{aligned}$$

Note that, using the fact that $\rho\in(0,1)$, in both cases $\phi\left(t\right)$ is decreasing, since:

$$\begin{aligned}\text{If }\phi &< 1 \implies \phi=\rho\phi-1<0\\ \text{If }\phi &> 1 \implies \frac{\phi}{\phi}=\rho-1<0\end{aligned}$$

Hence, we know that $\phi\left(t\right)$ is a decreasing function for all $t\in[t_0,T]$. Furthermore, since from (29) we have $\phi\left(T\right)=0<1$, there is a final interval, $\hat{t}\leq t\leq T$, during which $\phi<1$, $u^*\left(t\right)=0$ and $\phi=\rho\phi-1$. Thus, using (30):

$$\left.\begin{array}{c}u^*\left(t\right)=0\\ \phi^*\left(t\right)=\frac{1}{\rho}\left(1-e^{-\rho(T-t)}\right)\\ h^*\left(t\right)=h^*\left(\hat{t}\right)\end{array}\right\}\quad \hat{t}\leq t\leq T \tag{31}$$

The time $\hat{t}$ is determined by the equation $\phi^*\left(\hat{t}\right)=1$, that is,

$$\hat{t}=T+\frac{\ln\left(1-\rho\right)}{\rho}$$

provided that

$$\hat{t}>t_0 \implies T-t_0>-\frac{\ln\left(1-\rho\right)}{\rho}$$

Otherwise, the solution is given by (31) with $\hat{t} = t_0$. If $\hat{t} > t_0$, then there is an initial interval $t_0 \leq t < \hat{t}$ during which $\phi > 1$, $u^*(t) = 1$ and $\phi = (\rho - 1)\phi$. Using (30), the initial condition for $h(t)$ and the required continuity of $\phi(t)$ at $\hat{t}$:

$$\left.\begin{array}{c} u^*(t) = 1 \\ \phi^*(t) = e^{(\rho-1)\left(t-\hat{t}\right)} \\ h^*(t) = h_0 e^{t-t_0} \end{array}\right\} \quad t_0 \leq t < \hat{t} \tag{32}$$

Making use of (32) we can determine $h^*\left(\hat{t}\right)$

$$h^*\left(\hat{t}\right) = h_0 e^{\hat{t}-t_0}$$

and substituting for $\hat{t}$ we obtain

$$h^*\left(\hat{t}\right) = h_0 (1-\rho)^{\frac{1}{\rho}} e^{T-t_0}$$

Let us calculate the value function of this optimal[20] solution, $\mathcal{F}(h_0, t_0, T)$ (assuming $\hat{t} > t_0$):

$$\mathcal{F}(h_0, t_0, T) = \int_{\hat{t}}^{T} e^{-\rho(t-t_0)} h^*\left(\hat{t}\right) dt$$

$$\mathcal{F}(h_0, t_0, T) = h_0 (1-\rho)^{\frac{1-\rho}{\rho}} e^{(1-\rho)(T-t_0)}$$

Having established this, we can tackle the problem of the economy that starts with $h_0 < a$. Once again it is possible that the optimal solution does not involve a transition to the modern sector. Following an argument already made above, we know that amongst the paths that do not involve a transition there is a clear dominant path which is to never accumulate capital whatsoever choosing $u(t) = 0$ for all $t$.

We calculate the value function related to this path $\mathcal{F}_1$:

$$\begin{aligned} \mathcal{F}_1 &= \int_0^T e^{-\rho t} a dt \\ \mathcal{F}_1 &= \frac{a}{\rho}\left(1 - e^{-\rho T}\right) \end{aligned}$$

[20] Since $H(0, h, \lambda(t), t) = h$ and $H(1, h, \lambda(t), t) = \lambda(t) h$ are both obviously concave functions of $h$, the Arrow Theorem establishes sufficiency.

From now on, we focus on the search of the best path that involves a transition. For these paths we know that $h(t^*) = a$ for some $t^* \in (0, T)$. The Bellman Principle of Optimality implies that for all $t \geq t^*$ the solution will be the same as the one determined above. Thus making use of (31) e (32) we have:

$$\left.\begin{array}{c} u^*(t) = 0 \\ \phi^*(t) = \frac{1}{\rho}\left(1 - e^{-\rho(T-t)}\right) \\ h^*(t) = h^*\left(\hat{t}\right) \end{array}\right\} \quad \hat{t} \leq t \leq T$$

$$\left.\begin{array}{c} u^*(t) = 1 \\ \phi^*(t) = e^{(\rho-1)\left(t-\hat{t}\right)} \\ h^*(t) = h_0 e^{t-t_0} \end{array}\right\} \quad t^* \leq t < \hat{t}$$

assuming that $T$ is large enough, that is, $T > t^* + 1$.

Moreover, for $t \in [0, t^*]$ we can rewrite the agent's problem in the following manner:

$$\max_{\{u(t)\in[0,1]\}_{t=0}^{t^*}} \left\{ \begin{array}{cc} \int_0^{t^*} e^{-\rho t} a(1-u)\, dt & \\ & \dot{h} = hu \\ s.t. & h(0) = h_0 \\ & h(t^*) = a \end{array} \right\} + \max_{\{u(t)\in[0,1]\}_{t=t^*}^{T}} \left\{ \begin{array}{cc} \int_{t^*}^{T} e^{-\rho t} h(1-u)\, dt & \\ s.t. & \dot{h} = hu \\ & h(t^*) = a \end{array} \right\}$$

Observe that:

$$e^{-\rho t^*} F(a, t^*, T) = \max_{\{u(t)\in[0,1]\}_{t=t^*}^{T}} \left\{ \begin{array}{cc} \int_{t^*}^{T} e^{-\rho t} h(1-u)\, dt & \\ s.t. & \dot{h} = hu \\ & h(t^*) = a \end{array} \right\}$$

Therefore, the agent will solve

$$\max_{\{u(t)\in[0,1]\}_{t=0}^{t^*}} \int_0^{t^*} e^{-\rho t} a(1-u)\, dt + e^{-\rho t^*} F(a, t^*, T)$$

subject to

$$\dot{h} = hu, \quad h(0) = h_0, \quad h(t^*) = a, \quad t^* \text{ free}$$

The current value Hamiltonian is

$$H(u, h, \lambda, t) = \lambda_0 a(1-u) + \lambda h u$$

Then we have the following necessary conditions:

$$\lambda = \rho\lambda - H_h \implies \lambda = (\rho - u)\,\lambda \tag{33}$$

$$u \in \arg\max_u H \tag{34}$$

$$e^{-\rho t^*} H\left(u\left(t^*\right), h\left(t^*\right), \lambda\left(t^*\right), t^*\right) + \frac{d}{dt^*}\left(e^{-\rho t^*} F\left(a, t^*, T\right)\right) = 0 \tag{35}$$

$$\dot{h} = hu, \;\; h(0) = h_0, \;\; h\left(t^*\right) = a \tag{36}$$

From (33), (34) and (36) we have that:

$$\begin{aligned} \text{If } \lambda h \;\; &< \;\; \lambda_0 a \implies u^*(t) = 0,\; \frac{\lambda}{\lambda} = \rho \text{ and } \frac{\dot{h}}{h} = 0 \\ \text{If } \lambda h \;\; &> \;\; \lambda_0 a \implies u^*(t) = 1,\; \frac{\lambda}{\lambda} = (\rho - 1) \text{ and } \frac{\dot{h}}{h} = 1 \end{aligned} \tag{37}$$

Observe that in both cases the growth rate of $\lambda h$ is

$$\frac{\lambda}{\lambda} + \frac{\dot{h}}{h} = \rho$$

So $\lambda h$ is an increasing function. Consequently, one of these possibilities must be true: [1] $\lambda h < \lambda_0 a$ for $t \in [0, t^*]$ or [2] $\lambda h < \lambda_0 a$ for $t \in [0, \tilde{t})$ and $\lambda h > \lambda_0 a$ for $t \in (\tilde{t}, t^*]$ with $\tilde{t}$ being determined by $\lambda\left(\tilde{t}\right) h\left(\tilde{t}\right) = \lambda_0 a$ provided that $\tilde{t} > 0$, otherwise $\tilde{t} = 0$.

We can dismiss the first possibility since we are searching for a path that involves a transition. Seeing as if that was the case (37) implies that $u(t) = 0$ for all $t \in [0, t^*]$ and, in fact, this means that there will never be any capital accumulation.

Therefore, a transition can only occur if we are on the second case. Hence, we know that there is an interval $\left[\tilde{t}, t^*\right]$ during which $\lambda h > \lambda_0 a$ and, thus, $u^*(t) = 1$, $\frac{\lambda}{\lambda} = (\rho - 1)$ and $\frac{\dot{h}}{h} = 1$. In order to determine these functions explicitly we will, first, arrive at a terminal condition for the costate variable $\lambda(t)$. Using the fact that $h(t^*) = a$ and $u(t^*) = 1$ on (35) we obtain

$$\lambda\left(t^*\right) = (1 - \rho)^{\frac{1-\rho}{\rho}} e^{(1-\rho)(T - t^*)}$$

Notice that this condition is equivalent to $\lambda(t^*) = \phi(t^*)$. Then, using the terminal conditions for $\lambda(t)$ and $h(t)$ we get:

$$\left.\begin{array}{c} u^*(t) = 1 \\ \lambda^*(t) = e^{(\rho-1)\left(t - \hat{t}\right)} \\ h^*(t) = a e^{t - t^*} \end{array}\right\} \quad \tilde{t} \le t < t^* \tag{38}$$

If $\tilde{t} > 0$, it is determined by $\lambda^* \left(\tilde{t}\right) h^* \left(\tilde{t}\right) = \lambda_0 a$. Now, to carry on with the solution we will need the value of $\lambda_0$. From the Maximum Principle we know that $\lambda_0$ is either 0 or 1. Since our transversality condition is different than the usual one it is not straightforward that $\lambda_0 = 1$. Thus, we need to take under consideration the possibility that $\lambda_0 = 0$. We will, then, solve the problem for both values of $\lambda_0$. With the solutions at hand we will be able to compare the value functions and, finally, determine the value of $\lambda_0$.

We first solve with $\lambda_0 = 1$. If that is the case, we have that for $\tilde{t}_1 > 0$, the equation $\lambda^* \left(\tilde{t}_1\right) h^* \left(\tilde{t}_1\right) = a$ applies, therefore

$$\tilde{t}_1 = \frac{1}{\rho} t_1^* + \left(1 - \frac{1}{\rho}\right) \hat{t} \tag{39}$$

otherwise, the solution is given by (38) with $\tilde{t}_1 = 0$. Notice that $t_1^* < \hat{t}$ implies $\tilde{t}_1 < t_1^*$. For $t \in [0, \tilde{t}_1)$, we have that $\lambda h < a$ and, accordingly, $u^*(t) = 0$, $\lambda/\lambda = \rho$ and $\dot{h}/h = 0$. Using the initial condition for $h(t)$ and the required continuity of $\lambda(t)$ at $\tilde{t}_1$:

$$\left.\begin{array}{c} u^*(t) = 0 \\ \lambda^*(t) = e^{(\rho-1)\left(\tilde{t}_1 - \hat{t}\right)} e^{\rho\left(t - \tilde{t}_1\right)} \\ h^*(t) = h_0 \end{array}\right\} \quad 0 \leq t < \tilde{t}_1 \tag{40}$$

From the continuity of $h(t)$ at $\tilde{t}_1$ we know that $h^* \left(\tilde{t}_1\right) = h_0$. This provides another equation for $\tilde{t}_1$ and $t_1^*$:

$$t_1^* = \tilde{t}_1 - \ln\left(\frac{h_0}{a}\right) \tag{41}$$

Equations (39) and (41) compose a system that determines $\tilde{t}_1$ and $t_1^*$. Solving it we obtain:

$$\begin{aligned} \tilde{t}_1 &= T + \frac{1}{\rho} \ln(1-\rho) - \frac{1}{1-\rho} \ln\left(\frac{a}{h_0}\right) \\ t_1^* &= T + \frac{1}{\rho} \ln(1-\rho) - \frac{\rho}{1-\rho} \ln\left(\frac{a}{h_0}\right) \end{aligned}$$

The solution is established and we can calculate its value function:

$$\begin{aligned} F_2 &= \int_0^{\tilde{t}_1} e^{-\rho t} a dt + e^{-\rho t_1^*} F\left(a, t_1^*, T\right) \\ F_2 &= \frac{a}{\rho}\left(1 - \left(\frac{a}{h_0}\right)^{\frac{\rho}{1-\rho}} e^{-\rho T}\right) \end{aligned}$$

$$1 < \left(\frac{a}{h_0}\right)^{\frac{\rho}{1-\rho}}$$

We, now, solve for $\lambda_0 = 0$. Observe that, since $\lambda^*(t)\, h^*(t) = ae^{(\rho-1)\left(t-\hat{t}\right)}e^{t-t^*} > 0$ for all $t$ we have $\tilde{t}_2 = 0$. Then, the solution in this case will be given by (38):

$$\left.\begin{array}{c} u^*(t) = 1 \\ \lambda^*(t) = e^{(\rho-1)\left(t-\hat{t}\right)} \\ h^*(t) = ae^{t-t_2^*} \end{array}\right\} \quad 0 \le t < t_2^*$$

With $t_2^*$ being determined by the initial condition for $h(t)$

$$h^*(0) = ae^{t-t_2^*} \implies t_2^* = \ln\left(\frac{a}{h_0}\right)$$

Calculating the value function of this solution[21]

$$\begin{aligned} \digamma_3 &= e^{-\rho t_2^*}\digamma\left(a, t_2^*, T\right) \\ \digamma_3 &= a\left(1-\rho\right)^{\frac{1-\rho}{\rho}}\left(\frac{h_0}{a}\right)^{(1-\rho)} e^{(1-\rho)T} \end{aligned}$$

First notice that the assumption $\rho \in (0,1)$ implies $\digamma_2 < \digamma_1$. Therefore, a transition can only be optimal if $\digamma_3 > \digamma_2$ which implies $\lambda_0 = 0$. Hence, in order for a transition to be desirable we must have $\digamma_3 > \digamma_1$, and we obtain the following condition:

$$a\left(1-\rho\right)^{\frac{1-\rho}{\rho}}\left(\frac{h_0}{a}\right)^{1-\rho} e^{(1-\rho)T} > \frac{a}{\rho}\left(1-e^{-\rho T}\right)$$

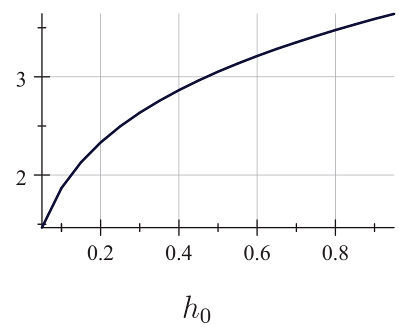
Observe that this is true if: [1] the traditional sector is not too productive (i.e. $a$ is small); [2] the level of initial human capital is high enough (i.e. $h_0$ is large); [3] the agent's time horizon is high enough (i.e. $T$ is large); and the agent does not discount too much the future (i.e. $\rho$ is small). This last affirmative is not obvious and to clarify it we provide the following table where we calculate $\digamma_3 - \digamma_1$ for $a = 1$ and $h_0 = 0.5$ (the condition $\hat{t} > t^*$ is

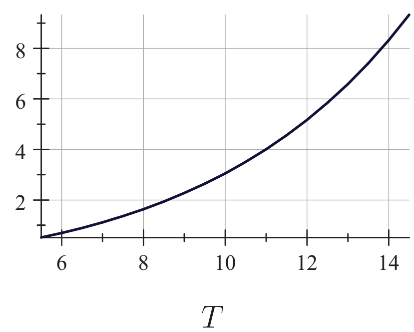
[21] Sufficiency is once again established since $H(0, h, \phi(t), t) = \phi_0 a$ and $H(u, h, \phi(t), t) = \phi(t)\, h$ are clearly concave functions of $h$.

satisfied for all of the parameters we use):

| | $\rho$=0.1 | $\rho$=0.2 | $\rho$=0.3 | $\rho$=0.4 | $\rho$=0.5 | $\rho$=0.6 | $\rho$=0.7 | $\rho$=0.8 | $\rho$=0.9 |
|---|---|---|---|---|---|---|---|---|---|
| T=2 | -0.56 | -0.48 | -0.42 | -0.36 | - | - | - | - | - |
| T=3 | 0.50 | 0.34 | 0.21 | 0.11 | 0.03 | -0.03 | -0.06 | -0.08 | - |
| T=4 | 4.30 | 3.02 | 2.07 | 1.38 | 0.88 | 0.52 | 0.27 | 0.10 | 0.00 |
| T=5 | 14.75 | 9.68 | 6.28 | 4.00 | 2.47 | 1.46 | 0.79 | 0.36 | 0.09 |
| T=6 | 41.46 | 25.09 | 15.08 | 8.95 | 5.20 | 2.91 | 1.53 | 0.69 | 0.21 |
| T=7 | 108.03 | 59.85 | 33.04 | 18.10 | 9.77 | 5.12 | 2.54 | 1.12 | 0.35 |
| T=8 | 272.58 | 137.60 | 69.39 | 34.86 | 17.34 | 8.44 | 3.92 | 1.64 | 0.50 |
| T=9 | 678.04 | 310.93 | 142.74 | 65.46 | 29.85 | 13.40 | 5.79 | 2.27 | 0.67 |
| T=10 | 1676.00 | 696.96 | 290.53 | 121.25 | 50.49 | 20.80 | 8.31 | 3.05 | 0.85 |

Notice that if $T$ is low enough and $\rho$ is high enough we might have $\mathcal{F}_1 > \mathcal{F}_3$. We also plot $\mathcal{F}_3 - \mathcal{F}_1$ taking $a = 1$, $T = 10$, $\rho = 0.8$ and $h_0 = 0.5$. For the graph of each parameter we vary only its value:




The intuition for each graph is quite obvious. For instance, the higher is initial level of human capital, $h_0$, the more it be worth to transition, thus, the greater will be the difference between the value functions $\mathcal{F}_3$ and $\mathcal{F}_1$.

# 4 Conclusions

We have solved a model that generates the usual convergence result for the countries that have a high enough level of human capital. If a country does not satisfy this prerequisite we provide conditions under which it will, at some point, enter the convergence group. This is in agreement with the fact brought up by Lucas (1988) that the convergence result does not apply to all countries; in particular that it does not apply for the poorest countries.

Though, without externality the model does not predict the pattern in Figure 1. That is why we introduce in a knowledge spillover in the human capital accumulation technology. Then the pattern in Figure 1 is generated for the countries that use the modern technology. The externality generated from the world's human capital accumulation pulls countries with low human capital into the growth stage; that is the result in Proposition 2. The level of human capital of a country that is not on the growth stage does not affect its output since the traditional technology uses only labor. Nevertheless, it will affect their transition date seeing as it will be more costly to make a transition.

We fancy this model-construction as a preliminary attempt to incorporate the dual-sector technology to endogenous growth models. The attractiveness of this would be that it could potentially provide a theory that would altogether reconcile the fact in Figure 1 and predict convergence only for a set of countries. This set being the ones that use the modern technology; the countries that can take advantage of the externality generated by the world's accumulation of human capital.

However, there is much to advance. For instance, we work with the simplifying assumption of linearity on labor in each sector's technology. This entitles a bang-bang type of solution; both sectors are never used at the same time. This is a baseless prediction and it would not apply had we assumed a more general technology function for each sector.

Also, the results on Propositions 1 and 2 are highly dependent on the strong assumptions that the traditional sector does not use any human capital and that there are no costs involved in moving labor from one sector to the other. It is hard to imagine a production sector that does not benefit from any type of knowledge. Moreover, as noted by Lucas (2009), there are significant reasons to believe that moving labor to a more industrialized sector imposes a learning cost.

Regarding the discussion in Lucas (1993) about the forces that generate a "miracle", this model provides a compelling intuition. The miracles would occur once a country transitions to the modern sector. If the world's level of human capital is high enough, during the period subsequent to the transition, then the country would experience extremely high growth rates until it converges to the leading economy's level of GDP. However, the model also predicts that during the transition period the country has a negative growth rate, a pattern that does not agree with the data.

Finally, we fully suppress physical capital from the analysis. The only variable that differentiates one country from the other is human capital. Introducing physical capital would provide another state variable that can be affected by the countries decisions. This might present new and interesting insights.

# References


[1] Bellman, Richard E., 1957. "Dynamic Programming", Princeton University Press, Princeton, New Jersey, 1957.

[2] Lucas, Robert E., Jr., 2009. "Trade and the Diffusion of the Industrial Revolution", American Economic Journal: Macroeconomics, American Economic Association, vol. 1(1), pages 1-25, January.

[3] Lucas, Robert E., Jr., 1993. "Making a Miracle", Econometrica, Econometric Society, vol. 61(2), pages 251-72, March.

[4] Lucas, Robert E., Jr., 1988. "On the mechanics of economic development", Journal of Monetary Economics, Elsevier, vol. 22(1), pages 3-42, July.

[5] Maddison, Angus, 2003. "The World Economy: Historical Statistics", Paris: OECD Development Centre.

[6] Malthus, Thomas R., 1798. "An essay on the principle of population", Chapter 1, p13 in Oxford World's Classics reprint.

[7] Parente, Stephen L., and Prescott, Edward C. 2002. "Barriers to Riches", MIT Press Books, The MIT Press, edition 1, volume 1, number 0262661306.

[8] Rodriguez, Alejandro, 2004. "Learning Externalities and Economic Growth", Mimeo - The U of C.

[9] Rodriguez, Alejandro, 2006. "Learning Externalities, Human Capital and Growth", University of Chicago doctoral dissertation.

[10] Romer, Paul M., 1990. "Endogenous Technological Change", Journal of Political Economy, University of Chicago Press, vol. 98(5), pages S71-102, October.

[11] Seierstad, Atle, and Sydsæter, Knut, 1999. "Optimal Control Theory with Economic Applications", Second reprint, North-Holland 1999.

[12] Tamura, Robert. 1991. "Income Convergence is an Endogenous Growth-Model", Journal of Political Economy, 99: 522-540.

[13] Tamura, Robert, 1996. "From decay to growth: A demographic transition to economic growth", Journal of Economic Dynamics and Control, Elsevier, vol. 20(6-7), pages 1237-1261.

[14] Uzawa, Hirofumi, 1965. "Optimum technical change in an aggregative model of economic growth", International Economic Review 6, 18-31.

# 5 Appendix

## 5.1 Appendix A - Solution of the Modern Economy Problem Without Externality and With Finite Horizon

We assume that together with Assumption 2.1 we have $T > -\rho^{-1} \ln \left(1 - \delta^{-1}\rho\right)$.

$$\max_{\{u(t)\in[0,1]\}_{t=0}^{T}} \int_0^T e^{-\rho t} \frac{\left(h\left(t\right)\left(1-u\left(t\right)\right)\right)^{1-\sigma}}{1-\sigma} dt$$

subject to

$$\dot{h}\left(t\right) = \delta h\left(t\right) u\left(t\right), \quad h\left(0\right) = h_0 > 0, \quad h\left(T\right) \text{ free} \tag{42}$$

We set the following strategy for solving the problem: We first search for *a* path $\left(u^*\left(t\right), h^*\left(t\right)\right)$ that satisfy the condition of the Maximum Principle. Having this path at hand, we conclude by checking that it satisfies the sufficiency conditions provided by the Arrow Theorem.

The Current Value Hamiltonian is

$$H\left(h, u, \lambda, t\right) = \lambda_0 \frac{\left(h\left(1-u\right)\right)^{1-\sigma}}{1-\sigma} + \lambda\delta h u$$

Note that since $h\left(T\right)$ is free we have $\lambda_0 = 1$. Assume that $\left(u^*\left(t\right), h^*\left(t\right)\right)$ is a solution to our problem, then $\lambda\left(t\right)$ satisfies

$$\dot{\lambda} = \rho\lambda - \frac{\partial H\left(h^*, u^*, \lambda, t\right)}{\partial h} = \rho\lambda - \left(h^*\right)^{-\sigma}\left(1-u^*\right)^{1-\sigma} - \lambda\delta u^*, \ \ \lambda\left(T\right) = 0 \tag{43}$$

Moreover, for each $t \in [0, T]$, $u^*\left(t\right)$ is a value of $u \in [0, 1]$ which maximizes

$$g\left(u\right) = H\left(h^*\left(t\right), u, \lambda\left(t\right), t\right) = \frac{\left(h^*\left(t\right)\left(1-u\right)\right)^{1-\sigma}}{1-\sigma} + \lambda\left(t\right)\delta h^*\left(t\right) u$$

Differentiating:

$$\begin{aligned} g'\left(u\right) &= -\left(h^*\left(t\right)\right)^{1-\sigma}\left(1-u\right)^{-\sigma} + \lambda\left(t\right)\delta h^*\left(t\right) \\ g''\left(u\right) &= -\sigma\left(h^*\left(t\right)\right)^{1-\sigma}\left(1-u\right)^{-\sigma-1} \end{aligned}$$

Observe that $g'(u) \to -\infty$ when $u \to 1$, thus $u = 1$ cannot maximize $g(u)$. Furthermore, $g(u)$ is strictly concave on $[0, 1)$. Then the following must hold:

$$u^*(t) = 0 \text{ maximizes } g(u) \Leftrightarrow g'(0) \leq 0 \Leftrightarrow \lambda(t) \leq \delta^{-1}(h^*(t))^{-\sigma} \tag{44a}$$

$$\begin{aligned} u^*(t) \in (0,1) \ \text{ maximizes } g(u) \Leftrightarrow g'(u^*(t)) = 0 \Leftrightarrow \\ \Leftrightarrow \lambda(t) = \delta^{-1}(h^*(t)(1 - u^*(t)))^{-\sigma} \end{aligned} \tag{44b}$$

Now, in order to determine the time interval in which $u^*(t) = 0$, let us define the function $\varphi(s) = \lambda(s) - \delta^{-1}(h^*(s))^{-\sigma}$ on $[0, T]$. From (43) we know that $\lambda(T) = 0$ which implies that $\varphi(T) < 0$. Let $\hat{t}$ be the smallest number $t$ such that $\varphi(s) < 0$ for all $s \in (t, T]$. Then if $\hat{t} > 0$ it is determined by the equation $\varphi\left(\hat{t}\right) = 0 \Leftrightarrow \lambda\left(\hat{t}\right) = \delta^{-1}\left(h^*\left(\hat{t}\right)\right)^{-\sigma}$ otherwise $\hat{t} = 0$. Thus, from (44$a$) we can see that there is a final interval $(\hat{t}, T]$ in which $u^*(t) = 0$.

Therefore, (42) and (43) imply that for $t \in (\hat{t}, T]$:

$$h^*(t) = h_T \tag{45a}$$

$$\lambda(t) = \rho^{-1}(h_T)^{-\sigma}\left(1 - e^{-\rho(T-t)}\right) \tag{45b}$$

with the constant $h_T$ yet to be determined. In particular,

$$\lambda\left(\hat{t}\right) = \rho^{-1}(h_T)^{-\sigma}\left(1 - e^{-\rho\left(T-\hat{t}\right)}\right)$$

and since we also know that $\lambda\left(\hat{t}\right) = \delta^{-1}(h_T)^{-\sigma}$, we can determine $\hat{t}$:

$$\hat{t} = T + \rho^{-1}\ln\left(1 - \delta^{-1}\rho\right) \tag{46}$$

Hence, since we have assumed $T > -\rho^{-1}\ln\left(1 - \delta^{-1}\rho\right)$, $\hat{t}$ is $> 0$ and there will be an initial interval $\left[0, \hat{t}\right]$ in which $u^*(t) \in (0, 1)$. Then, from (44$b$) we know that $\lambda(t) = \delta^{-1}(h^*(t)(1 - u^*(t)))^{-\sigma}$ in $\left[0, \hat{t}\right]$. Plugging this expression for $\lambda(t)$ into (43) we obtain $\dot{\lambda}(t) = (\rho - \delta)\lambda(t)$. Using the required continuity of $\lambda(t)$ at $\hat{t}$ we arrive at the following solution for this differential equation[22]:

$$\lambda(t) = \delta^{-1}(h_T)^{-\sigma} e^{(\rho-\delta)\left(t-\hat{t}\right)} \tag{47}$$

[22]Notice from (45$b$) and (47) that $\dot{\lambda} < 0$ for all $t \in [0, T]$. This, along with (44$a$) and (44$b$), justifies our search for a path that has $u \in (0, 1)$ in an initial interval and $u = 0$ in a final interval.

Moreover

$$\delta^{-1}\left(h^*(t)\left(1-u^*(t)\right)\right)^{-\sigma}=\delta^{-1}\left(h_T\right)^{-\sigma}e^{(\rho-\delta)\left(t-\hat{t}\right)} \tag{48}$$

By (42), $\dot{h}^*(t)=\delta h^*(t)\,u^*(t)$, substituting we get

$$\dot{h}^*(t)=\delta h^*(t)-\delta h_T e^{\frac{(\delta-\rho)}{\sigma}\left(t-\hat{t}\right)}$$

The solution to this differential equation, using the fact that $h^*\left(\hat{t}\right)=h_T$ is

$$h^*(t)=\frac{\rho-\delta\left(1-\sigma e^{-\frac{(\rho-\delta(1-\sigma))}{\sigma}\left(t-\hat{t}\right)}\right)}{\rho-\delta\left(1-\sigma\right)}h_T e^{\delta\left(t-\hat{t}\right)} \tag{49}$$

Using the initial condition $h^*(0)=h_0$, we determine the constant $h_T$:

$$h_T=\frac{\rho-\delta\left(1-\sigma\right)}{\rho-\delta\left(1-\sigma e^{\frac{(\rho-\delta(1-\sigma))}{\sigma}\hat{t}}\right)}h_0 e^{\delta\hat{t}} \tag{50}$$

Furthermore, using (48) we can find the following proposal for the optimal control for $t\in\left[0,\hat{t}\right]$:

$$u^*(t)=1-\left(h_T\right)\left(h^*(t)\right)^{-1}e^{\frac{\delta-\rho}{\sigma}\left(t-\hat{t}\right)}$$

Substituting for $h^*(t)$, using (49) we obtain

$$u^*(t)=1-\frac{\rho-\delta\left(1-\sigma\right)}{\rho-\delta\left(1-\sigma e^{-\frac{(\rho-\delta(1-\sigma))}{\sigma}\left(t-\hat{t}\right)}\right)}e^{-\frac{(\rho-\delta(1-\sigma))}{\sigma}\left(t-\hat{t}\right)} \tag{51}$$

At last we have our candidate for the optimal path. We propose the following control

$$u^*(t)=\begin{cases} u^*(t)\ \text{given in }(51) & t\in\left[0,\hat{t}\right] \\ 0 & t\in\left(\hat{t},T\right] \end{cases} \tag{52a}$$

the associated path for the state variable

$$h^*(t)=\begin{cases} h^*(t)\ \text{given in }(49) & t\in\left[0,\hat{t}\right] \\ h_T\ \text{given in }(50) & t\in\left(\hat{t},T\right] \end{cases} \tag{52b}$$

and for the costate variable

$$\lambda(t) = \begin{cases} \delta^{-1}(h_T)^{-\sigma} e^{(\rho-\delta)(t-\hat{t})} & t \in [0, \hat{t}] \\ \rho^{-1}(h_T)^{-\sigma}\left(1 - e^{-\rho(T-t)}\right) & t \in (\hat{t}, T] \end{cases} \tag{52c}$$

where $\hat{t}$ is given in (46) and $h_T$ in (50).

We are now in possession of a path that satisfies all the conditions in the Maximum Principle (demonstrating that this is true is straightforward and we leave it for the reader). Then we will carry on to the next undertaking which is to prove that this condition satisfies the sufficiency conditions in the Arrow Theorem[23].

We define for all $h \in [0, \infty)$,

$$\begin{aligned} \hat{H}(h, \lambda(t), t) &= \max_{u\in[0,1]} H(h, u, \lambda(t), t) \\ &= \max_{u\in[0,1]} \left( \frac{(h(1-u))^{1-\sigma}}{1-\sigma} + \lambda(t)\,\delta h u \right) \end{aligned}$$

Our aim is to prove that for all $t \in [0, T]$, $\hat{H}(h, \lambda(t), t)$ is a concave function for $h \in [0, \infty)$. Note that $\lambda(t)$ is given by $(52c)$.

Consider first those $t$ for which $\lambda(t) \le \delta^{-1}h^{-\sigma}$, that is, $h \le (\delta\lambda(t))^{-\frac{1}{\sigma}}$. In this case, we know from $(44a)$ that $u = 0$ maximizes $H$ for $u \in [0, 1]$, then

$$\hat{H}(h, \lambda(t), t) = \frac{h^{1-\sigma}}{1-\sigma} \qquad \text{for } h \le (\delta\lambda(t))^{-\frac{1}{\sigma}}$$

which is, obviously, a concave function of $h$.

Consider next those $t$ for which $\lambda(t) > \delta^{-1}h^{-\sigma}$, that is, $h > (\delta\lambda(t))^{-\frac{1}{\sigma}}$. In this case, we know from $(44b)$ that $H$ is maximized by some $u \in (0, 1)$ that satisfies $\lambda(t) = \delta^{-1}(h(1-u))^{-\sigma}$, i.e., $hu = h - (\delta\lambda(t))^{-\frac{1}{\sigma}}$ then

$$\hat{H}(h, \lambda(t), t) = \frac{\sigma}{1-\sigma}(\delta\lambda(t))^{\frac{\sigma-1}{\sigma}} + \delta\lambda(t)\,h \qquad \text{for } h > (\delta\lambda(t))^{-\frac{1}{\sigma}}$$

which is once again a clearly concave function of $h$.

All that that is left to establish the concavity of $\hat{H}$ for all $h \in [0, \infty)$ is to analyze how the function behaves at $h = (\delta\lambda(t))^{-\frac{1}{\sigma}}$. Observe that

[23] More specifically, we will use the Theorem 2.5 in Seierstad and Sydsaeter (1999), page 107, with the adjacent note 11.

$\hat{H}$ is continuous at $h = (\delta\lambda(t))^{-\frac{1}{\sigma}}$ taking the value of $(1-\sigma)^{-1}(\delta\lambda(t))^{\frac{\sigma-1}{\sigma}}$. Furthermore, the left-hand derivative of $\hat{H}$ at $h = (\delta\lambda(t))^{-\frac{1}{\sigma}}$ is $\delta\lambda(t)$ and it is equal to the right-hand derivative. Hence, $\hat{H}$ is differentiable at $h = (\delta\lambda(t))^{-\frac{1}{\sigma}}$. Therefore, we have demonstrated for all $t \in [0,T]$ that $\hat{H}$ is a concave function of $h$ for all $h \in [0,\infty)$.

Accordingly, the conditions in the Arrow Theorem are satisfied and we have verified that the path $(u^*(t), h^*(t))$ given by $(52a)$ and $(52b)$ is, in fact, optimal.

## 5.2 Appendix B - Proof of Proposition 1

**Proposition 1** *If Assumptions 1.1→1.4 are satisfied then the solution* $(u^*(t), h^*(t))$ *to Problem 1 with* $h_0 < a$ *will always involve capital accumulation and there exists a finite* $t^*$ *such that* $h^*(t^*) = a$.

**Proof.** Let $u^*(t)$ be the solution to Problem 1, and $h^*(t)$ the associated path for human capital. First notice that we will have $u^*(t) > 0$ for some $t$ only if $h^*(t) > a$ for some $t$. Given that if $u^*(t) > 0$ for some $t$ the country will spend some time accumulating capital instead of working and this can only be optimal if the accumulated human capital will be used to increase utility at some point[24]. Since human capital is an input only in sector 2, this means that in order to have $u^*(t) > 0$ for some $t$ it is necessary that $h^*(t) > a$ for some $t$.

This implies that if $h^*(t) \leq a$ for all $t$ then $u^*(t) = 0$ for all $t$, i.e. if there is not going to a transition to the modern sector then it is optimal never to accumulate human capital whatsoever.

Hence, if we can prove that $u(t) = 0$ for all $t$ is not the optimal path we

[24]Observe that any path $\hat{u}(t)$ in which the country accumulates human capital, choosing

$\begin{cases} \hat{u}(t) > 0 & , \; t \in \Omega \\ \hat{u}(t) = 0 & , \; t \in \mathbb{R}_+/\Omega \end{cases}$ where $\Omega \subseteq \mathbb{R}_+$ is a set with a nonzero measure, but in which $h(t)$ never reaches $a$, that is $\int_0^\infty \delta h^{1-\theta}\eta^\theta \hat{u}(t)\,dt < a - h_0$, cannot be better than the path with no human capital accumulation, since:

$$\int_\Omega e^{-\rho t}\frac{(a(1-\hat{u}(t)))^{1-\sigma}}{1-\sigma}dt + \int_{\mathbb{R}_+/\Omega} e^{-\rho t}\frac{a^{1-\sigma}}{1-\sigma}dt <$$
$$< \int_\Omega e^{-\rho t}\frac{a^{1-\sigma}}{1-\sigma}dt + \int_{\mathbb{R}_+/\Omega} e^{-\rho t}\frac{a^{1-\sigma}}{1-\sigma}dt = \int_{\mathbb{R}_+} e^{-\rho t}\frac{a^{1-\sigma}}{1-\sigma}dt$$

will have established that $h^*(t) > a$ for some $t$. Then, the continuity of $h(t)$ establishes the result.

First let us calculate the value function of the best path amongst those that do not involve capital accumulation, $W_1$:

$$W_1 = \int_0^\infty e^{-\rho t} \frac{a^{1-\sigma}}{1-\sigma} dt$$
$$W_1 = \frac{1}{\rho} \frac{a^{1-\sigma}}{1-\sigma}$$

We search for an arbitrary path that is better than this and that has a simple value function. We set the following path $\left(\tilde{u}(t), \tilde{h}(t)\right)$: [1] during an initial period $\left[0, \tilde{t}\right]$ the country chooses $\tilde{u}(t) = 0$ and only waits until the leader reaches a human capital level of $z \geq \eta_0$; [2] then there will be a period $(\tilde{t}, \hat{t}]$ during which it will choose $\tilde{u}(t) = \varepsilon \in (0,1)$ until its own human capital reaches $2a$ (we also choose $\tilde{l}_2(t) = 0$ for $t \in (\tilde{t}, \hat{t}]$); [3] after this, i.e. for $t \in \left(\hat{t}, \infty\right)$, he chooses $u(t) = 0$. That is:

$$u(t) = 0,\ t \in [0, \tilde{t})$$
$$u(t) = \varepsilon,\ t \in [\tilde{t}, \hat{t})$$
$$u(t) = 0,\ t \in [\hat{t}, \infty)$$

Therefore, $\tilde{t}$ and $\hat{t}$ are such that $\eta\left(\tilde{t}\right) = z$ and $h\left(\hat{t}\right) = 2a$. We first determine $\tilde{t}$:

$$\dot{\eta} = \gamma\eta,\ \eta(0) = \eta_0,\ \eta\left(\tilde{t}\right) = z$$
$$\eta(0) = \eta_0 \implies \eta = \eta_0 e^{\gamma t}$$
$$\eta\left(\tilde{t}\right) = z \implies \tilde{t} = \frac{1}{\gamma} \ln\left(\frac{z}{\eta_0}\right)$$

Next, we can determine $\hat{t}$:

$$\dot{h} = \delta\eta_0 e^{\gamma\theta t} h^{1-\theta}\varepsilon,\ h\left(\tilde{t}\right) = h_0,\ h\left(\hat{t}\right) = 2a$$
$$h^\theta = \frac{\delta\eta_0\varepsilon}{\gamma} e^{\gamma\theta t} + C$$
$$h\left(\tilde{t}\right) = h_0 \implies h^\theta = h_0^\theta + \frac{\delta\eta_0\varepsilon}{\gamma}\left[e^{\gamma\theta t} - \left(\frac{z}{\eta_0}a\right)^\theta\right]$$
$$h\left(\hat{t}\right) = 2a \implies \hat{t} = \frac{1}{\gamma\theta} \ln\left(\left(\frac{z}{\eta_0}\right)^\theta + \frac{\gamma}{\delta\eta_0\varepsilon}\left[(2a)^\theta - h_0^\theta\right]\right)$$

Hence, we can calculate the value function of this arbitrary path, $W_2$:

$$
\begin{aligned}
W_2 &= \int_0^{\tilde{t}} e^{-\rho t} \frac{a^{1-\sigma}}{1-\sigma} dt + \int_{\tilde{t}}^{\hat{t}} e^{-\rho t} \frac{((1-\varepsilon)\, a)^{1-\sigma}}{1-\sigma} dt + \int_{\hat{t}}^{\infty} e^{-\rho t} \frac{(a)^{1-\sigma}}{1-\sigma} dt \\
\frac{W_2}{W_1} &= 1 + \omega_1(z) \left[ (1-\varepsilon)^{1-\sigma} - 1 \right] + \omega_2(z) \left[ (2)^{1-\sigma} - (1-\varepsilon)^{1-\sigma} \right]
\end{aligned}
$$

where $\omega_1(z)$ and $\omega_2(z)$ are defined as follows:

$$
\begin{aligned}
\omega_1(z) &:= \left( \frac{z}{\eta_0} \right)^{-\frac{\rho}{\gamma}} \\
\omega_2(z) &:= \left( \left( \frac{z}{\eta_0} \right)^{\theta} + \frac{\gamma}{\delta \eta_0 \varepsilon} \left[ (2a)^{\theta} - h_0^{\theta} \right] \right)^{-\frac{\rho}{\gamma\theta}}
\end{aligned}
$$

We want to prove that there is a $z \geq \eta_0$ such that $W_2 > W_1$. Let us define

$$
\Pi(z) := \omega_1(z) \left[ (1-\varepsilon)^{1-\sigma} - 1 \right] + \omega_2(z) \left[ (2)^{1-\sigma} - (1-\varepsilon)^{1-\sigma} \right]
$$

We need to separate the proof for different values of $\sigma$:

∘ For $\sigma > 1$, since $W_1$ and $W_2$ are negative numbers, we need to show that $\Pi(z) < 0$. Note that $\left[ (1-\varepsilon)^{1-\sigma} - 1 \right] > 0$ and $\left[ (2)^{1-\sigma} - (1-\varepsilon)^{1-\sigma} \right] < 0$. Then, we have to choose $z \geq \eta_0$ such that $\Pi(z) < 0$, with some algebra we arrive at the following restriction for $z$:

$$
z > \eta_0 \left( \frac{\frac{\gamma}{\delta \eta_0 \varepsilon} \left( (2a)^{\theta} - h_0^{\theta} \right)}{\left( \frac{(2)^{1-\sigma} - (1-\varepsilon)^{1-\sigma}}{1-(1-\varepsilon)^{1-\sigma}} \right)^{\frac{\gamma\theta}{\rho}} - 1} \right)^{\frac{1}{\theta}} := \xi
$$

Therefore, taking $z = \eta_0 + \xi$ we establish the result.

∘ For $\sigma \in (0,1)$, $W_1$ and $W_2$ are positive numbers and we need to show that $\Pi(z) > 0$. Note that $\left[ (1-\varepsilon)^{1-\sigma} - 1 \right] < 0$ and $\left[ (2)^{1-\sigma} - (1-\varepsilon)^{1-\sigma} \right] > 0$. Then, we have to choose $z \geq \eta_0$ such that $\Pi(z) > 0$, with some algebra we arrive at the same restriction for $z$:

$$
z > \xi
$$

Thus, taking $z = \eta_0 + \xi$ we establish the result once again.

∘ For $\sigma = 1$ the result follows by continuity. Using L'Hôpital's rule we can determine the restriction on $z$ taking the limit of $\xi$ as $\sigma \to 1$:

$$z > \lim_{\sigma \to 1} \xi = \eta_0 \left( \frac{\frac{\gamma}{\delta \eta_0 \varepsilon} \left( (2a)^{\theta} - h_0^{\theta} \right)}{\left(1 - \frac{\ln(2)}{\ln(1-\varepsilon)}\right)^{\frac{\gamma\theta}{\rho}} - 1} \right)^{\frac{1}{\theta}} := \xi_2$$

So, choosing $z = \eta_0 + \xi_2$ we again achieve the result.

Observe that if Assumptions 1.1→1.4 are valid then $\xi$ and $\xi_2$ are finite numbers. However, if $\theta \to 0$, $\xi$ and $\xi_2$ tend to infinity. Thus, the presence of externality is crucial to establish the result. ■

## 5.3 Appendix C - Proof of Proposition 2

**Proposition 2** *If Assumptions 2.1→2.3 are satisfied then there exists an exogenous and unique $h^{\#} < a$ such that if $h_0 \in \left(h^{\#}, a\right)$ the solution $(u^*(t), h^*(t))$ to Problem 2 with $h_0 < a$ involves capital accumulation and there exists a finite $t^*$ such that $h^*(t^*) = a$.*

**Proof.** Let $u^*(t)$ be the solution to Problem 2, and $h^*(t)$ the associated path for human capital. Following the same reasoning as Proposition 1 we get that if $h^*(t) \leq a$ for all $t$ then $u^*(t) = 0$ for all $t$, i.e. if there is not going to a transition to the modern sector then it is optimal never to accumulate human capital whatsoever[25].

Hence, if we can prove that $u(t) = 0$ for all $t$ is not the optimal path for $h_0$ large enough (that is $h_0 \in \left(h^{\#}, a\right)$ for some $h^{\#} < a$) we will have established that $h^*(t) > a$ for some $t$ if $h_0$ is large enough. Then, the continuity of $h(t)$ establishes the result.

[25] Observe that any path $\tilde{u}(t)$ in which the country accumulates human capital, choosing $\begin{cases} \tilde{u}(t) > 0 & , \ t \in \Omega \\ \tilde{u}(t) = 0 & , \ t \in \mathbb{R}_+/\Omega \end{cases}$ where $\Omega \subseteq \mathbb{R}_+$ is a set with a nonzero measure, but in which $h(t)$ never reaches $a$, that is $\int_0^{\infty} \delta h(t) \tilde{u}(t) dt < a - h_0$, cannot be better than the path with no human capital accumulation, since:

$$\int_{\Omega} e^{-\rho t} \frac{\left(a\left(1 - \tilde{u}(t)\right)\right)^{1-\sigma}}{1-\sigma} dt + \int_{\mathbb{R}_+/\Omega} e^{-\rho t} \frac{a^{1-\sigma}}{1-\sigma} dt <$$

$$< \int_{\Omega} e^{-\rho t} \frac{a^{1-\sigma}}{1-\sigma} dt + \int_{\mathbb{R}_+/\Omega} e^{-\rho t} \frac{a^{1-\sigma}}{1-\sigma} dt = \int_{\mathbb{R}_+} e^{-\rho t} \frac{a^{1-\sigma}}{1-\sigma} dt$$

With that in mind, we first calculate the value function of the path with no human capital accumulation, $V_1$:

$$V_1 = \int_0^{\infty} e^{-\rho t} \frac{a^{1-\sigma}}{1-\sigma} dt$$
$$V_1 = \frac{1}{\rho} \frac{a^{1-\sigma}}{1-\sigma}$$

Now, we search for an arbitrary path that is better than the one with no human capital accumulation and one that we can easily calculate the value function. For an $\bar{u} \in (0,1)$ we set $u^{\#}(t) = \bar{u}$ for all $t \geq 0$. In order to calculate the value function, we first determine the path for $h^{\#}(t)$ related to this path for $u^{\#}(t)$ and the time $\hat{t}$ in which $h^{\#}(t) = a$. Using the law of motion for the human capital:

$$\dot{h}^{\#}(t) = \delta h^{\#}(t) \bar{u}$$
$$h^{\#}(0) = h_0 \implies h^{\#}(t) = h_0 e^{\delta \bar{u} t}$$
$$h^{\#}\left(\hat{t}\right) = a \implies \hat{t} = \frac{1}{\delta \bar{u}} \ln\left(\frac{a}{h_0}\right)$$

We can, now, calculate the value function of this second path, $V_3$:

$$V_3 = \int_0^{\hat{t}} e^{-\rho t} \frac{\left(a\left(1-\bar{u}\right)\right)^{1-\sigma}}{1-\sigma} dt + \int_{\hat{t}}^{\infty} e^{-\rho t} \frac{\left(h\left(1-\bar{u}\right)\right)^{1-\sigma}}{1-\sigma} dt$$
$$V_3 = \frac{\left(a\left(1-\bar{u}\right)\right)^{1-\sigma}}{\left(1-\sigma\right)\rho}\left(1-e^{-\rho\hat{t}}\right) + \frac{\left(1-\bar{u}\right)^{1-\sigma} h_0^{1-\sigma}}{\left(1-\sigma\right)\left(\rho-\left(1-\sigma\right)\delta\bar{u}\right)} e^{((1-\sigma)\delta\bar{u}-\rho)\hat{t}}$$
$$V_3 = \frac{\left(a\left(1-\bar{u}\right)\right)^{1-\sigma}}{\left(1-\sigma\right)\rho}\left(1-\left(\frac{a}{h_0}\right)^{-\frac{\rho}{\delta\bar{u}}}\right) + \frac{\left(1-\bar{u}\right)^{1-\sigma} h_0^{1-\sigma}}{\left(1-\sigma\right)\left(\rho-\left(1-\sigma\right)\delta\bar{u}\right)} \left(\frac{a}{h_0}\right)^{\frac{((1-\sigma)\delta\bar{u}-\rho)}{\delta\bar{u}}}$$
$$V_3 = \frac{1}{\rho}\frac{a^{1-\sigma}}{1-\sigma}\left[\left(1-\bar{u}\right)^{1-\sigma}\left(1+\left(\frac{a}{h_0}\right)^{-\frac{\rho}{\delta\bar{u}}}\left(\frac{\left(1-\sigma\right)\delta\bar{u}}{\rho-\left(1-\sigma\right)\delta\bar{u}}\right)\right)\right]$$
$$\frac{V_3}{V_1} = \left(1-\bar{u}\right)^{1-\sigma}\left(1+\left(\frac{a}{h_0}\right)^{-\frac{\rho}{\delta\bar{u}}}\left(\frac{\left(1-\sigma\right)\delta\bar{u}}{\rho-\left(1-\sigma\right)\delta\bar{u}}\right)\right)$$

If we take $\bar{u}$ to be the balanced growth path solution to the problem of an economy with $h_0 > a$, that is[26]:

$$\bar{u} = \frac{\delta - \rho}{\delta\sigma}$$

[26] Assumption 1.3 implies $\bar{u} > 0$.

We get:

$$\frac{V_3}{V_1} = \left(\frac{\rho-(1-\sigma)\,\delta}{\delta\sigma}\right)^{1-\sigma}\left(1+\left(\frac{a}{h_0}\right)^{-\frac{\sigma\rho}{(\delta-\rho)}}\left(\frac{(1-\sigma)\,(\delta-\rho)}{\rho-(1-\sigma)\,\delta}\right)\right)$$

Let we define $\Pi\left(\frac{a}{h_0}\right) := \frac{V_3}{V_1}$. We want to show that $V_3 > V_1$, then, for $\sigma \in (0,1)$ we need $\Pi\left(\frac{a}{h_0}\right) > 1$ and for $\sigma > 1$ we need $\Pi\left(\frac{a}{h_0}\right) < 1$. First notice that:

$$\Pi'\left(\frac{a}{h_0}\right) = (\sigma-1)\left(\frac{\rho-(\sigma-1)\,\delta}{\delta\sigma}\right)^{1-\sigma}\left(\frac{\sigma\rho}{\rho-(1-\sigma)\,\delta}\right)\left(\frac{a}{h_0}\right)^{-\frac{\sigma\rho}{(\delta-\rho)}-1}$$

By inspection (remember that Assumption 2.2 implies $(1-\sigma)\,\delta-\rho<0$) it is easy to see that $\Pi'\left(\frac{a}{h_0}\right) < 0$ if $\sigma \in (0,1)$ and $\Pi'\left(\frac{a}{h_0}\right) > 0$ if $\sigma > 1$. Let $h^{\#}$ be such that $\Pi\left(\frac{a}{h^{\#}}\right) = 1$. Solving this equation for $h^{\#}$ we get the following:

$$h^{\#} = a\left(\left(\frac{\rho-(1-\sigma)\,\delta}{(1-\sigma)\,(\delta-\rho)}\right)\left(\left(\frac{\rho-(1-\sigma)\,\delta}{\delta\sigma}\right)^{\sigma-1}-1\right)\right)^{\frac{\delta-\rho}{\sigma\rho}}$$

Consequently, for $h_0 \in \left(h^{\#}, a\right)$ we have $V_3 > V_1$.

In theory we could have $h^{\#} \geq a$ and in this case the interval would be degenerate and there would be no $h_0$ that would make the transition to sector 2 desirable. Though, we can we can show that we will always have $h^{\#} < a$. First, notice that $h^{\#} < a$ is equivalent to:

$$\left(\left(\frac{\rho-(1-\sigma)\,\delta}{(1-\sigma)\,(\delta-\rho)}\right)\left(\left(\frac{\rho-(1-\sigma)\,\delta}{\delta\sigma}\right)^{\sigma-1}-1\right)\right)^{\frac{\delta-\rho}{\sigma\rho}} < 1$$

Observe that this is equivalent to:

$$\left(\frac{\rho-(1-\sigma)\,\delta}{(1-\sigma)}\right)\left(\left(\frac{\rho-(1-\sigma)\,\delta}{\delta\sigma}\right)^{\sigma-1}-1\right) < (\delta-\rho) \tag{53}$$

Now, we have to separate in three cases. If $\sigma \in (0,1)$, then the inequality (53) is equivalent to:

$$\begin{gathered}
(\rho-(1-\sigma)\delta)\left(\left(\frac{\rho-(1-\sigma)\delta}{\delta\sigma}\right)^{\sigma-1}-1\right)<(1-\sigma)(\delta-\rho) \\
(\rho-(1-\sigma)\delta)^{\sigma}(\delta\sigma)^{1-\sigma}-(\rho-(1-\sigma)\delta)<(1-\sigma)(\delta-\rho) \\
\frac{(\delta\sigma+\rho-\delta)^{\sigma}}{(\delta\sigma)^{\sigma}}<\frac{\rho}{\delta}
\end{gathered}$$

Since $\sigma \in (0,1)$ the function $f(x) = x^{\sigma}$ is concave, then:

$$\begin{aligned}
f(x+k) &< f(x)+f'(x)k \\
(x+k)^{\sigma} &< (x)^{\sigma}+\sigma(x)^{\sigma-1}k
\end{aligned}$$

Taking $x = \delta\sigma$ and $k = \rho - \delta$ we have:

$$\begin{gathered}
(\delta\sigma+\rho-\delta)^{\sigma}<(\delta\sigma)^{\sigma}+\sigma(\delta\sigma)^{\sigma-1}(\rho-\delta) \\
\frac{(\delta\sigma+\rho-\delta)^{\sigma}}{(\delta\sigma)^{\sigma}}<\frac{\rho}{\delta}
\end{gathered}$$

For $\sigma > 1$ the logic is the same. In this case the inequality (53) is equivalent to:

$$\begin{gathered}
(\rho-(1-\sigma)\delta)\left(\left(\frac{\rho-(1-\sigma)\delta}{\delta\sigma}\right)^{\sigma-1}-1\right)>(1-\sigma)(\delta-\rho) \\
(\rho-(1-\sigma)\delta)^{\sigma}(\delta\sigma)^{1-\sigma}-(\rho-(1-\sigma)\delta)>(1-\sigma)(\delta-\rho) \\
\frac{(\delta\sigma+\rho-\delta)^{\sigma}}{(\delta\sigma)^{\sigma}}>\frac{\rho}{\delta}
\end{gathered}$$

For $\sigma > 1$ the function $f(x) = x^{\sigma}$ is convex, then:

$$\begin{aligned}
f(x+k) &> f(x)+f'(x)k \\
(x+k)^{\sigma} &> (x)^{\sigma}+\sigma(x)^{\sigma-1}k
\end{aligned}$$

Again, taking $x = \delta\sigma$ and $k = \rho - \delta$ we have:

$$\begin{gathered}
(\delta\sigma+\rho-\delta)^{\sigma}>(\delta\sigma)^{\sigma}+\sigma(\delta\sigma)^{\sigma-1}(\rho-\delta) \\
\frac{(\delta\sigma+\rho-\delta)^{\sigma}}{(\delta\sigma)^{\sigma}}>\frac{\rho}{\delta}
\end{gathered}$$

For $\sigma = 1$ the result follows by continuity. We first rearrange the inequality (53):

$$\frac{(\rho - (1-\sigma)\delta)^{\sigma}(\delta\sigma)^{1-\sigma} - (\rho - (1-\sigma)\delta)}{(1-\sigma)} < (\delta - \rho)$$

Notice that only the left side is dependent on $\sigma$. Taking the limit for $\sigma \to 1$, using L'Hopital, we have that:

$$\lim_{\sigma \to 1} \frac{(\rho - (1-\sigma)\delta)^{\sigma}(\delta\sigma)^{1-\sigma} - (\rho - (1-\sigma)\delta)}{(1-\sigma)} = 0$$

Since $\rho < \delta$ by assumption, the inequality is once again suited. This establishes the result. ■

## 5.4 Appendix D - Sufficiency

We first establish sufficiency for our solution of the Modern Economy Problem with externality:

Define for all $x \in [0, \infty)$,

$$\begin{aligned} \hat{H}(x, \phi(t), t) &= \max_{u \in [0,1]} H(u, x, \phi(t), t) \\ &= \max_{u \in [0,1]} \left( \eta_0 \frac{(x(1-u))^{1-\sigma}}{1-\sigma} + \phi(t)\left(\delta x^{1-\theta} u - \gamma x\right) \right) \end{aligned}$$

Our aim is to prove that for all $t \in [0, T]$, $\hat{H}(x, \phi(t), t)$ is a concave function for $x \geq 0$.

First observe that if we define $g(u) = H(u, x, \phi(t), t)$ we have:

$$\begin{aligned} g(u) &= \eta_0 \frac{(x(1-u))^{1-\sigma}}{1-\sigma} + \phi(t)\left(\delta x^{1-\theta} u - \gamma x\right) \\ g'(u) &= -\eta_0 x^{1-\sigma}(1-u)^{-\sigma} + \phi(t)\delta x^{1-\theta} \\ g''(u) &= -\sigma\eta_0 x^{1-\sigma}(1-u)^{-\sigma} \end{aligned}$$

Therefore, $g(u)$ is strictly concave and since $g'(u) \to -\infty$ as $u \to 1$, $u = 1$ cannot be a solution at any $t$. Then, we have:

$$\begin{aligned} &u^*(t) = 0 \text{ maximizes } g(u) \Leftrightarrow g'(0) \leq 0 \Leftrightarrow \phi(t) \leq \delta^{-1}\eta_0 x^{\theta-\sigma} \\ &u^*(t) \in (0,1) \text{ maximizes } g(u) \Leftrightarrow g'(u^*(t)) = 0 \Leftrightarrow \\ &\qquad \Leftrightarrow \phi(t) = \delta^{-1}\eta_0 x^{\theta-\sigma}(1-u)^{-\sigma} \end{aligned}$$

For $t$ such that $\phi(t) \leq \delta^{-1}\eta_0 x^{\theta-\sigma}$:

$$\begin{aligned}
\hat{H} &= \eta_0 \frac{x^{1-\sigma}}{1-\sigma} - \phi(t)\gamma x \\
\hat{H}_x &= \eta_0 x^{-\sigma} - \phi(t)\gamma \\
\hat{H}_{xx} &= -\sigma\eta_0 x^{-\sigma}
\end{aligned}$$

So $\hat{H}$ is concave on $x$. While for $t$ such that $\phi(t) > \delta^{-1}\eta_0 x^{\theta-\sigma}$, we have $\phi(t) = \delta^{-1}\eta_0 x^{\theta-\sigma}(1-u)^{-\sigma}$ then:

$$\begin{aligned}
\hat{H} &= \frac{\sigma}{1-\sigma}\eta_0^{\frac{1}{\sigma}}(\delta\phi(t))^{\frac{\sigma-1}{\sigma}} x^{-\frac{\theta}{\sigma}(\sigma-1)} + \phi(t)\delta x^{1-\theta} - \phi(t)\gamma x \\
\hat{H}_x &= \theta\eta_0^{\frac{1}{\sigma}}(\delta\phi(t))^{\frac{\sigma-1}{\sigma}} x^{-\frac{(\sigma-\theta+\theta\sigma)}{\sigma}} + (1-\theta)\phi(t)\delta x^{-\theta} - \phi(t)\gamma \\
\hat{H}_{xx} &= \theta\left(\frac{\theta}{\sigma} - (1+\theta)\right)\eta_0^{\frac{1}{\sigma}}(\delta\phi(t))^{\frac{\sigma-1}{\sigma}} x^{-\frac{(\sigma-\theta+\theta\sigma)}{\sigma}} - \\
&\quad -\theta(1-\theta)\phi(t)\delta x^{-(1+\theta)}
\end{aligned}$$

Hence since from Assumption 1.5 we have $\sigma > \frac{\theta}{1+\theta}$ it is proven that $\hat{H}$ is concave on $x$. Then $\hat{H}$ is concave for all $x \geq 0$ and the solution satisfies all the conditions in the Arrow Theorem. Thus we have demonstrated sufficiency.

Next we establish sufficiency for our solution of the Traditional Economy Problem with externality:

Define for all $h \in [0, \infty)$,

$$\begin{aligned}
\hat{H}(h, \lambda(t), t) &= \max_{u\in[0,1]} H(u, h, \lambda(t), t) \\
&= \max_{u\in[0,1]} \left( e^{-\rho t}\frac{(a(1-u))^{1-\sigma}}{1-\sigma} + \lambda(t)\left(\delta\eta_0^{\theta}e^{\gamma\theta t}h^{1-\theta}u\right)\right)
\end{aligned}$$

Again, we intend to prove that for all $t \in [0, T]$, $\hat{H}(x, \phi(t), t)$ is a concave function for the state variable $h \geq 0$.

We start by recalling equations $(13a)$ and $(13b)$:

$$\begin{aligned}
&u^*(t) = 0 \text{ maximizes } g(u) \Leftrightarrow g'(0) \leq 0 \Leftrightarrow \lambda(t) \leq \delta^{-1}\eta_0^{-\theta}h^{\theta-1}e^{-(\rho+\gamma\theta)t}a^{1-\sigma} \\
&u^*(t) \in (0,1) \text{ maximizes } g(u) \Leftrightarrow g'(u^*(t)) = 0 \Leftrightarrow \\
&\qquad\qquad \Leftrightarrow \lambda(t) = \delta^{-1}\eta_0^{-\theta}h^{\theta-1}e^{-(\rho+\gamma\theta)t}a^{1-\sigma}(1-u^*(t))^{-\sigma}
\end{aligned}$$

For $t$ such that $\lambda(t) \leq \delta^{-1}\eta_0^{-\theta}h^{\theta-1}e^{-(\rho+\gamma\theta)t}a^{1-\sigma}$:

$$\hat{H} = e^{-\rho t}\frac{a^{1-\sigma}}{1-\sigma}$$

So $\hat{H}$ is obviously concave on $x$. While for $t$ such that $\lambda(t) > \delta^{-1}\eta_0^{-\theta}$ $h^{\theta-1}e^{-(\rho+\gamma\theta)t}a^{1-\sigma}$, we have $\lambda(t) = \delta^{-1}\eta_0^{-\theta}h^{\theta-1}e^{-(\rho+\gamma\theta)t}a^{1-\sigma}(1-u)^{-\sigma}$ then:

$$\begin{aligned}
\hat{H} &= \left(\frac{\sigma}{1-\sigma}\right)e^{-\frac{(\rho+\theta\gamma-\theta\sigma\gamma)}{\sigma}t}a^{\frac{1-\sigma}{\sigma}}\left(\lambda\delta\eta_0^{\theta}\right)^{\frac{\sigma-1}{\sigma}}h^{-\frac{(1-\theta)(1-\sigma)}{\sigma}} + \lambda\delta\eta_0^{\theta}e^{\gamma\theta t}h^{1-\theta} \\
\hat{H}_h &= -(1-\theta)\,e^{-\frac{(\rho+\theta\gamma-\theta\sigma\gamma)}{\sigma}t}a^{\frac{1-\sigma}{\sigma}}\left(\lambda\delta\eta_0^{\theta}\right)^{\frac{\sigma-1}{\sigma}}h^{-\frac{(\theta\sigma-\theta+1)}{\sigma}} + (1-\theta)\,\lambda\delta\eta_0^{\theta}e^{\gamma\theta t}h^{-\theta} \\
\hat{H}_{hh} &= \frac{(1-\theta(1-\sigma))(1-\theta)}{\sigma}e^{-\frac{(\rho+\theta\gamma-\theta\sigma\gamma)}{\sigma}t}a^{\frac{1-\sigma}{\sigma}}\left(\lambda\delta\eta_0^{\theta}\right)^{\frac{\sigma-1}{\sigma}}h^{-\frac{(\sigma-\theta+\theta\sigma+1)}{\sigma}} - \\
&\quad -\theta(1-\theta)\,\lambda\delta\eta_0^{\theta}e^{\gamma\theta t}h^{-(\theta+1)}
\end{aligned}$$

Finally, from Assumption 1.5 we have $\sigma < \frac{1+\theta}{\theta}$ and it is proven once again that $\hat{H}$ is concave on $h$. Then $\hat{H}$ is concave for all $h \geq 0$ and using the Arrow Theorem we establish sufficiency. Observe that since we use the solution of the Modern Economy Problem to find the solution to the Traditional Economy Problem, we actually need $\theta/(1+\theta) < \sigma < (1+\theta)/\theta$ to establish the sufficiency for the whole Traditional Economy Problem.

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