Instituto Nacional de Matemática Pura e Aplicada

Etereldes Gonçalves Júnior

FETI-Mortar-Multilevel Preconditioners for a Linear-Quadratic Elliptic Control Problem

PhD Thesis

Thesis presented to IMPA's doctoral program in Applied Mathematics as part of the requirement to the degree of Doctor in Sciences.

Adviser: Prof. Marcus Vinicius Sarkis

Rio de Janeiro
March 2009

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Abstract

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In this thesis we develop and analyze multilevel preconditioners for a linear-quadratic elliptic control problem (LQECP). In the problem considered here the control variable corresponds to the Neumann data on the boundary of a convex polygonal domain. The control variable is chosen such that the harmonic extension approximates a specified target (possibly a discontinuous function). After a finite element discretization, the LQECP leads to a large scale symmetric indefinite linear system. The approach considered here is: First, reduce this large system to a symmetric positive Hessian system for determining the discrete control variable. Second, design a robust preconditioner for this very ill-conditioned Hessian system. In order to propose a preconditioner for the Hessian System we discuss the following preconditioning issues: (1) Preconditioners for product of operators, (2) Multilevel preconditioners for singularly perturbed sum of operators of negative order, and (3) Negative-norm splitting for nonconforming finite elements. Finally, we introduce robust Finite Element Tearing and Interconnecting-Mortar-Multilevel preconditioners and provide condition number estimates. We present also numerical experiments which agree with the theoretical results.

Keywords

Preconditioners. Elliptic control problems. Schur complement. Multilevel methods. Wavelets. FETI. Mortar projection.

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”Quando alguém compreende que é contrário à sua dignidade de homem obedecer a leis injustas, nenhuma tirania pode escravizá-lo”

Mahatma Gandhi

1
Introduction

The problem of solving linear systems is central in numerical analysis. Systems arising from the discretization of PDEs and control problems have received special attention, since they appear in many applications, such as fluid dynamics and structural mechanics; see [22] and references therein. Usually one can take advantage of special properties of these systems to build efficient numerical schemes to solve them. In this thesis we will present and analyze algorithms for solving the system arising from a finite element discretization of a linear-quadratic elliptic control problem (LQECP). Typically, as the dimension of the discretization space increases, the resulting problem is not only larger, but also more ill-conditioned.

Here we propose and analyze a preconditioner for the system resulting from the application of the Schur complement method to a discrete LQECP. The Schur complement method is a powerful tool to solve linear systems.

In order to derive our preconditioner, we first show that the Schur complement matrix is associated to a linear combination of Sobolev norms. Next we use this fact to propose a preconditioner based on multilevel methods.

We also present numerical experiments. The numerical results agree with the theoretical results.

The rest of this chapter is organized as follows. In Section 1.1 we introduce the LQECP and the two discretizations considered for this problem. We present the systems arising of these discretizations and the Schur complement matrices with respect to the control variables. In Section 1.2 we describe how to associate these Schur complement matrices to a linear combination of Sobolev norms. We describe also how to design a preconditioner for one of these Schur complements using a preconditioner for the other one. Then we present multilevel preconditioners for one of these Schur complements in Section 1.3. In Section 1.4 we present further applications of the preconditioners of these Schur complements. In Section 1.5 we describe how this thesis is organized and we present a diagram to illustrate this issue.

1.1 Setting Out The Problem

In this thesis we are interested in a numerical scheme for the following LQECP:

$$\begin{aligned} &\text{minimize} \quad \|y - y_*\|^2_{L^2(\Omega)} + \tfrac{\alpha}{2}\|\lambda\|^2_{H^{-1/2}(\partial\Omega)} + \tfrac{\beta}{2}\|\lambda\|^2_{L^2(\partial\Omega)} \\ &\text{subject to} \quad \begin{cases} -\Delta y(x) = f(x) & \text{in} \quad \Omega \subset \mathbb{R}^2, \\ \gamma\frac{\partial y}{\partial \eta}(s) = \lambda(s) & \text{on} \quad \Gamma := \partial\Omega, \\ \int_\Omega y(x)dx = 0, \end{cases} \end{aligned} \tag{1.1}$$

where $y_*, f \in L^2(\Omega)$ are given functions, γ is the trace operator on Γ and α and β are nonnegative given parameters. Here, the norm $H^{-1/2}(\Gamma)$ is defined as follows:

$$\|\lambda\|^2_{H^{-1/2}(\Gamma)} := \|y_\lambda\|^2_{H^1(\Omega)}, \tag{1.2}$$

where y_λ is the harmonic extension of λ with mean zero. More precisely, y_λ is the solution of

$$\begin{cases} -\Delta y(x) = 0 & \text{in} \quad \Omega \subset \mathbb{R}^2, \\ \gamma\frac{\partial y}{\partial \eta}(s) = \lambda(s) & \text{on} \quad \Gamma := \partial\Omega, \\ \int_\Omega y(x)dx = 0. \end{cases} \tag{1.3}$$

We assume Ω to be a convex domain, this hypothesis gives H^2-regularity of y. Discussion on well-posedness of the problem (1.1) can be found in [22] and references therein.

We consider two discretizations for the problem (1.1):

1. Standard P_1 (piecewise linear and continuous functions) finite element space $V_h(\Omega)$ for the state variable y and the standard P_1 finite element space $W_h(\Gamma)$ for the control variable λ; see Figure 1.1.

2. Standard P_1 finite element space $V_h(\Omega)$ for the state variable y and the Bernardi-Maday-Patera space $\widehat{W}_h(\Gamma)$, defined in [5], for the control variable; see Section 4.2 for details of this discretization. The functions of space $\widehat{W}_h(\Gamma)$ are generally discontinuous at the corners of the polygon; see Figure 1.2.

We observe that the normal derivative of a function defined on a polygon is generally discontinuous at the corners of the boundary of the polygonal domain. Hence, in this case $\widehat{W}_h(\Gamma)$ is a more suitable space than $W_h(\Gamma)$ to approximate the normal derivative λ. For discussion on convergence rate of these discretizations we refer to [10] and references therein.

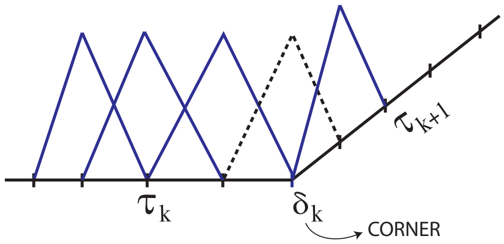


Figure 1.1: Basis of $W_h(\Gamma)$

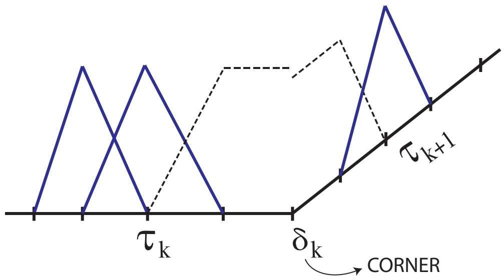


Figure 1.2: Basis of $\widehat{W}_h(\Gamma)$

Using a Lagrangean formulation to impose the PDE constraints, the two discretizations of the problem (1.1) yields the following linear systems:

$$1. \quad \begin{bmatrix} M & 0 & A^T \\ 0 & G & Q_{ext}^T \\ A & Q_{ext} & 0 \end{bmatrix} \begin{bmatrix} \mathrm{y} \\ \lambda \\ \mathrm{p} \end{bmatrix} = \begin{bmatrix} \mathrm{f}_1 \\ \mathrm{f}_2 \\ \mathrm{f}_3 \end{bmatrix}, \tag{1.4}$$

$$2. \quad \begin{bmatrix} M & 0 & A^T \\ 0 & \widehat{G} & B_{ext}^T \\ A & B_{ext} & 0 \end{bmatrix} \begin{bmatrix} \mathrm{y} \\ \hat{\lambda} \\ \mathrm{p} \end{bmatrix} = \begin{bmatrix} \mathrm{f}_1 \\ \mathrm{f}_2 \\ \mathrm{f}_3 \end{bmatrix}, \tag{1.5}$$

where, the matrices M, A, Q_{ext}, B_{ext}, G and $\widehat{G}$ are defined as follows. Let ϕ_i, φ_j and ψ_l basis functions such that $V_h(\Omega) = span\{\phi_i\}_{i=1}^n$, $W_h(\Gamma) = span\{\varphi_j\}_{j=1}^m$ and $\widehat{W}_h(\Gamma) = span\{\psi_l\}_{l=1}^{\hat{m}}$, respectively. Then we define

- $M \in \mathbb{R}^{n\times n}$: $M_{ij} := (\phi_i, \phi_j)_{L^2(\Omega)}$. Mass matrix in Ω.
- $A \in \mathbb{R}^{n\times n}$: $A_{ij} := (\nabla\phi_i, \nabla\phi_j)_{L^2(\Omega)}$. Stiffness matrix of the Laplacian operator.
- $Q_{ext} \in \mathbb{R}^{n\times m}$: $Q_{ext_{ij}} := (\phi_i, \varphi_j)_{L^2(\Gamma)}$; $\phi_i \in V_h(\Omega)$ and $\varphi_j \in W_h(\Gamma)$.
- $B_{ext} \in \mathbb{R}^{n\times \hat{m}}$: $B_{ext_{ij}} := (\phi_i, \psi_j)_{L^2(\Gamma)}$; $\phi_i \in V_h(\Omega)$ and $\psi_j \in \widehat{W}_h(\Gamma)$.

We now define the matrices G and $\widehat{G}$. We say that a symmetric positive definite matrix X is associated to a Sobolev norm $\|\cdot\|_{H^s}^2$ in a space W when there exists positive constants c_1 and c_2 such that $c_1\, w^T X w \le \|w\|_{H^s}^2 \le c_2\, w^T X w$ for all $w \in W$. We define $G \in \mathbb{R}^{m\times m}$ and $\widehat{G} \in \mathbb{R}^{\hat{m}\times\hat{m}}$ to be the matrices associated to the norm $\frac{\alpha}{2}\|\cdot\|_{H^{-1/2}(\Gamma)}^2 + \frac{\beta}{2}\|\cdot\|_{L^2(\Gamma)}^2$ in $W_h(\Gamma)$ and $\widehat{W}_h(\Gamma)$, respectively. More specifically, using (1.2) we have

- $G \in \mathbb{R}^{m\times m}$: $G := \alpha(Q_{ext}^T A^\dagger)\, A\, (A^\dagger Q_{ext}) + \beta Q = \alpha Q_{ext}^T A^\dagger Q_{ext} + \beta Q$,
- $\widehat{G} \in \mathbb{R}^{\hat{m}\times\hat{m}}$: $G := \alpha(B_{ext}^T A^\dagger)\, A\, (A^\dagger B_{ext}) + \beta\widehat{Q} = \alpha B_{ext}^T A^\dagger B_{ext} + \beta\widehat{Q}$,

where

- $Q \in \mathbb{R}^{m\times m}$: $Q_{ij} := (\varphi_i, \varphi_j)_{L^2(\Gamma)}$; $\varphi_i \in W_h(\Gamma)$ and $\varphi_j \in W_h(\Gamma)$,
- $\widehat{Q} \in \mathbb{R}^{\hat{m}\times \hat{m}}$: $\widehat{Q}_{ij} := (\psi_i, \psi_j)_{L^2(\Gamma)}$; $\psi_i \in \widehat{W}_h(\Gamma)$ and $\psi_j \in \widehat{W}_h(\Gamma)$.

Here and the following $A^\dagger$ is a pseudo inverse of A. Note that by construction $m = \hat{m}+K$, where K is the number of corners of the polygon Γ.

The discrete forcing are defined by $(\mathrm{f}_1)_i = \int_\Omega y_*(x)\,\phi_i(x)\,dx$, for $1 \le i \le n$ with $\mathrm{f}_2 = 0$, and $(\mathrm{f}_3)_i = \int_\Omega f(x)\,\phi_i(x)\,dx$. The coefficient matrices of (1.4) and (1.5) are known by KKT (Karush-Kuhn-Tucker) matrices.

A well known technique to solve linear systems, such as (1.4) and (1.5), is the block Gaussian elimination or Schur complement; see [17]. The Schur complement with respect the control variable λ in (1.4) is defined as follows. Eliminating the variables y and p from equation (1.4), we have:

$$My + Ap = \mathrm{f}_1 \quad \Rightarrow \quad p = -A^\dagger M y + A^\dagger \mathrm{f}_1, \tag{1.6}$$

$$G\lambda + Q_{ext}^T p = 0 \quad \Rightarrow \quad G\lambda = Q_{ext}^T A^\dagger M y - Q_{ext}^T A^\dagger \mathrm{f}_1, \tag{1.7}$$

$$Ay + Q_{ext}\lambda = \mathrm{f}_3 \quad \Rightarrow \quad y = -A^\dagger Q_{ext}\lambda + A^\dagger \mathrm{f}_3. \tag{1.8}$$

Substituting (1.8) into (1.7) we obtain

$$(G + Q_{ext}^T A^\dagger M A^\dagger Q_{ext})\,\lambda = Q_{ext}^T A^\dagger M A^\dagger \mathrm{f}_3 - Q_{ext}^T A^\dagger \mathrm{f}_1. \tag{1.9}$$

The matrix $C_\mathrm{c} := G + Q_{ext}^T A^\dagger M A^\dagger Q_{ext}$ is known as the *Schur complement* (Reduced Hessian) with respect to the discrete control variable λ. We observe that the state variable y can be obtained by solving (1.9) and using (1.8). Analogously, we can obtain the Schur complement with respect the control variable $\hat{\lambda}$ of (1.5). Eliminating the variables y and p in the equation (1.5) we have

$$(\widehat{G} + B_{ext}^T A^\dagger M A^\dagger B_{ext})\,\hat{\lambda} = B_{ext}^T A^\dagger M A^\dagger \mathrm{f}_3 - B_{ext}^T A^\dagger \mathrm{f}_1. \tag{1.10}$$

Hence, the *Schur complement* with respect to the discrete control variable $\hat{\lambda}$ is $C_\mathrm{d} := \widehat{G} + B_{ext}^T A^\dagger M A^\dagger B_{ext}$.

The matrices C_c and C_d can be rewritten in the following way. Let $B \in \mathbb{R}^{m\times\hat{m}}$ defined as $B_{ij} := (\varphi_i, \psi_j)_{L^2(\Gamma)}$; $\varphi_i \in W_h(\Gamma)$ and $\psi_j \in \widehat{W}_h(\Gamma)$. Let $E \in \mathbb{R}^{n\times m}$ the extension by zero operator defined as $E := [0_{n,m-n} \;\; Id_n]$. Define $S_1^\dagger := E A^\dagger E^T$ and $S_3^\dagger := E A^\dagger M A^\dagger E^T$, then

$$\begin{aligned} C_\mathrm{c} &= G + Q^T S_3^\dagger Q &\asymp\; & Q\,(\alpha S_1^\dagger + \beta Q^{-1} + S_3^\dagger)\,Q, \\ C_\mathrm{d} &= \widehat{G} + B^T S_3^\dagger B &\asymp\; & B^T(\alpha S_1^\dagger + \beta Q^{-1} + S_3^\dagger)\,B. \end{aligned} \tag{1.11}$$

Here we use the notation $A \preceq (\succeq)\ B$ to indicate that $A \leq (\geq)\, c\, B$, where the positive constant c depends only on the shape of Ω. When $A \preceq B \preceq A$, we say $A \asymp B$.

Direct methods like Gaussian elimination can be used for solving the system (1.10). However, direct methods are computationally very expensive and require knowing explicitly all entries of the coefficient matrix C_d. To circumvent these issues, iterative methods are often used to solve such a problem; a complete explanation on iterative methods can be found in [17].

We observe that C_c and C_d are symmetric semi-positive definite matrices. Hence, the *Preconditioned Conjugate Gradient* (PCG) is a good choice to solve the systems (1.9) and (1.10). The main goal of this thesis is to develop and analyze preconditioners for the matrices C_c and C_d. Our preconditioner for C_d is based on a preconditioner for C_c.

1.2 Analyzing the matrices C_c and C_d

In this section we associate the matrix C_c to a linear combination of Sobolev norms. Based on this fact we will construct a preconditioner for the matrix C_c using multilevel methods. We also explain, how we can use a preconditioner for C_c to precondition C_d. We first introduce some Sobolev norms and spaces. Define

$$H^{3/2}_{t,00}(\Gamma) := \left\{ \lambda \in H^{1/2}(\Gamma);\ \frac{\partial \lambda}{\partial \tau_k} \in H^{1/2}_{00}(\Gamma_k),\ 1 \leq k \leq K \right\}, \tag{1.12}$$

where Γ_k is an edge of the polygon $\Gamma = \partial\Omega$ and τ_k is the tangent vector on Γ_k; see Figure 1.3. For a definition of standard Sobolev spaces like H^1, H^1_0, $H^{1/2}$, $H^{1/2}_{00}$ and its duals see [1, 18, 28]. The norm of $H^{3/2}_{t,00}(\Gamma)$ is defined as

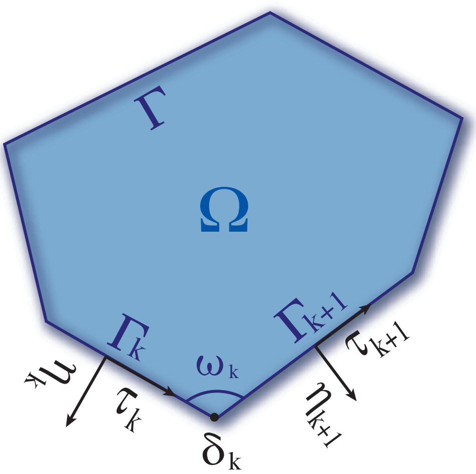


Figure 1.3: A convex polygonal domain in $\mathbb{R}^2$.

$$\|v\|_{H^{3/2}_{t,00}(\Gamma)} := \|v\|^2_{H^{1/2}(\Gamma)} + \sum_{k=1}^{K} \left\| \frac{\partial v}{\partial \tau_k} \right\|^2_{H^{1/2}_{00}(\Gamma_k)}, \tag{1.13}$$

and $H^{-3/2}_{t,00}(\Gamma)$ its dual endowed with the norm

$$\|v\|^2_{H^{-3/2}_{t,00}(\Gamma)} := \sup_{\varphi \in H^{3/2}_{t,00}(\Gamma)} \frac{(v,\varphi)_{L^2}}{\|\varphi\|^2_{H^{3/2}_{t,00}(\Gamma)}}. \tag{1.14}$$

Note that the functions of $H^{3/2}_{t,00}(\Gamma)$ are not zero at the corners of the boundary Γ; see Figure 1.4. See Chapter 3 of this thesis for details and more properties of these spaces. The space $H^{3/2}_{t,00}(\Gamma)$ appear because the trace of a function of $H^2(\Omega)$, which the normal

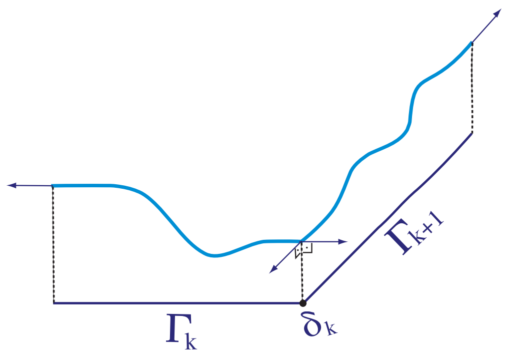


Figure 1.4: Illustration of a function in the space $H^{3/2}_{t,00}(\Gamma)$.

derivative is equals to zero on Γ, belongs to $H^{3/2}_{t,00}(\Gamma)$; see Lemma 3.2.

To construct a preconditioner for C_c we first prove that C_c is a matrix associated to a linear combination of Sobolev norms. This result is based on the following fact; see Theorem 3.10. The $L^2(\Omega)$-norm of the harmonic extension function y_λ, with Neumann data λ, is equivalent to the $H^{-3/2}_{t,00}(\Gamma)$-norm of its Neumann data λ; see (1.3). More precisely,

$$\|\lambda\|_{H^{-3/2}_{t,00}(\Gamma)} \preceq \|y_\lambda\|_{L^2(\Omega)} \preceq \|\lambda\|_{H^{-3/2}_{t,00}(\Gamma)}. \tag{1.15}$$

Using (1.15) we prove Theorem 4.1, which states

$$\lambda^T Q S_3^{\dagger} Q \lambda \asymp \|\lambda\|^2_{H^{-3/2}_{t,00}(\Gamma)} + h^2 \|\lambda\|^2_{H^{-1/2}(\Gamma)}; \quad \forall \lambda \in W_h(\Gamma). \tag{1.16}$$

Hence, by definition of C_c and (1.16) we conclude that C_c is associated to the following linear combination of Sobolev norms

$$C_c \asymp (\alpha + h^2)\|\cdot\|^2_{H^{-1/2}(\Gamma)} + \beta\|\cdot\|^2_{L^2(\Gamma)} + \|\cdot\|^2_{H^{-3/2}_{t,00}(\Gamma)}. \tag{1.17}$$

We now explain how to precondition $C_d \in \mathbb{R}^{\hat{m}\times\hat{m}}$ using a preconditioner for $C_c \in \mathbb{R}^{m\times m}$. Defining $H := \alpha S_1^{\dagger} + \beta Q^{-1} + S_3^{\dagger}$ and using (1.11) we obtain that $C_c \asymp QHQ$ and $C_d \asymp B^T H B$. We use the Finite Element Tearing and Interconnecting

(FETI) method to propose a preconditioner for C_d based on a preconditioner for C_c; see [13, 14, 15, 24, 32]. More precisely, let PC a preconditioner for C_c, i. e. PC is spectrally equivalent to $C_\mathrm{c}^\dagger$. Using the FETI method we define the matrix $D \in \mathbb{R}^{m\times\hat{m}}$ such that the preconditioner for C_d has the form $D^T PCD$. We observe that the matrix D performs the adjustment of dimensions. Inspired in [21], we choose $D = B(B^T Q^{-1} B)^{-1}$ and prove, see Theorem 5.1, the following result:

$$(\hat{\lambda}, \hat{\lambda})_{\ell^2} \;\preceq\; ([D^T PCD] C_\mathrm{d} \hat{\lambda}, \hat{\lambda})_{\ell^2} \;\preceq\; (\hat{\lambda}, \hat{\lambda})_{\ell^2}, \quad \forall \hat{\lambda} \in \widehat{W}_h(\Gamma). \tag{1.18}$$

Hence, from (1.18) we conclude that $D^T PCD$ is spectrally equivalent to $C_\mathrm{d}^\dagger$.

1.3 Preconditioners Based on Multilevel Methods

In this section we first describe a standard multilevel method to precondition the matrix C_c. Next we introduce a new multilevel method yielding another preconditioner for C_c, which is easier to compute. Multilevel methods, such as multigrid methods, are among the most efficient methods to precondition matrices associated to Sobolev norms; see [2, 7, 8, 30, 31, 34, 35].

A standard multilevel method is described as follows. Let $\mathcal{T}_h$ be the restriction of $\mathcal{T}_h(\Omega)$ to Γ. We assume that the triangulation $\mathcal{T}_h$ of Γ has a *multilevel* structure. More precisely, $\mathcal{T}_h$ is obtained from $(L-1)$ successive refinements of an initial coarse triangulation $\mathcal{T}_0$ with initial grid size h_0. Let $h_\ell = h_{\ell-1}/2$ be the grid size of the ℓ-*th* triangulation $\mathcal{T}_\ell$ and denote by V_ℓ the standard space generated by continuous and piecewise linear functions on $\mathcal{T}_h$. By definition we have

$$V_1 \subset V_2 \subset \cdots \subset V_L \cdots \subset L^2. \tag{1.19}$$

Let P_ℓ denotes the L^2-orthogonal projection onto V_ℓ and $P_0 = 0$, that is, $P_\ell v$ is a unique vector in V_ℓ such that

$$(P_\ell v, \varphi_i^\ell)_{L^2} = (v, \varphi_i^\ell)_{L^2} \text{ for all } 1 \le i \le m_\ell, \tag{1.20}$$

where $\{\varphi_i^\ell\}_{i=1}^{m_\ell}$ are basis functions of V_ℓ. In matrix term,

$$P_\ell := R_\ell^T Q_\ell^{-1} R_\ell Q, \tag{1.21}$$

where R_ℓ is defined such that its i-*th* row is obtained by interpolating the basis function φ_i^ℓ at the nodes of the finest triangulation $\mathcal{T}_L := \mathcal{T}_h$, and $Q_\ell := R_\ell Q R_\ell^T$ is a mass matrix. The following equivalence of norms holds; see [7] :

$$\|\mathrm{u}\|^2_{H^s(\Gamma)} \asymp \sum_{\ell=1}^{L} h_\ell^{-2s} \|P_\ell \mathrm{u} - P_{\ell-1}\mathrm{u}\|^2_{L^2(\Gamma)}, \text{ for all } \mathrm{u} \in V_L \text{ and } -3/2 < s < 3/2. \tag{1.22}$$

The restriction on s comes from the fact that $V_L(\Gamma) \not\subset H^s(\Gamma)$ if $s \geq 3/2$. In this thesis we extend (1.22) to the critical exponent $s = -3/2$; see Theorem 6.1. To prove this theorem we explore some of the ideas presented in [31]. In Theorem 6.1 we prove that for the critical exponent $s = -3/2$ the equivalence (1.22) is quasi-optimal. More precisely, we prove

$$\|\mathrm{u}\|^2_{H^{-3/2}_{t,00}(\Gamma)} \preceq \sum_{\ell=1}^{L} h_\ell^3 \, \|P_\ell \mathrm{u} - P_{\ell-1}\mathrm{u}\|^2_{L^2(\Gamma)} \preceq (L+1)^2 \, \|\mathrm{u}\|^2_{H^{-3/2}_{t,00}(\Gamma)}, \quad \text{for all } \mathrm{u} \in V_L. \qquad (1.23)$$

Defining $\mu_\ell := \left((\alpha + h_L^2)h_\ell + \beta + h_\ell^3\right)$ and using (1.17), (1.22) and (1.23) we obtain

$$\sum_{\ell=1}^{L} \mu_\ell \, \|P_\ell \mathrm{u} - P_{\ell-1}\mathrm{u}\|^2_{L^2(\Gamma)} \ \preceq \ C_{\mathrm{c}} \ \preceq \ (L+1)^2 \sum_{\ell=1}^{L} \mu_\ell \, \|P_\ell \mathrm{u} - P_{\ell-1}\mathrm{u}\|^2_{L^2(\Gamma)}. \qquad (1.24)$$

Based on (1.24) we prove that (see Section 6.1):

$$\frac{1}{(L+1)^2} \sum_{\ell=1}^{L} \mu_\ell^{-1} \, \Delta_\ell \ \preceq \ C_{\mathrm{c}}^{-1} \ \preceq \ \sum_{\ell=1}^{L} \mu_\ell^{-1} \Delta_\ell, \qquad (1.25)$$

where $\Delta_\ell := (P_\ell - P_{\ell-1})Q^{-1}$. Hence, a standard preconditioner for C_{c} is $\sum_{\ell=1}^{L} \mu_\ell^{-1} \Delta_\ell$.

The splitting $\sum_{\ell=1}^{L} h_\ell^3 \, \|P_\ell \mathrm{u} - P_{\ell-1}\mathrm{u}\|^2_{L^2(\Gamma)}$ depends on the exact L^2-projection P_ℓ onto the ℓ-*th* levels. These projections require mass matrix inversions; see (1.21). To avoid these inversions we propose another multilevel preconditioner. This preconditioner is obtained by replacing the projections P_ℓ by a more suitable one $\bar{P}_\ell$; see Chapter 7.

We now define the projection $\bar{P}_\ell$. We first introduce the spaces $\bar{V}_\ell := span\{\bar{\psi}_i^\ell\}_{i=1}^{m_\ell}$ for $\ell < L$, where $\bar{\psi}_i^\ell := -1/2\varphi_{j-1}^{\ell+1} + 3\varphi_j^{\ell+1} - 1/2\varphi_{j+1}^{\ell+1}$; see Figure 1.5, and $\bar{V}_L := V_L$. We have a biorthogonality property between $\bar{V}_\ell$ and V_ℓ for $\ell < L$, that is,

$$(\varphi_{i_1}^\ell, \bar{\psi}_{i_2}^\ell) = 0 \ \text{ if } \ i_1 \neq i_2. \qquad (1.26)$$

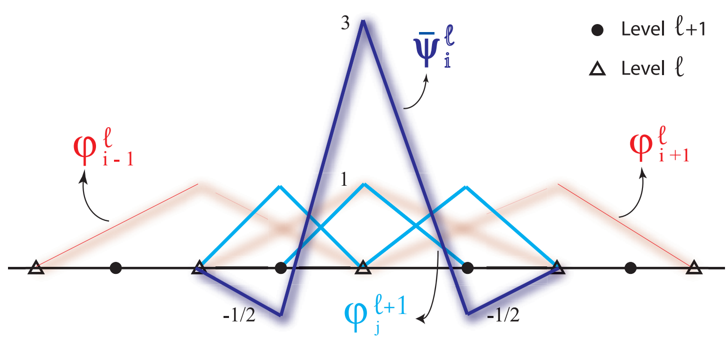


Figure 1.5: Illustration of basis function $\bar{\psi}_i^\ell$.

We define $\bar{P}_\ell : L^2 \longmapsto V_\ell$ as a mortar projection; see [4, 5]. More specifically, for each $v \in L^2$, $\bar{P}_\ell v$ is the unique vector belonging to V_ℓ such that

$$(\bar{P}_\ell v, \bar{\psi}_i^\ell)_{L^2} = (v, \bar{\psi}_i^\ell)_{L^2} \text{ for all } 1 \le i \le m_\ell. \tag{1.27}$$

Due to the biorthogonality property, see (1.26), $\bar{P}_\ell$ can be easily calculated, requiring no matrix inversion.

Theorem 7.1 states that for all $s > 0$ the following equivalence relation holds

$$\sum_{\ell=0}^{L} h_\ell^{-2s} \|\bar{P}_\ell \mathrm{v} - \bar{P}_{\ell-1}\mathrm{v}\|_{L^2}^2 \preceq \sum_{\ell=0}^{L} h_\ell^{-2s} \|P_\ell \mathrm{v} - P_{\ell-1}\mathrm{v}\|_{L^2}^2 \preceq \sum_{\ell=0}^{L} h_\ell^{-2s} \|\bar{P}_\ell \mathrm{v} - \bar{P}_{\ell-1}\mathrm{v}\|_{L^2}^2. \tag{1.28}$$

An important corollary of this Theorem is (see Collorary 7.1):

$$\sum_{\ell=0}^{L} \|\bar{P}_\ell \mathrm{v} - \bar{P}_{\ell-1}\mathrm{v}\|_{L^2}^2 \preceq \sum_{\ell=0}^{L} \|P_\ell \mathrm{v} - P_{\ell-1}\mathrm{v}\|_{L^2}^2 \preceq L \sum_{\ell=0}^{L} \|\bar{P}_\ell \mathrm{v} - \bar{P}_{\ell-1}\mathrm{v}\|_{L^2}^2. \tag{1.29}$$

In the next section we present further applications to the preconditioner of the matrix C_c developed in this thesis.

1.4 Further Applications for the Preconditioner of the Matrix C_c

In this section we present other two applications to the preconditioner of the matrix C_c:

1) The block preconditioner for (1.4) presented in [25] needs of a preconditioner for C_c.
2) The matrix C_c is the Schur complement with respect to the vorticity in a mixed finite element method (FEM) to solve the biharmonic problem.

Elaborating on 1). In [25] it is proposed a different approach to solve (1.4). This approach is based on GMRES with a block preconditioner. However the proposed block preconditioner depends on a preconditioner for C_c, which was not presented in [25].

Elaborating on 2). Consider a mixed FEM to solve the biharmonic problem

$$\begin{cases} \Delta^2 w & = \; f \quad \text{in} \quad \Omega, \\ w = \dfrac{\partial w}{\partial \eta} & = \; 0 \quad \text{on} \quad \Gamma, \end{cases} \tag{1.30}$$

where $\Omega \subset \mathbb{R}^2$ is a convex polygonal domain. In a mixed method, the problem (1.30) is decomposed in two problems: $-\Delta y = f$ and $-\Delta w = y$; see [12, 16]. The discrete problem resulting from a mixed FEM applied to (1.30) has the form

$$\begin{bmatrix} 0 & A & -Q_{ext}^T \\ A & -M & 0 \\ -Q_{ext} & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{w} \\ \mathrm{y} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathrm{f} \\ 0 \\ 0 \end{bmatrix}. \tag{1.31}$$

We observe that the Schur complement with respect to the variable λ is $Q^T S_3^{\dagger} Q$; see (1.11). Therefore, the multilevel preconditioner for the matrix C_c described in Section 1.3, can be used to solve (1.31).

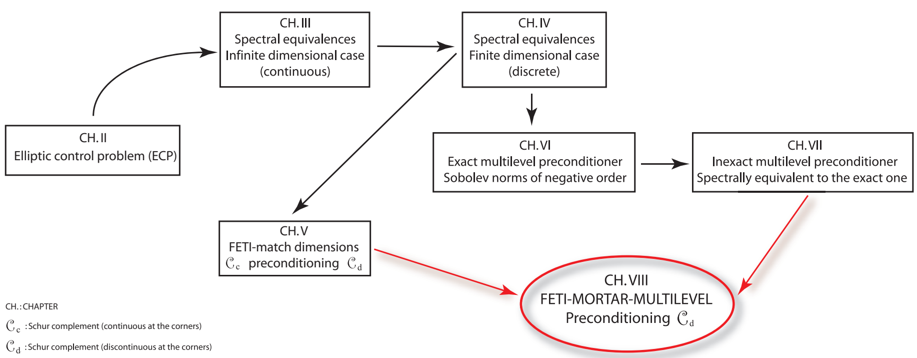
1.5 Structure of this Thesis

This thesis is organized as follows. In the second chapter we introduce the Neumann control problem considered and we provide figures and tables to compare two discretizations of this problem. In the third chapter we discuss the properties of the Steklov-Poincaré operator which are useful to analyze the preconditioners. The main result of the third chapter is Theorem 3.10, which establishes (1.15). In the fourth chapter we provides a discrete version of Theorem 3.10. The most important result of the fourth chapter is Theorems 4.1, which establishes (1.17). Chapter 5 is devoted to prove (1.18); see Theorem 5.1.

In Chapter 6 we extend Theorem 2 of [31] for piecewise linear and continuous functions and norms of order $-3/2$. In the sixth chapter we provides an exact multilevel preconditioner for C_c. The main result of the sixth chapter is Theorem 6.1, which establishes (1.23). In Chapter 7 we develop an inexact multilevel preconditioner for C_c based on biorthogonal wavelets spaces and mortar projections. In the seventh chapter we prove (1.28); see Theorem 7.1.

Finally, in the last chapter we make a compendium of the theory developed in the previous chapters. We provide also numerical experiments which support the theoretical results. The conclusions of this thesis are presented also.

The diagram below illustrates the structure of this thesis



2
Introduction to Neumann Control Problem

In this chapter we present the linear quadratic elliptic control problem (LQECP) considered in this thesis and two discretizations of this problem. We present also figures to illustrate the solution of the LQECP in a model problem. Are presented also tables to illustrate the rate of convergence of these discretizations.

The LQECP introduced in this chapter is the following: Find the Neumann control data $\lambda(\cdot)$ on Γ such that the solution $y(\cdot)$ to the problem

$$\left\{\begin{array}{rcll} -\Delta y & = & f & \text{in } \Omega, \\ \gamma \dfrac{\partial y}{\partial \eta} & = & \lambda & \text{on } \Gamma, \\ \int_{\Omega} y\, ds & = & 0 & \end{array}\right. \tag{2.1}$$

minimizes the performance functional

$$J(y, \lambda) := \frac{1}{2}\left(\|y - y_*\|^2_{L^2(\Omega)} + \alpha \, \|\lambda\|^2_{H^{-1/2}(\Gamma)} + \beta \, \|\lambda\|^2_{L^2(\Gamma)} \right), \tag{2.2}$$

where $y_*(\cdot) \in L^2(\Omega)$ is a given target data, $\alpha \geq 0$ and $\beta \geq 0$. Here, γ is the trace operator on Γ. This problem is referred as LQECP.

The weak formulation of LQECP is defined as follows. we first introduce the function space $H^1(\Omega)/\mathbb{R}$ for $y(\cdot)$. We identify $H^1(\Omega)/\mathbb{R}$ with $\{v \in H^1(\Omega); \int_{\Omega} v\, dx = 0\}$; see Chapter 3. We also introduce the space

$$H_f^{-1/2}(\Gamma) := \left\{ \lambda \in H^{-1/2}(\Gamma); \int_{\Omega} f\, dx + \int_{\Gamma} \lambda\, ds = 0 \right\} \tag{2.3}$$

for given $f \in L^2(\Omega)$. We abuse of the notation writing $\int_{\Gamma} \lambda\, ds$ as the linear functional $\lambda \in H^{-1/2}(\Gamma)$ applied at the function $1 \in H^{1/2}(\Gamma)$. Define the constraint set $\mathcal{X}_f \subset \mathcal{X} \equiv H^1(\Omega)/\mathbb{R} \times H_f^{-1/2}(\Gamma)$:

$$\mathcal{X}_f := \left\{ (y, \lambda) \in \mathcal{X} \,:\, a(y, w) = (f, w) + < \lambda, w >, \;\; \forall w \in H^1(\Omega)/\mathbb{R} \right\}, \tag{2.4}$$

where the forms are defined by:

$$\begin{cases} a(y,w) & := & \int_\Omega \nabla y \cdot \nabla w \, dx, & \text{for } y,w \in H^1(\Omega)/\mathbb{R} \\ (f,w) & := & \int_\Omega f\, w\, dx, & \text{for } w \in H^1(\Omega)/\mathbb{R} \\ <\lambda, w> & := & \int_\Gamma \lambda\, w\, ds, & \text{for } \lambda \in H_f^{-1/2}(\Gamma),\ w \in H^{1/2}(\Gamma)/\mathbb{R}. \end{cases} \tag{2.5}$$

The weak solution of LQECP is $(y^\star, \lambda^\star) \in \mathcal{X}_f$ the solution of:

$$J(y^\star, \lambda^\star) = \min_{(y,\lambda)\in \mathcal{X}_f} J(y,\lambda). \tag{2.6}$$

We now present the Lagrangian formulation of (2.6). Introducing $p \in H^1(\Omega)/\mathbb{R}$ as a Lagrange multiplier function to enforce the constraints, we have a saddle point formulation of (2.6). Define the following Lagrangian functional $\mathcal{L}(\cdot,\cdot,\cdot)$:

$$\mathcal{L}(y,\lambda,p) \equiv J(y,\lambda) + (a(y,p) - (f,p) - <\lambda, p>), \tag{2.7}$$

for $(y,\lambda,p) \in \mathcal{X} \times H^1(\Omega)/\mathbb{R}$. Then, the constrained minimum $(y^\star, u^\star)$ can be obtained from the saddle point $(y^\star,\lambda^\star,p^\star)$ of $\mathcal{L}(\cdot,\cdot,\cdot)$, where $(y^\star,\lambda^\star,p^\star) \in \mathcal{X} \times H^1(\Omega)/\mathbb{R}$ satisfies :

$$\sup_q \mathcal{L}(y^\star,\lambda^\star,q) = \mathcal{L}(y^\star,\lambda^\star,p^\star) = \inf_{(y,\lambda)} \mathcal{L}(y,\lambda,p^\star). \tag{2.8}$$

For discussions on well-posedness of the saddle point problem (2.8) see [22].

We now introduce two discretizations of (2.8). The most common discretization use the space of piecewise linear and continuous functions $X_h(\Omega)$ for the state function $y \in H^1(\Omega)$ and the restriction of $X_h(\Omega)$ to the boundary $V_h(\Gamma)$ for the control variable $\lambda \in L^2(\Gamma)$. Here $X_h(\Omega) \subset H^1(\Omega)$ is the regular P_1-conforming finite element space associated with a quasi-uniform triangulation $\tau_h(\Omega)$ with mesh size h, and let $V_h(\Gamma)$ be its restriction to Γ. See [6] for the definition of quasi-uniformity and mesh size of a triangulation. However, a more appropriate space for the Neumann condition requires discontinuities at the corners of Γ. We introduce a discontinuous at the corners discretization for the control variable λ.

Let $\{\phi_1(x),\ldots,\phi_n(x)\}$ and $\{\varphi_1(x),\ldots,\varphi_m(x)\}$ denote the standard piecewise linear nodal basis functions for $X_h(\Omega)$ and $V_h(\Gamma)$, respectively. We define the discrete Neumann space $W_h(\Gamma) := span\{\varphi_i\}_{i=1}^m$. We introduce the discontinuous discrete Neumann space $\widehat{W}_h(\Gamma)$ as follows; see [5]. The triangulation $\tau_h(\Omega)$ induces a partition by elements on Γ, that is, $\Gamma_k = \cup_{j=1}^{n_k}[x_j^k, x_{j+1}^k]$. Let $\psi_{n_k-1}^k$ be the piecewise linear and continuous function on $[x_{n_k-2}^k, x_{n_k}^k]$ such that $\psi_{n_k-1}^k(x_{n_k-2}^k) = 0$, $\psi_{n_k-1}^k(x_{n_k-1}^k) = 1$, $\psi_{n_k-1}^k(x_{n_k}^k) = 1$, and let $\psi_{n_k-1}^k = 0$ on $\Gamma \backslash [x_{n_k-2}^k, x_{n_k}^k]$; see Figure 2.1. Let $\psi_{n_k-2}^k$ be the piecewise linear and continuous function on $[x_{n_k-3}^k, x_{n_k-1}^k]$ such that $\psi_{n_k-2}^k(x_{n_k-3}^k) = 0$, $\psi_{n_k-2}^k(x_{n_k-2}^k) = 1$,

$\psi^k_{n_k-2}(x^k_{n_k-1}) = 0$, and let $\psi^k_{n_k-2} = 0$ on $\Gamma\backslash[x^k_{n_k-3}, x^k_{n_k-1}]$; see Figure 2.1. Define ψ^k_1 to be the piecewise linear and continuous function on $[x^k_1, x^k_3]$ such that $\psi^k_1(x^k_1) = 1$, $\psi^k_1(x^k_2) = 1$, $\psi^k_1(x^k_3) = 0$, and let $\psi^k_1 = 0$ on $\Gamma\backslash[x^k_1, x^k_3]$. Define also ψ^k_2 to be the piecewise linear and continuous function on $[x^k_2, x^k_4]$ such that $\psi^k_2(x^k_2) = 0$, $\psi^k_2(x^k_3) = 1$, $\psi^k_2(x^k_4) = 0$, and let $\psi^k_2 = 0$ on $\Gamma\backslash[x^k_2, x^k_4]$. For $3 \leq j \leq n_k - 3$ define ψ^k_j to be the standard piecewise linear and continuous function such that $\psi^k_j(x^k_j) = 1$ and zero on the remaining nodes; see Figure 2.1. Note that $x^k_1 = x^{k-1}_{n_k-1} = \delta_{k-1}$ and the space $\widehat{W}_h(\Gamma) := span\left\{\psi^k_j; 1 \leq k \leq K,\ \ 1 \leq j \leq n_k - 1\right\}$ has dimension $m - K$. We can write $\widehat{W}_h(\Gamma) = span\{\psi_\ell, 1 \leq \ell \leq m - K\}$ where $\psi_\ell = \psi^k_j$ for $\ell = n_1 + \cdots + n_{k-1} + j$ if $k \geq 2$ and $\ell = j$ if $k = 1$.

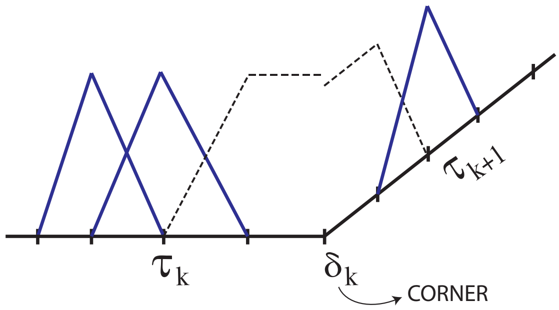


Figure 2.1: Illustration of the basis of $\widehat{W}_h(\Gamma)$.

The continuous at the corners finite element discretization of (2.6) will seek $(y^\star, \lambda^\star) \in X_h(\Omega) \times W_h(\Gamma)$ such that:

$$J(y^\star, \lambda^\star) = \min_{(y,\lambda) \in \mathcal{X}_{h,f}} J(y, \lambda), \tag{2.9}$$

where the discrete constraint space $\mathcal{X}_{h,f} \subset \mathcal{X}_h \equiv X_h(\Omega) \times W_h(\Gamma)$ is defined by:

$$\mathcal{X}_{h,f} = \{(y, \lambda) \in \mathcal{X}_h \ :\ a(y, w) = (f, w) + <\lambda, w>, \ \ \forall w \in X_h(\Omega)\}. \tag{2.10}$$

Similarly the discontinuous at corners discretization of (2.6) will seek $\left(\hat{y}^\star, \hat{\lambda}^\star\right) \in X_h(\Omega) \times \widehat{W}_h(\Gamma)$ such that:

$$J(\hat{y}^\star, \hat{\lambda}^\star) = \min_{(\hat{y},\hat{\lambda}) \in \widehat{\mathcal{X}}_{h,f}} J(\hat{y}, \hat{\lambda}), \tag{2.11}$$

where the discrete constraint space $\widehat{\mathcal{X}}_{h,f} \subset \widehat{\mathcal{X}}_h \equiv X_h(\Omega) \times \widehat{W}_h(\Gamma)$ is defined by:

$$\widehat{\mathcal{X}}_{h,f} = \left\{(\hat{y}, \hat{\lambda}) \in \widehat{\mathcal{X}}_h \ :\ a(\hat{y}, w) = (f, w) + <\hat{\lambda}, w>, \ \ \forall w \in X_h(\Omega)\right\}. \tag{2.12}$$

To obtain the solution of (2.9) and (2.11) we have to solve the linear systems (1.4) and (1.5) respectively.

Next we provide some figures to illustrate the discrete solutions of a particular choices of Ω, f and y_* in LQECP associated to the discontinuous discretization and the continuous one. We provide also some tables to illustrate the rate of convergence of these discretizations. We remember that our goal in this thesis is not comparing or analyzing these discretizations. The figures and tables were generated in MATLAB using the following mesh in $\Omega = [0,1]^2$. We divide each edge of $\Gamma := \partial\Omega$ into 2^N parts of equal length, where N is an integer denoting the number of refinements. In all figures shown in this chapter the target function is $y_* = 8(y - 3/4)$ in $[0,1/4]\times[3/4,1]$, $y_* = -1/4$ in $[3/4,1]\times[0,1]$ and $y_* = 0$ elsewhere and the right side function f is given by $f(x,y) = 2\pi^2\ \sin(\pi x)\cos(\pi y)$. Each figure is associated to a $\beta \in \{0, 10^{-6}, 10^{-3}\}$ and $\alpha = 0$. We show in each figure three functions: discrete solution $\hat{\mathrm{y}}^\star$ of LQECP associated to the discontinuous discretization of $L^2(\Gamma)$, the discrete solution $\mathrm{y}^\star$ of LQECP associated to the continuous discretization of $L^2(\Gamma)$ and the target function y_*.

In Table 2.1 we denote $\epsilon_N = \frac{1}{2}||\mathrm{y}^\star_N - \mathrm{y}_*||^2_{L^2(\Omega)}$, where $\mathrm{y}^\star_N$ is the discrete solution of LQECP associated to the continuous discretization when the mesh parameter is $h = 2^{-N}$. Analogously we denote $\hat{\epsilon}_N = \frac{1}{2}||\hat{\mathrm{y}}^\star_N - \mathrm{y}_*||^2_{L^2(\Omega)}$. In Table 2.2 we show

$$E_N = \frac{||\mathrm{y}^\star_N - \mathrm{y}^\star_{N-1}||^2_{L^2(\Omega)}}{||\mathrm{y}^\star_{N+1} - \mathrm{y}^\star_N||^2_{L^2(\Omega)}} \quad \text{and} \quad \hat{E}_N = \frac{||\hat{\mathrm{y}}^\star_N - \hat{\mathrm{y}}^\star_{N-1}||^2_{L^2(\Omega)}}{||\hat{\mathrm{y}}^\star_{N+1} - \hat{\mathrm{y}}^\star_N||^2_{L^2(\Omega)}}, \tag{2.13}$$

for the mesh parameter $4 \le N \le 6$. The results of Table 2.2 suggest that the rate of convergence is $O(h)$ for both discretizations; see [10]. In Table 2.3 we show

$$\mathcal{E}_N = \frac{||\mathrm{y}^\star_N - \mathrm{y}^\star_{N-1}||^2_{H^1(\Omega)}}{||\mathrm{y}^\star_{N+1} - \mathrm{y}^\star_N||^2_{H^1(\Omega)}} \quad \text{and} \quad \hat{\mathcal{E}}_N = \frac{||\hat{\mathrm{y}}^\star_N - \hat{\mathrm{y}}^\star_{N-1}||^2_{H^1(\Omega)}}{||\hat{\mathrm{y}}^\star_{N+1} - \hat{\mathrm{y}}^\star_N||^2_{H^1(\Omega)}}. \tag{2.14}$$

In Table 2.4 we show

$$\mathbb{E}_N = \frac{||\lambda_N - \lambda_{N-1}||^2_{L^2(\Gamma)}}{||\lambda_{N+1} - \lambda_N||^2_{L^2(\Gamma)}} \quad \text{and} \quad \hat{\mathbb{E}}_N = \frac{||\hat{\lambda}_N - \hat{\lambda}_{N-1}||^2_{L^2(\Gamma)}}{||\hat{\lambda}_{N+1} - \hat{\lambda}_N||^2_{L^2(\Gamma)}}, \tag{2.15}$$

where λ_N, $\hat{\lambda}_N$ are discrete Neumann controls data associated to the mesh parameter $h = 2^{-N}$.

	$\alpha = 0$ and $\beta = 0$	$\alpha = 0$ and $\beta = 0$	$\alpha = 0$ and $\beta = 0$
N ↓	ϵ_N	$\hat{\epsilon}_N$	$\epsilon_N - \hat{\epsilon}_N$
3	9.9168e-003	9.4051e-003	5.1170e-004
4	7.3417e-003	7.3401e-003	1.6000e-006
5	6.6331e-003	6.6343e-003	-1.1857e-006
6	6.5700e-003	6.5700e-003	-4.0797e-008
7	6.6515e-003	6.6515e-003	2.1648e-008

Table 2.1: $\epsilon_N = \frac{1}{2}\|\mathrm{y}_N^\star - \mathrm{y}_*\|^2_{L^2(\Omega)}$ and $\hat{\epsilon}_N = \frac{1}{2}\|\hat{\mathrm{y}}_N^\star - \mathrm{y}_*\|^2_{L^2(\Omega)}$ when the mesh parameter is 2^{-N}.

	$\alpha = 0$ and $\beta = 0$	$\alpha = 0$ and $\beta = 0$
N ↓	E_N	$\hat{E}_N$
4	3.2024	4.0003
5	3.5438	3.5516
6	3.7786	3.7728

Table 2.2: $E_N = \frac{\|\mathrm{y}_N^\star - \mathrm{y}_{N-1}^\star\|^2_{L^2(\Omega)}}{\|\mathrm{y}_{N+1}^\star - \mathrm{y}_N^\star\|^2_{L^2(\Omega)}}$ and $\hat{E}_N = \frac{\|\hat{\mathrm{y}}_N^\star - \hat{\mathrm{y}}_{N-1}^\star\|^2_{L^2(\Omega)}}{\|\hat{\mathrm{y}}_{N+1}^\star - \hat{\mathrm{y}}_N^\star\|^2_{L^2(\Omega)}}$ when the mesh parameter is 2^{-N}.

	$\alpha = 0$ and $\beta = 0$	$\alpha = 0$ and $\beta = 0$
N ↓	$\mathcal{E}_N$	$\hat{\mathcal{E}}_N$
4	3.3385	3.6545
5	2.6552	2.4912
6	1.1566	1.1534

Table 2.3: $\mathcal{E}_N = \frac{\|\mathrm{y}_N^\star - \mathrm{y}_{N-1}^\star\|^2_{H^1(\Omega)}}{\|\mathrm{y}_{N+1}^\star - \mathrm{y}_N^\star\|^2_{H^1(\Omega)}}$ and $\hat{\mathcal{E}}_N = \frac{\|\hat{\mathrm{y}}_N^\star - \hat{\mathrm{y}}_{N-1}^\star\|^2_{H^1(\Omega)}}{\|\hat{\mathrm{y}}_{N+1}^\star - \hat{\mathrm{y}}_N^\star\|^2_{H^1(\Omega)}}$ when the mesh parameter is 2^{-N}.

	$\alpha = 0$ and $\beta = 0$	$\alpha = 0$ and $\beta = 0$
N ↓	$\mathbb{E}_N$	$\hat{\mathbb{E}}_N$
4	2.2213	2.3426
5	2.1935	2.2124
6	1.9198	1.9468

Table 2.4: $\mathbb{E}_N = \frac{\|\lambda_N - \lambda_{N-1}\|^2_{L^2(\Gamma)}}{\|\lambda_{N+1} - \lambda_N\|^2_{L^2(\Gamma)}}$ and $\hat{\mathbb{E}}_N = \frac{\|\hat{\lambda}_N - \hat{\lambda}_{N-1}\|^2_{L^2(\Gamma)}}{\|\hat{\lambda}_{N+1} - \hat{\lambda}_N\|^2_{L^2(\Gamma)}}$ when the mesh parameter is 2^{-N}.

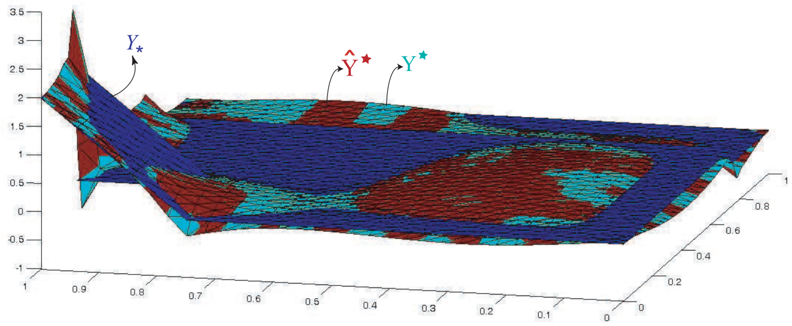


Figure 2.2: Target function $Y_*(x, y) = 8(y - 3/4)$ if $(x, y) \in [0, 1/4] \times [3/4, 1]$ and $Y_*(x, y) = -1/4$ if $(x, y) \in [3/4, 1] \times [0, 1]$ (blue), the discrete solution $\mathrm{Y}^\star$ (green) of the control associated to continuous at the corners discretization and the discrete solution $\hat{\mathrm{Y}}^\star$ (red) of the control associated to discontinuous at the corners discretization. The parameter of regularization $\beta = 0$ and the mesh parameter is $N = 5$.

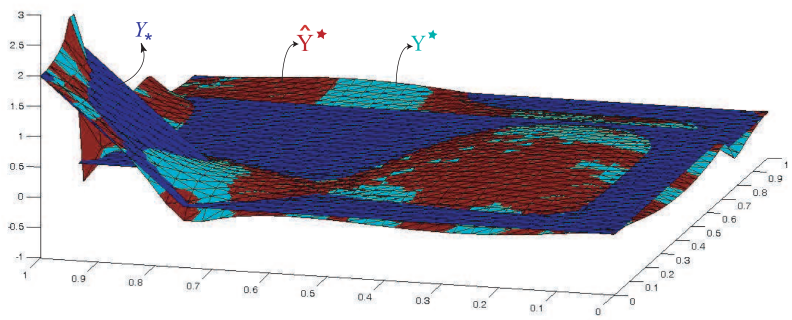


Figure 2.3: Target function $Y_*(x, y) = 8(y - 3/4)$ if $(x, y) \in [0, 1/4] \times [3/4, 1]$ and $Y_*(x, y) = -1/4$ if $(x, y) \in [3/4, 1] \times [0, 1]$ (blue), the discrete solution $\mathrm{Y}^\star$ (green) of the control associated to continuous at the corners discretization and the discrete solution $\hat{\mathrm{Y}}^\star$ (red) of the control associated to discontinuous at the corners discretization. The parameter of regularization $\beta = 10^{-6}$ and the mesh parameter is $N = 5$.

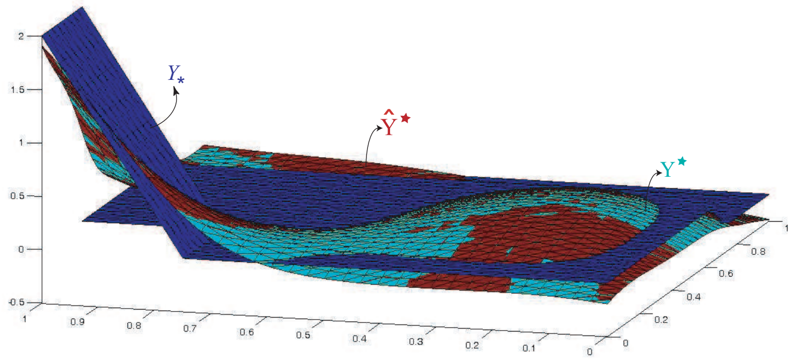


Figure 2.4: Target function $Y_*(x,y) = 8(y - 3/4)$ if $(x,y) \in [0,1/4] \times [3/4,1]$ and $Y_*(x,y) = -1/4$ if $(x,y) \in [3/4,1] \times [0,1]$ (blue), the discrete solution $\mathrm{Y}^\star$ (green) of the control associated to continuous at the corners discretization and the discrete solution $\hat{\mathrm{Y}}^\star$ (red) of the control associated to discontinuous at the corners discretization. The parameter of regularization $\beta = 10^{-3}$ and the mesh parameter is $N = 5$.

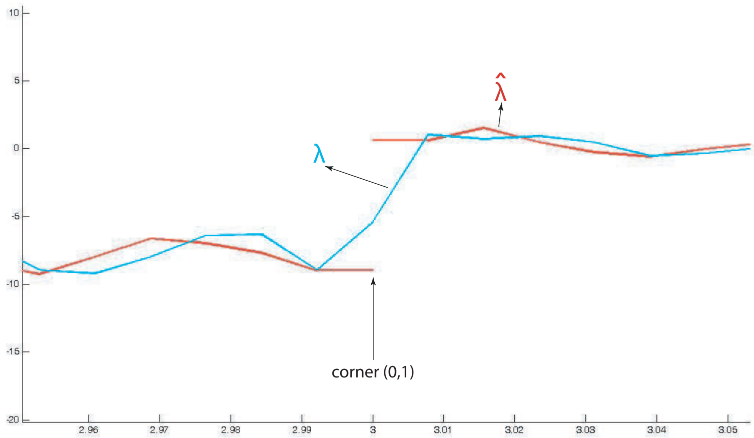


Figure 2.5: Neumann boundary controls data, λ and $\hat{\lambda}$, next to the corner (0,1). Here $h = 2^{-7}$, $\alpha = 0$ and $\beta = 0$.

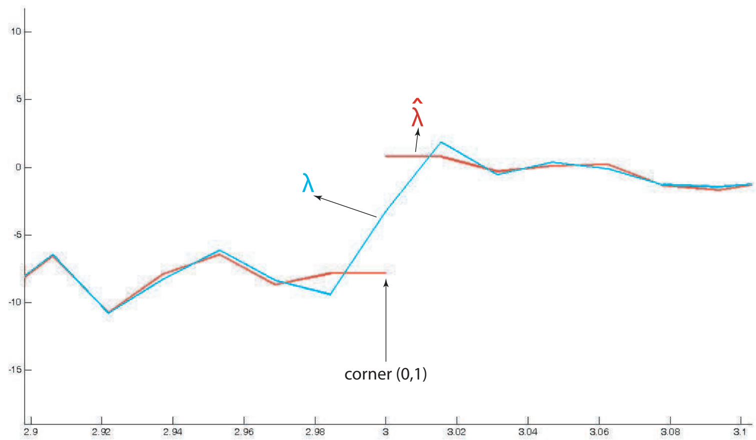


Figure 2.6: Neumann boundary controls data, λ and $\hat{\lambda}$, next to the corner (0,1). Here $h = 2^{-6}$, $\alpha = 0$ and $\beta = 0$.

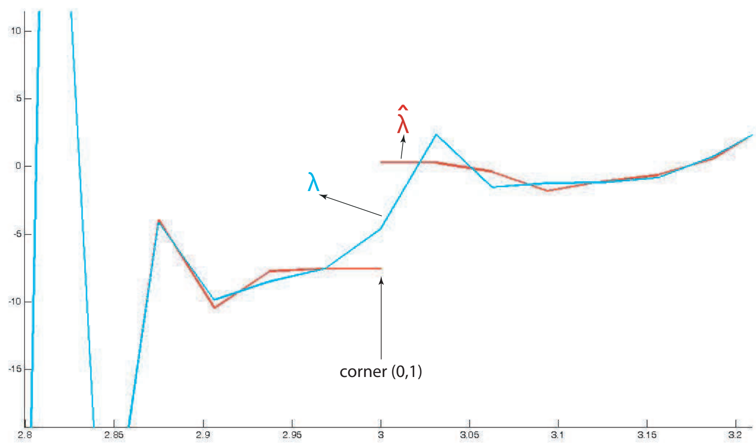


Figure 2.7: Neumann boundary controls data, λ and $\hat{\lambda}$, next to the corner (0,1). Here $h = 2^{-5}$, $\alpha = 0$ and $\beta = 0$.

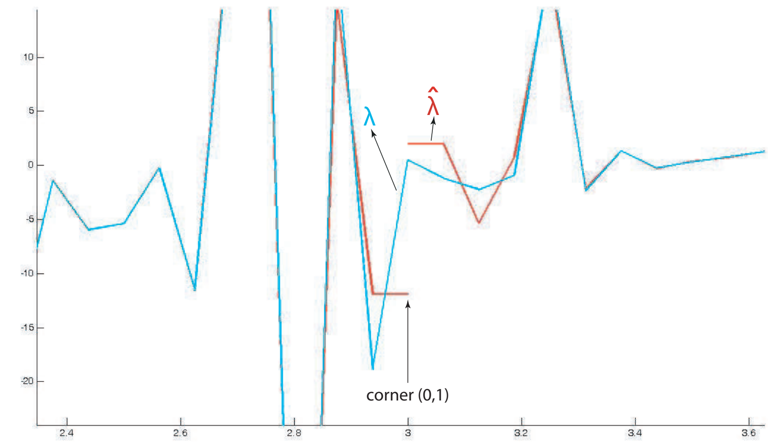


Figure 2.8: Neumann boundary controls data, λ and $\hat{\lambda}$, next to the corner (0,1). Here $h = 2^{-4}$, $\alpha = 0$ and $\beta = 0$.

3
The Steklov-Poincaré Operator

In this chapter we study the Steklov-Poincaré operator associated to the Laplacian. This operator, denoted by $\mathcal{S}$, maps Dirichlet data to Neumann data (D2N). We divide this chapter in three sections: in the first section we introduce some assumptions and notations. In Section 3.2 we study the operator $\mathcal{S}$ on Sobolev spaces of order 1 and -1. Finally, in the last and most important section, we consider the operator $\mathcal{S}$ on Sobolev spaces of order $3/2$ and $-3/2$. In this section we prove Theorem 3.10, which states that the $L^2(\Omega)$-norm of the harmonic extension function y_λ, with Neumann data λ, is equivalent to the $H^{-3/2}_{t,00}(\Gamma)$-norm of its Neumann data λ.

3.1 Assumptions and Notations

Let $\Omega \subset \mathbb{R}^2$ be a convex polygonal domain. We denote each face of $\Gamma := \partial\Omega$ by $\overline{\Gamma}_k$ where Γ_k is an open segment where $1 \leq k \leq K$ for some integer K. The segments are numbered in such a way that Γ_{k+1} follows Γ_k according to an anticlockwise orientation; see Figure 3.1. The vertex which is the end point of Γ_k and Γ_{k+1} is denoted by δ_k. Let ω_k be the interior angle at δ_k . The outward normal (resp. tangent) vector on Γ_k is denoted by $\boldsymbol{\eta}_k$ (resp. $\boldsymbol{\tau}_k$) . We identify Γ_{K+1} with Γ_1 and δ_K with δ_0.

Let $x_k(t)$ be the arc-length parameterization of $O_{\delta_k} \cap \Gamma$, where O_{δ_k} is a neighborhood of δ_k. For small enough $|t|$, (say $|t| < \theta_k$), we let $x_k(t)$ to be the distance of t to δ_k along Γ where $x_k(t) \in \Gamma_k$ when $t < 0$ and $x_k(t) \in \Gamma_{k+1}$ when $t > 0$.

We say that two functions φ_k and φ_{k+1} defined on Γ_k and Γ_{k+1}, respectively, are equivalent at δ_k, and write $\varphi_k \equiv \varphi_{k+1} \quad \text{at} \quad \delta_k$, if

$$\int_0^{\theta_k} \frac{|\varphi_k(x_k(-t)) - \varphi_{k+1}(x_k(+t))|^2}{t} \mathrm{dt} < \infty. \tag{3.1}$$

If $\varphi_k \in H^{1/2}(\Gamma_k)$, $\varphi_{k+1} \in H^{1/2}(\Gamma_{k+1})$ and $\varphi_k \equiv \varphi_{k+1}$ at δ_k, and defining $\varphi := \varphi_k$ on Γ_k and $\varphi := \varphi_{k+1}$ on Γ_{k+1}, it follows that φ belongs to $H^{1/2}(\Gamma_k \cup \Gamma_{k+1})$.

Given a function $u \in H^{1/2}(\Gamma)$, define

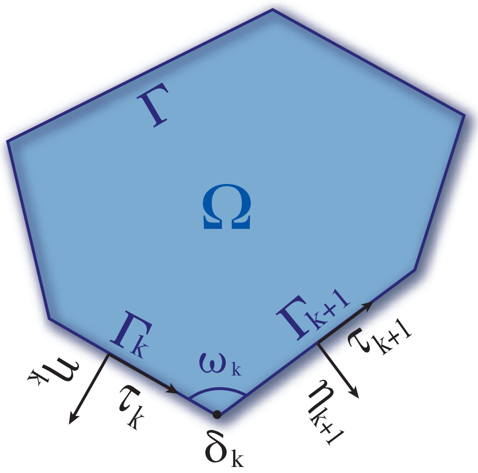


Figure 3.1: A convex polygonal domain in $\mathbb{R}^2$.

$$\mathcal{S}u = \gamma \frac{\partial y}{\partial \eta}, \tag{3.2}$$

where $y \in H^1(\Omega)$ is the solution of the Dirichlet problem

$$\begin{cases} -\Delta y = 0 & \text{in } \Omega, \\ \gamma y = u & \text{on } \Gamma. \end{cases} \tag{3.3}$$

Here, and in the following, γ denotes the trace operator on Γ and $\partial y/\partial \eta$ denotes the normal derivative on Γ. In order to define adequately the operators presented above, we introduce few results about trace operators, harmonic extensions and Sobolev spaces. We use standard definitions and notations of the Sobolev spaces; see e.g. [18].

We next study the action of the operator $\mathcal{S}$ in Sobolev spaces of order 1 and -1.

3.2 The Operator $\mathcal{S}$ on Sobolev Spaces of Order 1 and -1

In this section we study some properties of the Steklov-Poincaré operator $\mathcal{S}$ on Sobolev spaces of order 1 and -1. For this end we need the next two theorems, which are slightly modifications of the Theorems 2.1, 2.2, 2.3 and 2.4 from Necas [28]. See [27] for more general statements of these theorems. For a broader discussion on this subject, see also [20, 11, 9, 26, 27].

Theorem 3.1 *Let Ω be a domain with Lipschitz boundary, $f \in L^2(\Omega)$, $\lambda \in L^2(\Gamma)$ and $c \in \mathbb{R}$, and assume that*

$$\int_\Omega f\, dx + \int_\Gamma \lambda\, ds = 0. \tag{3.4}$$

Then there exists a unique solution of

$$\begin{cases} -\Delta y = f & in \quad \Omega, \\ \gamma \dfrac{\partial y}{\partial \eta} = \lambda & on \quad \Gamma, \end{cases} \tag{3.5}$$

satisfying $\int_\Gamma \gamma\, y\, dx = c$*, where* c *is a fixed constant. Furthermore*

$$\|\gamma y\|_{H^1(\Gamma)} \preceq \left(\|f\|_{L^2(\Omega)} + \|\lambda\|_{L^2(\Gamma)}\right), \tag{3.6}$$

where γ *is the trace operator on* Γ*.*

When $f = 0$ and $c = 0$ in Theorem 3.1, denote

$$\begin{aligned} \mathcal{K}: \quad L^2(\Gamma)\backslash\mathbb{R} &\longmapsto H^1(\Gamma)\backslash\mathbb{R} \\ \lambda &\longmapsto \gamma y. \end{aligned} \tag{3.7}$$

Note that we have used quotient spaces since $c = 0$ and the compatibility relation (3.4) holds.

Theorem 3.2 *Let* Ω *be as in the previous theorem and* y *be the solution of the Dirichlet problem with right-hand data* $f \in L^2(\Omega)$ *and Dirichlet data* $u \in H^1(\Gamma)$*. Then* $\gamma \dfrac{\partial y}{\partial \eta} \in L^2(\Gamma)$ *and*

$$\left\|\gamma\, \frac{\partial y}{\partial \eta}\right\|_{L^2(\Gamma)} \preceq (\|f\|_{L^2(\Omega)} + \|u\|_{H^1(\Gamma)}). \tag{3.8}$$

We denote by

$$\begin{aligned} \mathcal{S}: \quad H^1(\Gamma) &\longmapsto L^2(\Gamma)\backslash\mathbb{R} \\ u &\longmapsto \gamma \frac{\partial y}{\partial \eta}. \end{aligned} \tag{3.9}$$

We can identify $H^1(\Gamma)\backslash\mathbb{R}$ with the set $\mathcal{X} = \{u \in H^1(\Gamma); \int_\Gamma u\, ds = 0\}$. In addition, the standard topology of $H^1(\Gamma)/\mathbb{R}$ is equivalent to the $H^1(\Gamma)$-induced topology of $\mathcal{X}$. Indeed, given a class $u + \mathbb{R} \in H^1(\Gamma)/\mathbb{R}$, and choosing $\tilde{u} := u - \int_\Gamma u\, ds$ to represent the class $u + \mathbb{R} = \tilde{u} + \mathbb{R} \in H^1(\Gamma)/\mathbb{R}$, then

$$\begin{aligned} \|\tilde{u}\|^2_{H^1(\Gamma)/\mathbb{R}} &= \inf_{c\in\mathbb{R}} \|\tilde{u} - c\|^2_{H^1(\Gamma)} \\ &= \inf_{c\in\mathbb{R}} \left(\|\tilde{u} - c\|^2_{L^2(\Gamma)} + |\tilde{u}|^2_{H^1(\Gamma)}\right) \\ &= \inf_{c\in\mathbb{R}} \left(\|\tilde{u}\|^2_{L^2(\Gamma)} + |\tilde{u}|^2_{H^1(\Gamma)} - 2\, c \int_\Gamma \tilde{u}\, ds + c^2 \int_\Gamma ds\right) \\ &= \inf_{c\in\mathbb{R}} \left(\|\tilde{u}\|^2_{L^2(\Gamma)} + |\tilde{u}|^2_{H^1(\Gamma)} + c^2 \int_\Gamma ds\right) \\ &= \|\tilde{u}\|^2_{H^1(\Gamma)}. \end{aligned}$$

In the same way we can identify $L^2(\Gamma)\backslash\mathbb{R}$ with the set $\left\{\lambda \in L^2(\Gamma);\ \int_\Gamma \lambda\, ds = 0\right\}$ and $H^2(\Omega)/\mathbb{R}$ with the set $\left\{y \in H^2(\Omega);\ \int_\Omega y\, ds = 0\right\}$. Thus, the Theorems 3.1 and 3.2, with $f = 0$ and $c = 0$, imply that $\mathcal{K}$ is a pseudo-inverse of $\mathcal{S}$ denoted by $\mathcal{S}^\dagger$. Furthermore $\mathcal{S}^\dagger$ and $\mathcal{S} : H^1(\Gamma)\backslash\mathbb{R} \longmapsto L^2(\Gamma)\backslash\mathbb{R}$ are norm-isomorphisms, that is,

$$\|u\|_{H^1(\Gamma)} \preceq \|\mathcal{S}u\|_{L^2(\Gamma)} \preceq \|u\|_{H^1(\Gamma)}, \quad \text{for all} \quad u \in H^1(\Gamma)\backslash\mathbb{R}. \tag{3.10}$$

We can resume this discussion with the following theorem:

Theorem 3.3 *Suppose* Ω *is a convex polygonal domain. The operator*

$$\mathcal{S} : H^1(\Gamma)\backslash\mathbb{R} \longmapsto L^2(\Gamma)\backslash\mathbb{R},$$

is a norm isomorphism.

Now we are ready to prove an important theorem for $\mathcal{S}$.

Theorem 3.4 *Let* Ω *be a convex polygonal domain. Then* $\mathcal{S}^\dagger$ *is an isomorphism from* $H^{-1}(\Gamma)\backslash\mathbb{R}$ *onto* $L^2(\Gamma)\backslash\mathbb{R}$.

Proof: By the symmetry of the $\mathcal{S}^\dagger$ and Cauchy inequality we have for all $\lambda \in \hat{H}^{-1/2}(\Gamma)\backslash\mathbb{R}$ and $\mu \in L^2(\Gamma)\backslash\mathbb{R}$

$$\langle\lambda, \mathcal{S}^\dagger\mu\rangle \;=\; (\mathcal{S}^\dagger\lambda, \mu)_{L^2(\Gamma)} \;\leq\; \|\mathcal{S}^\dagger\lambda\|_{L^2(\Gamma)}\|\mu\|_{L^2(\Gamma)}, \tag{3.11}$$

therefore, by Theorem 3.3 and inequality (3.11) we obtain

$$\|\lambda\|_{H^{-1}} \;=\; \sup_{0\neq\mu\in H^1(\Gamma)\backslash\mathbb{R}} \frac{\langle\lambda,\mu\rangle}{\|\mu\|_{H^1(\Gamma)}} \;=\; \sup_{0\neq\mu\in L^2(\Gamma)\backslash\mathbb{R}} \frac{\langle\lambda, \mathcal{S}^\dagger\mu\rangle}{\|\mathcal{S}^\dagger\mu\|_{H^1(\Gamma)}} \;\preceq\; \|\mathcal{S}^\dagger\lambda\|_{L^2(\Gamma)}, \tag{3.12}$$

where to obtain the last inequality we have used the inequality $\|\mathcal{S}^\dagger\mu\|_{H^1(\Gamma)} \succeq \|\mu\|_{L^2(\Gamma)}$; see Theorem 3.3.

For the other side, consider $\mu \in \hat{H}^{1/2}(\Gamma)\backslash\mathbb{R}$ and y the solution of

$$\begin{cases} -\Delta y &= 0 \quad \text{in} \quad \Omega, \\ \gamma\dfrac{\partial y}{\partial\eta} &= \mu \quad \text{on} \quad \Gamma. \end{cases} \tag{3.13}$$

By Theorem 3.3,

$$\|\gamma y\|_{H^1(\Gamma)} = \|\mathcal{S}^\dagger\mu\|_{H^1(\Gamma)} \preceq \|\mu\|_{L^2(\Gamma)}, \tag{3.14}$$

furthermore,

$$\int_\Omega \nabla y \cdot \nabla w\, dx = \int_\Gamma (\mu)\,(\gamma w)\, ds, \quad \forall w \in H^1(\Omega).$$

Given $\lambda \in L^2(\Gamma)\mathbb{R}$ choose w such that

$$\begin{cases} -\Delta w &= 0 \quad \text{in} \quad \Omega, \\ \gamma \dfrac{\partial w}{\partial \eta} &= \lambda \quad \text{on} \quad \Gamma. \end{cases} \tag{3.15}$$

Then

$$\int_\Gamma (\lambda)\,(\gamma y)\,ds = \int_\Gamma (\mu)\,(\gamma w)\,ds,$$

and therefore,

$$\begin{aligned} \|\mu\|_{H^{-1}(\Gamma)} &= \sup_{0\neq\varphi\in H^1(\Gamma)\setminus\mathbb{R}} \frac{\langle \mu, \varphi\rangle}{\|\varphi\|_{H^1(\Gamma)}} \geq \frac{\int_\Gamma (\mu)\,(\gamma w)\,ds}{\|\gamma w\|_{H^1(\Gamma)}} = \frac{\int_\Gamma (\lambda)\,(\gamma y)\,ds}{\|\mathcal{S}^\dagger \lambda\|_{H^1(\Gamma)}} \\ &\geq \frac{\int_\Gamma (\lambda)\,(\gamma y)\,ds}{\|\lambda\|_{L^2(\Gamma)}}. \end{aligned} \tag{3.16}$$

Since λ is arbitrary we have

$$\|\mu\|_{H^{-1}(\Gamma)} \geq \|\gamma y\|_{L^2(\Gamma)} = \|\mathcal{S}^\dagger \mu\|_{L^2(\Gamma)}. \tag{3.17}$$

By density we can extend continuously $\mathcal{S}^\dagger$ to $H^{-1}(\Gamma)$. □

3.3 The Operator $\mathcal{S}$ on Sobolev Spaces of Order 3/2 and −3/2

In this section we study some properties of the Steklov-Poincaré operator $\mathcal{S}$ on Sobolev spaces of order 3/2 and −3/2. The goal of this section is proving the Theorem 3.10, main result of this chapter. For this end we need the next two theorems. See [19] page 31, and pages 45, 54–59, respectively, for a complete proof.

Theorem 3.5 *Let $\Omega \subset \mathbb{R}^2$ be a convex polygonal domain. Assume that $\lambda_k \in H^{1/2}(\Gamma_k)$ for all $k \in \{1, ..., K\}$ and $\sum_k \int_{\Gamma_k} \lambda_k ds = 0$. Then, there exists a unique function $y \in H^2(\Omega)\setminus\mathbb{R}$ such that*

$$\begin{cases} -\Delta y &= 0 \quad in \quad \Omega, \\ \gamma \dfrac{\partial y}{\partial \eta_k} &= \lambda_k \quad on \quad \Gamma_k \subset \Gamma. \end{cases} \tag{3.18}$$

Theorem 3.6 (Trace on Polygon) *Let Ω be a polygonal domain of $\mathbb{R}^2$. Then the mapping*

$$y \longmapsto \left\{ (u_k := \gamma\, y\,,\ \lambda_k := \gamma \frac{\partial y}{\partial \eta_k}), \ \ 1 \leq k \leq K \right\}$$

is linear continuous from $H^2(\Omega)$ onto the subspace of

$$\prod_{1\le k\le K} H^{3/2}(\Gamma_k)\times H^{1/2}(\Gamma_k)$$

defined by: $\left\{(u_1,\lambda_1,\dots,u_K,\lambda_K)\in\prod_{1\le k\le K} H^{3/2}(\Gamma_k)\times H^{1/2}(\Gamma_k)\right\}$ *satisfying the following conditions:*

1. $u_k(\delta_k)=u_{k+1}(\delta_k)$ *for every* k *,*

2. $u'_k\equiv-\cos(\omega_k)\,u'_{k+1}+\sin(\omega_k)\,\lambda_{k+1}$ *at* δ_k *for every* k*,*

3. $\lambda_k\equiv-\cos(\omega_k)\,\lambda_{k+1}-\sin(\omega_k)\,u'_{k+1}$ *at* δ_k *for every* k*.*

Here $(u_{K+1},\lambda_{K+1}):=(u_1,\lambda_1)$*.*

Inspired in the Theorem 3.6 we define the space

$$\hat{H}^{3/2}(\Gamma)=\left\{u\in L^2(\Gamma);u\in H^{3/2}(\Gamma_k)\quad\text{and}\quad u\quad\text{is continuous on}\ \ \Gamma\right\}. \tag{3.19}$$

The following lemma characterizes $\hat{H}^{3/2}(\Gamma)$.

Lemma 3.1 *Define*

$$\mathcal{A}_1=\{u\in H^1(\Gamma);u\in H^{3/2}(\Gamma_k),\ 1\le k\le K\}, \tag{3.20}$$

and

$$\mathcal{A}_2=\{u\in H^{1/2}(\Gamma);u\in H^{3/2}(\Gamma_k),\ 1\le k\le K\}. \tag{3.21}$$

Then we have

$$\mathcal{A}_1=\hat{H}^{3/2}(\Gamma)=\mathcal{A}_2,$$

where the space $\hat{H}^{3/2}(\Gamma)$ *is given by (3.19).*

Proof: It is enough to consider $\Gamma_1=(0,1)$ and $\Gamma_2=(1,2)$, and make the analysis in a neighborhood of the corner δ_1 associated with $x=1$. By Sobolev embedding theorem we have $\mathcal{A}_1\subset\hat{H}^{3/2}(\Gamma)$; see [28]. In order to prove $\hat{H}^{3/2}(\Gamma)\subset\mathcal{A}_1$ consider the functions $u_1\in H^1(]0,1[)$ and $u_2\in H^1(]1,2[)$. By Sobolev embedding theorem, there exist the limits $\lim_{x\to1^-}u_1(x)=a\in\mathbb{R}$ and $\lim_{x\to1^+}u_2(x)=b\in\mathbb{R}$. Hence, define the function $u(x)$ by

$$u(x)=\begin{cases}u_1(x) & \text{if}\quad x\in]0,1[\\ u_2(x) & \text{if}\quad x\in]1,2[\\ u(1):=a.\end{cases} \tag{3.22}$$

To conclude that $\hat{H}^{3/2}(\Gamma) \subset \mathcal{A}_1$ it is sufficient to show that if $a = b$ then u belongs to $H^1(]0,2[)$. To see this, note that $\frac{d}{dx}u(x)$ is given by

$$\frac{d}{dx}u(x) = \left(\lim_{x\to 1^-} u_1(x) - \lim_{x\to 1^+} u_2(x)\right)\boldsymbol{\delta}(x-1) + \frac{d}{dx}u_1(x) + \frac{d}{dx}u_2(x)$$

in the sense of distributions. Here $\boldsymbol{\delta}$ is the *Dirac* function. Since $a = b$, $\dfrac{d}{dx}u(x) \in L^2(]0,2[)$, it follows that $u \in H^1(]0,2[)$.

Clearly $\mathcal{A}_1 \subset \mathcal{A}_2$, therefore, we have $\hat{H}^{3/2}(\Gamma) \subset \mathcal{A}_2$. In order to conclude the proof it is sufficient to show that $\mathcal{A}_2 \subset \hat{H}^{3/2}(\Gamma)$, that is, if u defined in (3.22) belongs to $H^{1/2}(]0,2[)$ then u is continuous, that is, $a = b$. Let us prove by contradiction, that is, we claim that if $a \neq b$ then $u \notin H^{1/2}(]0,2[)$. Since u_1 and u_2 are continuous, there exists $\delta > 0$ such that

$$|u_1(x) - a| < \frac{|a-b|}{4} \quad \text{and} \quad |u_2(x) - b| < \frac{|a-b|}{4} \quad \text{if} \quad |x-1| < \delta.$$

By definition $u \notin H^{1/2}(]0,2[)$ is the same as

$$\int_0^2 \int_0^2 \frac{|u(x) - u(y)|^2}{|x-y|^2} dx\, dy > \infty.$$

Since

$$\int_{1-\delta}^{1} \int_{1}^{1+\delta} \frac{|u_1(x) - u_2(y)|^2}{|x-y|^2} dx\, dy > \frac{|a-b|^2}{4} \int_{1-\delta}^{1} \int_{1}^{1+\delta} \frac{1}{|x-y|^2} dx\, dy > \infty,$$

then $u \notin H^{1/2}(]0,2[)$. The proof is complete. □

According to Lemma 3.1, a natural norm on $\hat{H}^{3/2}(\Gamma)$ is:

$$||u||^2_{\hat{H}^{3/2}(\Gamma)} := ||u||^2_{H^{1/2}(\Gamma)} + \sum_{k=1}^{K} \Big|\Big| \frac{\partial u}{\partial \tau_k} \Big|\Big|^2_{H^{1/2}(\Gamma_k)}. \tag{3.23}$$

We denote by $\hat{H}^{-3/2}(\Gamma)$ the standard dual space of $\hat{H}^{3/2}(\Gamma)$ endowed with the norm

$$||\lambda||_{\hat{H}^{-3/2}} = \sup_{0\neq\varphi\in\hat{H}^{3/2}(\Gamma)} \frac{\langle \lambda, \varphi \rangle}{||\varphi||_{\hat{H}^{3/2}(\Gamma)}}. \tag{3.24}$$

The next two theorems are particular cases of the results found in [19], Ch. 2 pages 45, and 54–59.

Theorem 3.7 *Let $\Omega \subset \mathbb{R}^2$ be a convex polygonal domain. The Dirichlet extension operator*

$$\begin{aligned} \mathcal{E}_{\mathcal{D}} \ : \ \hat{H}^{3/2}(\Gamma)\backslash\mathbb{R} &\longmapsto H^2(\Omega)\backslash\mathbb{R} \\ u &\longmapsto \mathcal{E}_{\mathcal{D}}\, u := y \end{aligned} \tag{3.25}$$

defined via

$$\begin{cases} -\Delta y = 0 & in \quad \Omega, \\ \gamma y = u & on \quad \Gamma, \end{cases}$$

is continuous and injective.

Denote

$$\widehat{H}^{1/2}(\Gamma) := \{\lambda \in L^2(\Gamma); \lambda \in H^{1/2}(\Gamma_k),\ 1 \le k \le K\} \tag{3.26}$$

endowed with the natural norm

$$\|\lambda\|^2_{\widehat{H}^{1/2}(\Gamma)} := \sum_{k=1}^{K} \|\lambda\|^2_{H^{1/2}(\Gamma_k)}.$$

We denote by $\widehat{H}^{-1/2}(\Gamma)$ the standard dual of $\widehat{H}^{1/2}(\Gamma)$.

Theorem 3.8 *Let $\Omega \subset \mathbb{R}^2$ be a convex polygonal domain. The Neumann extension operator*

$$\begin{aligned} \mathcal{E}_{\mathcal{N}} \ : \ \hat{H}^{1/2}(\Gamma)\backslash\mathbb{R} &\longmapsto H^2(\Omega)\backslash\mathbb{R} \\ \lambda &\longmapsto \mathcal{E}_{\mathcal{N}}\, \lambda := y \end{aligned} \tag{3.27}$$

defined via

$$\begin{cases} -\Delta y = 0 & in \quad \Omega, \\ \gamma \dfrac{\partial y}{\partial \eta} = \lambda & on \quad \Gamma, \end{cases}$$

is continuous and injective.

Combining Theorems 3.7, 3.8 and 3.6 we deduce the following result:

Theorem 3.9 *Let $\Omega \subset \mathbb{R}^2$ be a convex polygonal domain. The operator*

$$\begin{aligned} \mathcal{S} \ : \ \hat{H}^{3/2}(\Gamma) &\longmapsto \hat{H}^{1/2}(\Gamma)\backslash\mathbb{R} \\ u &\longmapsto \gamma \frac{\partial}{\partial \eta}(\mathcal{E}_{\mathcal{D}}\, u), \end{aligned} \tag{3.28}$$

is linear surjective and its kernel is one dimensional; ker$(\mathcal{S})$ = span$\{1\}$.

For all $\lambda \in \hat{H}^{1/2}(\Gamma)\backslash\mathbb{R}$ there exists a unique $u \in \hat{H}^{3/2}(\Gamma)\backslash\mathbb{R}$ such that $\mathcal{S}u = \lambda$. We denote $\mathcal{S}^{\dagger}\lambda := u$. The following technical proposition will be useful in the proof of the Theorem 3.10 below.

Proposition 3.1 *Let $\lambda \in L^2(\Gamma)\backslash\mathbb{R}$ and $X \subset L^2(\Gamma)\backslash\mathbb{R}$ a Hilbert space densely defined. Then there exists $\rho \in X$ such that $\|\rho\|_X = 1$ and*

$$\|\lambda\|_{X'} \leq 2\int_\Gamma \lambda\rho\, ds. \tag{3.29}$$

Proof: By definition

$$\|\lambda\|_{X'} = \sup_{\|\rho\|_X=1}\int_\Gamma \lambda\rho\, ds,$$

then there exists ρ with $\|\rho\|_X = 1$ satisfying

$$\int_\Gamma \lambda\rho\, ds \geq \frac{\|\lambda\|_{X'}}{2}.$$

and the proposition follows. □

We now define some spaces and norms. Define $C^\infty_{t,00}(\Gamma_k) := \{v \in C^\infty(\Gamma_k); \partial v/\partial\tau_k \in C^\infty_0(\Gamma_k)\}$. Denote $H^2_{t,00}(\Gamma_k)$ by the closure of $C^\infty_{t,00}(\Gamma_k)$ in the $H^2(\Gamma_k)$-norm, that is,

$$H^2_{t,00}(\Gamma_k) := \{v \in H^2(\Gamma_k); \frac{\partial v}{\partial\tau_k}(\delta_{k-1}) = \frac{\partial v}{\partial\tau_k}(\delta_k) = 0\}. \tag{3.30}$$

We now define the following spaces using interpolation theory; see [3, 23]:

1. $H^{1/2}_{00}(\Gamma_k) := \left[H^1_0(\Gamma_k), L^2(\Gamma_k)\right]_{1/2}$,
2. $H^{3/2}_{t,00}(\Gamma_k) := \left[H^2_{t,00}(\Gamma_k), H^1(\Gamma_k)\right]_{1/2}$,
3. $H^{-3/2}_{t,00}(\Gamma_k) := (H^{3/2}_{t,00}(\Gamma_k))'$.

By Theorem 14.3 page 107 of [23] we can characterize $H^{3/2}_{t,00}(\Gamma_k)$ as follows

$$\begin{aligned}\left[H^2_{t,00}(\Gamma_k), H^1(\Gamma_k)\right]_{1/2} &\overset{Th.14.3}{=} \left\{v \in \left[H^2(\Gamma_k), H^1(\Gamma_k)\right]_{1/2}; \partial v/\partial\tau_k \in \left[H^1_0(\Gamma_k), L^2(\Gamma_k)\right]_{1/2}\right\}\\ &= \left\{v \in H^{3/2}(\Gamma_k); \partial v/\partial\tau_k \in H^{1/2}_{00}(\Gamma_k)\right\}.\end{aligned}$$

Therefore, the natural norm in $H^{3/2}_{t,00}(\Gamma_k)$ is defined by

$$\|v\|^2_{H^{3/2}_{t,00}(\Gamma_k)} := \|v\|^2_{H^{3/2}(\Gamma_k)} + \left\|\frac{\partial v}{\partial\tau_k}\right\|^2_{H^{1/2}_{00}(\Gamma_k)}. \tag{3.31}$$

Define also $H^{3/2}_{t,00}(\Gamma) := H^{1/2}(\Gamma) \cap \prod_{k=1}^K H^{3/2}_{t,00}(\Gamma_k)$ endowed with the norm

$$\|v\|_{H^{3/2}_{t,00}(\Gamma)} := \|v\|^2_{H^{1/2}(\Gamma)} + \sum_{k=1}^K \left\|\frac{\partial v}{\partial\tau_k}\right\|^2_{H^{1/2}_{00}(\Gamma_k)}, \tag{3.32}$$

and $H^{-3/2}_{t,00}(\Gamma)$ its dual endowed with the norm

$$\|v\|^2_{H^{-3/2}_{t,00}(\Gamma)} := \sup_{\varphi\in H^{3/2}_{t,00}(\Gamma)} \frac{(v,\varphi)_{L^2}}{\|\varphi\|^2_{H^{3/2}_{t,00}(\Gamma)}}; \tag{3.33}$$

see Figure 3.2 for illustration of the functions in the space $H^{3/2}_{t,00}(\Gamma)$.

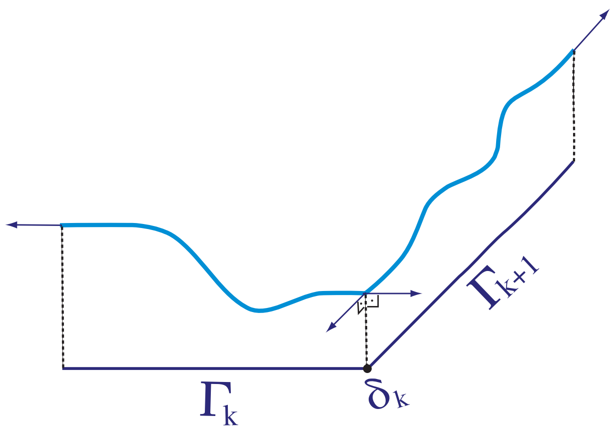


Figure 3.2: Illustration of a function in the space $H^{3/2}_{t,00}(\Gamma)$.

We now present another characterization of the space $H^{3/2}_{t,00}(\Gamma)$. By Theorem 12.3 page 79 of [23] we have $\left[H^1_0(\Gamma_k), H^{-1}(\Gamma_k)\right]_{3/4} = H^{1/2}_{00}(\Gamma_K)$. Using again the Theorem 14.3 of [23] we have

$$\begin{aligned}\left[H^2_{t,00}(\Gamma_k), L^2(\Gamma_k)\right]_{3/4} &= \left\{v \in \left[H^2(\Gamma_k), L^2(\Gamma_k)\right]_{3/4}; \partial v/\partial\tau_k \in \left[H^1_0(\Gamma_k), H^{-1}(\Gamma_k)\right]_{3/4}\right\} \\ &= \left\{v \in H^{3/2}(\Gamma_k); \partial v/\partial\tau_k \in H^{1/2}_{00}(\Gamma_k)\right\} = H^{3/2}_{t,00}(\Gamma_k).\end{aligned}$$

This observation will be important in Chapter 6. Note that, in general, the functions of $H^{3/2}_{t,00}(\Gamma)$ are not zero at the corners of the boundary Γ. In the same way, we can define also $H^{3/2}(\Gamma) := \{v \in L^2(\Gamma); v, \partial v/\partial\tau \in H^{1/2}(\Gamma)\}$. Here $\partial v/\partial\tau$ is the tangential derivative of v on Γ. We equip $H^{3/2}(\Gamma)$ with the norm

$$\|v\|^2_{H^{3/2}(\Gamma)} := \|v\|^2_{H^{1/2}(\Gamma)} + \left\|\frac{\partial v}{\partial\tau}\right\|^2_{H^{1/2}(\Gamma)}. \tag{3.34}$$

Note that $H^{3/2}_{t,00}(\Gamma) \subset H^{3/2}(\Gamma)$ algebraically and topologically.

In the proof of Theorem 3.10, most important result of this chapter, we will use the trace lemma proved next.

Lemma 3.2 *Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain and $w \in H^2(\Omega)$ such that $\gamma\frac{\partial w}{\partial\eta} = 0$ on Γ. Then $\gamma w \in H^{3/2}_{t,00}(\Gamma)$ and $\gamma\frac{\partial w}{\partial\tau_k} \in H^{1/2}_{00}(\Gamma_k)$ for all $k \in \{1, 2, \ldots, K\}$. Moreover, we have*

$$\left\|\gamma\frac{\partial w}{\partial\tau_k}\right\|_{H^{1/2}_{00}(\Gamma_k)} \leq c(\omega_{k-1},\omega_k)\|w\|_{H^2(\Omega)}, \tag{3.35}$$

Where $c(\omega_{k-1},\omega_k) = \max\{1/|\sin(\omega_{k-1})|, 1/|\sin(\omega_k)|\}$.

Proof: Since $w \in H^2(\Omega)$, then for any vector $v \in \mathbb{R}^2$ we have $\frac{\partial w}{\partial v} \in H^1(\Omega)$. By the Trace Theorem we have, $\gamma\frac{\partial w}{\partial v} \in H^{1/2}(\Gamma)$ and

$$\left\|\gamma\frac{\partial w}{\partial v}\right\|_{H^{1/2}(\Gamma)} \leq \|\nabla w\|_{H^1(\Omega)} \leq \|w\|_{H^2(\Omega)}. \tag{3.36}$$

By 3. compatibility condition of Theorem 3.6 we have $\sin(\omega_{k-1})\gamma\frac{\partial w}{\partial\tau_k} \equiv 0$ at δ_{k-1} and $\sin(\omega_k)\gamma\frac{\partial w}{\partial\tau_k} \equiv 0$ at δ_k. Let $\mathrm{E}_0 : H^{1/2}_{00}(\Gamma_k) \longmapsto H^{1/2}(\Gamma)$ be the extension by zero operator. It is known that $\|E_0 v\|_{H^{1/2}(\Gamma)} \asymp \|v\|_{H^{1/2}_{00}(\Gamma_k)}$; see [33, 28]. Using this with (3.36) we obtain

$$\left\|\gamma\frac{\partial w}{\partial\tau_k}\right\|_{H^{1/2}_{00}(\Gamma_k)} \leq \left\|E_0\gamma\frac{\partial w}{\partial\tau_k}\right\|_{H^{1/2}(\Gamma)} \leq \left\|\gamma\frac{\partial w}{\partial\tau_k}\right\|_{H^{1/2}(\Gamma)} \leq \|\nabla w\|_{H^1(\Omega)} \leq \|w\|_{H^2(\Omega)}.$$

This conclude the proof. □

Theorem 3.10 *Let $\Omega \subset \mathbb{R}^2$ be a convex polygonal domain and $\lambda \in \widehat{H}^{1/2}(\Gamma)\backslash\mathbb{R}$. Then the $L^2(\Omega)$-norm of the harmonic extension function y_λ, with Neumann data λ, is equivalent to the $H^{-3/2}_{t,00}(\Gamma)$-norm of its Neumann data λ. More precisely,*

$$\|\lambda\|_{H^{-3/2}_{t,00}(\Gamma)} \leq \|y_\lambda\|_{L^2(\Omega)} \leq \|\lambda\|_{H^{-3/2}_{t,00}(\Gamma)}. \tag{3.37}$$

Proof: Since $\lambda \in \widehat{H}^{1/2}(\Gamma)/\mathbb{R}$ we have $y_\lambda \in H^2(\Omega)/\mathbb{R}$. By Green's formula, (see e.g. [19] page 24), we have for all $w \in H^2(\Omega)$

$$\int_\Omega y_\lambda\,\Delta w\,dx - \int_\Omega w\,\Delta y_\lambda\,dx = \sum_{k=1}^{K}\int_{\Gamma_k}(\gamma\,w)(\lambda)\,ds - \sum_{k=1}^{K}\int_{\Gamma_k}(\gamma\,\frac{\partial w}{\partial\eta_k})(\gamma\,y_\lambda)\,ds. \tag{3.38}$$

Let $w_\lambda \in H^2(\Omega)/\mathbb{R}$ be the solution of

$$\left\{\begin{array}{rcll} \Delta w_\lambda & = & y_\lambda & \text{in} \quad \Omega, \\ \gamma\frac{\partial w_\lambda}{\partial\eta} & = & 0 & \text{on} \quad \Gamma. \end{array}\right. \tag{3.39}$$

The regularity theory, (see [19] pages 54–59), ensures that

$$\|w_\lambda\|_{H^2(\Omega)} \leq \|y_\lambda\|_{L^2(\Omega)}. \tag{3.40}$$

By Lemma 3.2 we have $\gamma w_\lambda \in H^{3/2}_{t,00}(\Gamma)$ and

$$\left\|\gamma w_\lambda\right\|_{H^{3/2}_{t,00}(\Gamma)} \leq \|w_\lambda\|_{H^2(\Omega)}. \tag{3.41}$$

Choosing $w = w_\lambda$ in (3.38) and using (3.40) and (3.41) we obtain

$$\begin{aligned}
\|y_\lambda\|^2_{L^2(\Omega)} &= \int_\Omega (\Delta w_\lambda) y_\lambda \, dx &&= \int_\Gamma \gamma w_\lambda \lambda ds \\
&\leq \|\lambda\|_{H^{-3/2}_{t,00}(\Gamma)} \|\gamma w_\lambda\|_{H^{3/2}_{t,00}(\Gamma)} &&\leq \|\lambda\|_{H^{-3/2}_{t,00}(\Gamma)} \|w_\lambda\|_{H^2(\Omega)} \\
&\leq \|\lambda\|_{H^{-3/2}_{t,00}(\Gamma)} \|y_\lambda\|_{L^2(\Omega)}.
\end{aligned}$$

The last inequality provides the second inequality of (3.37).

We next show the first inequality of (3.37). By Proposition 3.1 we can choose $\rho \in H^{3/2}_{t,00}(\Gamma)\backslash\mathbb{R}$ such that $\|\rho\|_{H^{3/2}_{t,00}(\Gamma)} = 1$ and

$$\|\lambda\|_{H^{-3/2}_{t,00}(\Gamma)} \leq 2 \int_\Gamma \lambda \rho \, ds. \tag{3.42}$$

By Theorem 3.6 and Lemma 3.2, there exists $w \in H^2(\Omega)$ such that $\gamma \dfrac{\partial w}{\partial \eta} = 0$, $\gamma w = \rho$ and

$$\|w\|_{H^2(\Omega)} \leq \|\rho\|_{H^{3/2}_{t,00}(\Gamma)}. \tag{3.43}$$

From (3.38) we get

$$\int_\Gamma \rho \, \lambda \, ds \;=\; \int_\Omega (\Delta w) \, y_\lambda \mathrm{dx} \;\leq\; \|w\|_{H^2(\Omega)} \|y_\lambda\|_{L^2(\Omega)}. \tag{3.44}$$

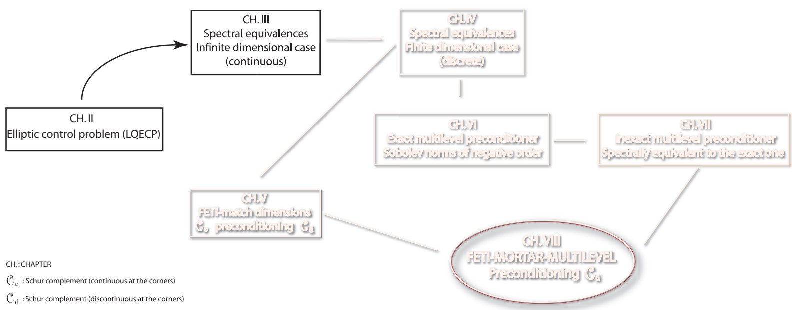
Inserting (3.44) and (3.43) into (3.42), we have the other side inequality of equivalence (3.37), that is,

$$\|\lambda\|_{H^{-3/2}_{t,00}(\Gamma)} \leq 2 \int_\Gamma \lambda \rho \, ds \leq \|y_\lambda\|_{L^2(\Omega)}. \tag{3.45}$$

This complete the proof. □

Theorem 3.10 is used in the proof of Theorem 4.1, which allows us to associate the matrix C_c defined in (1.9) to a linear combination of Sobolev norms.

The diagram below illustrates the steps completed



4
Discrete Equivalences

In this chapter we prove discrete versions of the Theorem 3.10. We consider two discretizations of the Neumann space $H^{-1/2}(\Gamma)$: In Section 4.1 a continuous discretization is considered; and in Section 4.2 we consider a discretization which is discontinuous at the corners of the polygonal domain Ω. The most important result of this chapter Theorem 4.1. This theorem allows us to associate the matrix C_{c} to a linear combination of Sobolev norms.

4.1 A Continuous Discretization of $H^{-1/2}(\Gamma)$

In this section we prove a discrete version of the Theorem 3.10. The discretization considered here is continuous at corners of the polygonal domain $\Omega \subset \mathbb{R}^2$. We use the same notation of the Chapter 2. Remember that $X_h(\Omega) \subset H^1(\Omega)$ is the regular P_1-conforming finite element space associated with a quasi-uniform triangulation $\tau_h(\Omega)$ with mesh size h, and $V_h(\Gamma)$ is its restriction to Γ. We denote also $\{\phi_1(x), \ldots, \phi_n(x)\}$ and $\{\varphi_1(x), \ldots, \varphi_m(x)\}$ as the standard piecewise linear nodal basis functions for $X_h(\Omega)$ and $V_h(\Gamma)$, respectively. We define the discrete Neumann space $W_h(\Gamma) := span\{\varphi_i\}_{i=1}^m$. The stiffness matrix A and mass matrix M associated to the bilinear forms

$$\begin{cases} a(y,w) & := & \int_\Omega \nabla y \cdot \nabla w \, dx, & \text{for } y, w \in H^1(\Omega) \\ (y,w)_{L^2(\Omega)} & := & \int_\Omega y(x)\, w(x)\, dx, & \text{for } y, w \in L^2(\Omega), \end{cases} \tag{4.1}$$

are given by:

$$\begin{cases} (A)_{ij} & \equiv & \int_\Omega \nabla\phi_i(x) \cdot \nabla\phi_j(x)\, dx, & \text{for } 1 \le \; i \, , \, j \; \le n \\ (M)_{ij} & \equiv & \int_\Omega \phi_i(x)\, \phi_j(x)\, dx, & \text{for } 1 \le \; i \, , \, j \; \le n. \end{cases} \tag{4.2}$$

We also define the mass matrix Q on Γ given by $(Q)_{i,j} := (\varphi_i, \varphi_j)_{L^2(\Gamma)}$, for $1 \le i, j \le m$.

Given functions $\mathrm{y} \in X_h(\Omega)$, $\mathrm{v} \in V_h(\Gamma)$ and $\lambda \in W_h(\Gamma)$ we associate vectors $y \in \mathbb{R}^n$ and $\mathrm{v}, \lambda \in \mathbb{R}^m$ as follows

$$\mathrm{y}(x) = \sum_{i=1}^{n} \mathrm{y}_i\, \phi_i(x), \quad \mathrm{v}(x) = \sum_{j=1}^{m} \mathrm{v}_j\, \varphi_j(x) \quad \text{and} \quad \lambda(x) = \sum_{j=1}^{m} \lambda_j\, \varphi_j(x). \tag{4.3}$$

Note that we do not make distinction between a function $\mathrm{y} \in X_h(\Omega)$ and $\mathrm{y} \in \mathbb{R}^n$, $\lambda \in W_h(\Gamma)$ and $\lambda \in \mathbb{R}^m$, and so on.

We now define some spaces. We define $X\backslash_M\mathbb{R} := \{\mathrm{y} \in X_h(\Omega); (\mathrm{y}, 1)_{L^2(\Omega)} = (M\mathrm{y}, 1_n)_{\ell^2} = 0\}$. In the same way, define $W\backslash_Q\mathbb{R} := \{\lambda \in W_h(\Gamma); (\lambda, 1)_{L^2(\Gamma)} = (Q\lambda, 1_m)_{\ell^2} = 0\}$. The space $V\backslash_{\ell^2}\mathbb{R}$ is defined as

$$V\backslash_{\ell^2}\mathbb{R} := \{\mathrm{v} \in V_h(\Gamma); (\mathrm{v}, 1_m)_{\ell^2} = 0\}, \tag{4.4}$$

where 1_m is the vector of ones of $\mathbb{R}^m$ and $\mathrm{v} := \sum_{j=1}^{m} \mathrm{v}_j\, \varphi_j$.

Next we prove Theorem 4.1 which is a discrete version of Theorem 3.10 and the most important result of this chapter.

Theorem 4.1 *Let $\Omega \subset \mathbb{R}^2$ be a convex polygonal domain. Let $\phi_\lambda := A^\dagger E\,\lambda \in H^1(\Omega)\backslash\mathbb{R}$ be the discrete harmonic function with Neumann data $\lambda \in W\backslash_Q\mathbb{R}$. Then*

$$\sqrt{\lambda^T Q S_3^\dagger Q\lambda} := \|\phi_\lambda\|_{L^2(\Omega)} \asymp \|\lambda\|_{H^{-3/2}_{t,00}(\Gamma)} + h\,\|\lambda\|_{H^{-1/2}(\Gamma)}. \tag{4.5}$$

Proof: Let y_λ be the continuous harmonic extension with Neumann data $\lambda \in W\backslash_Q\mathbb{R}$; see (1.3). By approximation and regularity theory we obtain

$$\|\phi_\lambda - y_\lambda\|_{L^2(\Omega)} \preceq h\|y_\lambda\|_{H^1(\Omega)} \preceq h\|\lambda\|_{H^{-1/2}(\Gamma)}. \tag{4.6}$$

By a triangle inequality, Theorem 3.10 and (4.6), we have

$$\begin{aligned} \|\phi_\lambda\|_{L^2(\Omega)} &\preceq \|y_\lambda\|_{L^2(\Omega)} + \|\phi_\lambda - y_\lambda\|_{L^2(\Omega)} \\ &\preceq \|\lambda\|_{H^{-3/2}_{t,00}(\Gamma)} + h\|\lambda\|_{H^{-1/2}(\Gamma)}, \end{aligned} \tag{4.7}$$

and one of the inequalities of (4.5) follows.

Next we prove the other side inequality of (4.5). By an inverse inequality and a well known minimal energy property of y_λ and Poincaré inequality we have

$$\|\phi_\lambda\|_{L^2(\Omega)} \succeq h\|\phi_\lambda\|_{H^1(\Omega)} \asymp h\|y_\lambda\|_{H^1(\Omega)}. \tag{4.8}$$

Using that $\mathcal{S}$ is a norm isomorphism from $H^{1/2}(\Gamma)\backslash\mathbb{R}$ onto $H^{-1/2}(\Gamma)\backslash\mathbb{R}$, we obtain

$$\|y_\lambda\|_{H^1(\Omega)} \asymp \|\mathcal{S}^\dagger\lambda\|_{H^{1/2}(\Gamma)} \asymp \|\lambda\|_{H^{-1/2}(\Gamma)}. \tag{4.9}$$

Hence, we obtain $\|\phi_\lambda\|_{L^2(\Omega)} \geq c_1 h\|\lambda\|_{H^{-1/2}(\Gamma)}$ where $c_1 > 0$ is independent of h and λ. From Theorem 3.10 and (4.6), there exist constants $c_2 > 0$ and $c_3 > 0$, independent of h and λ, satisfying

$$\begin{aligned}\|\phi_\lambda\|_{L^2(\Omega)} &\geq \|y_\lambda\|_{L^2(\Omega)} - \|\phi_\lambda - y_\lambda\|_{L^2(\Omega)} \\ &\geq c_3\|\lambda\|_{H^{-3/2}_{t,00}(\Gamma)} - c_2 h\|\lambda\|_{H^{-1/2}(\Gamma)} \\ &= (c_3 - c_2\beta)\|\lambda\|_{H^{-3/2}_{t,00}(\Gamma)},\end{aligned} \tag{4.10}$$

where $\beta := h\,\dfrac{\|\lambda\|_{H^{-1/2}(\Gamma)}}{\|\lambda\|_{H^{-3/2}_{t,00}(\Gamma)}}$. Consequently,

$$\|\phi_\lambda\|_{L^2(\Omega)} \geq \max\{c_1\beta,\ c_3 - c_2\beta\}\|\lambda\|_{H^{-3/2}_{t,00}(\Gamma)}. \tag{4.11}$$

Since

$$\min_{\beta>0} \max\{c_1\beta,\ c_3 - c_2\beta\} = \frac{c_3 c_1}{c_2 + c_1} > 0, \tag{4.12}$$

we obtain

$$\|\phi_\lambda\|_{L^2(\Omega)} \succeq \|\lambda\|_{H^{-3/2}_{t,00}(\Gamma)} + h\|\lambda\|_{H^{-1/2}(\Gamma)}. \tag{4.13}$$

This concludes the proof. □

4.2 A Discontinuous Discretization of $H^{-1/2}(\Gamma)$

In this section we prove that the Theorem 4.1 remains true if we change $W_h(\Gamma)$ by $\widehat{W}_h(\Gamma)$. The space $W_h(\Gamma)$ is continuous at corners of Γ. Remember that a more suitable space for a better approximation of the Neumann condition requires discontinuities at the corners of the polygonal domain. For the definition of the discontinuous discrete Neumann space $\widehat{W}_h(\Gamma)$ see Chapter 2.

Now we enunciate the version of Theorem 4.1 using the space $\widehat{W}\backslash_B\mathbb{R}$ instead of $W\backslash_Q\mathbb{R}$ for the discrete Neumann space. First we define

$$\widehat{W}\backslash_B\mathbb{R} := \left\{\hat{\lambda} \in \widehat{W}_h(\Gamma); (\hat{\lambda}, 1)_{L^2(\Gamma)} = (B\hat{\lambda}, 1_m)_{\ell^2} = 0\right\}, \tag{4.14}$$

where B is the $m \times (m - K)$ matrix with entries $(B)_{i,j} = (\phi_i, \psi_j)_{L^2(\Gamma)}$, $1 \leq i \leq m$ and $1 \leq j \leq m - K$. The proof of Theorem 4.2 is essentially the same of Theorem 4.1.

Theorem 4.2 *Let $\Omega \subset \mathbb{R}^2$ be a convex polygonal domain. Let $\phi_{\hat{\lambda}} \in X\backslash_M\mathbb{R}$ be the discrete harmonic function with Neumann data $\hat{\lambda} \in \widehat{W}\backslash_B\mathbb{R}$. Then*

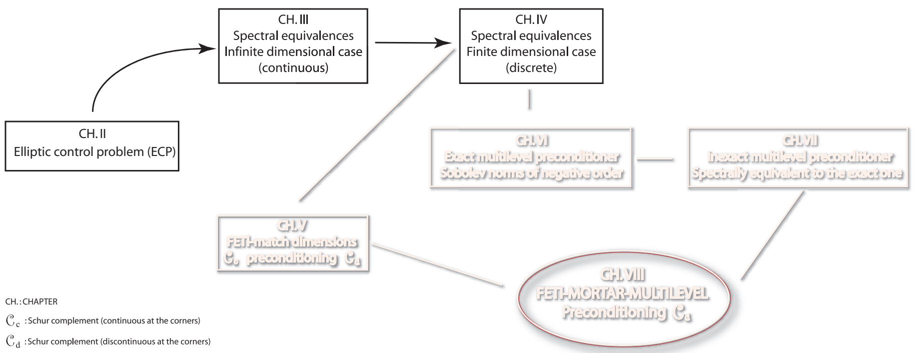
$$\sqrt{\lambda^T B^T S_3^\dagger B \lambda} := \|\phi_{\hat{\lambda}}\|_{L^2(\Omega)} \asymp \|\hat{\lambda}\|_{H^{-3/2}_{t,00}(\Gamma)} + h\|\hat{\lambda}\|_{H^{-1/2}(\Gamma)}. \tag{4.15}$$

Using Theorems 4.1 and 4.2 we conclude that

$$\begin{aligned} C_{\mathrm{c}} &\asymp (\alpha + h^2)\|\cdot\|^2_{H^{-1/2}(\Gamma)} + \beta\|\cdot\|^2_{L^2(\Gamma)} + \|\cdot\|^2_{H^{-3/2}_{t,00}(\Gamma)} \quad \text{in} \quad W\backslash_Q\mathbb{R}, \\ C_{\mathrm{d}} &\asymp (\alpha + h^2)\|\cdot\|^2_{H^{-1/2}(\Gamma)} + \beta\|\cdot\|^2_{L^2(\Gamma)} + \|\cdot\|^2_{H^{-3/2}_{t,00}(\Gamma)} \quad \text{in} \quad \widehat{W}\backslash_B\mathbb{R}. \end{aligned} \tag{4.16}$$

In the next chapter we describe how to precondition C_{d} using a precondition for C_{c} using a FETI formulation; see (1.18).

The diagram below illustrates the steps completed



5
C_c Preconditioning C_d: FETI Method

In this chapter we develop an optimal preconditioner for C_d using a preconditioner for C_c. Remember that $C_\mathrm{d} \in \mathbb{R}^{\hat{m}\times\hat{m}}$ and $C_\mathrm{c} \in \mathbb{R}^{m\times m}$. We use the FETI method to adjust the dimensions; see Section 1.2.

The matrices C_c and C_d are spectrally equivalent to $Q\,(\alpha S_1^\dagger + \beta Q^{-1} + S_3^\dagger)\,Q$ and $B^T(\alpha S_1^\dagger + \beta Q^{-1} + S_3^\dagger)\,B$ respectively; see Section 1.1 for the definition of the matrices Q, B, $S_1^\dagger$ and $S_3^\dagger$. To develop a preconditioner for C_d using a preconditioner for C_c we first develop a preconditioner for $\widehat{T} := B^T S_3^\dagger B$ using a preconditioner for $T := QS_3^\dagger Q$. By compatibility conditions we have $S_3^\dagger : W\backslash_{\ell^2}\mathbb{R} \longmapsto V\backslash_{BB^T}\mathbb{R}$, that is, given $\mathrm{u} \in V\backslash_{BB^T}\mathbb{R} := \{\mathrm{v} \in V_h(\Gamma); (BB^T\mathrm{v}, 1_m)_{\ell^2} = 0\}$, there exists a unique $\lambda \in W\backslash_Q\mathbb{R} = \{\lambda \in W_h(\Gamma); (Q\lambda, 1_m)_{\ell^2=0}\}$ such that $S_3^\dagger Q\lambda = \mathrm{u}$. We denote by $S_3 : V\backslash_{BB^T}\mathbb{R} \longmapsto W\backslash_{\ell^2}\mathbb{R}$ the operator defined by $S_3\mathrm{u} := Q\lambda$. Inspired on the FETI method we propose the following preconditioner for $\widehat{T}$

$$PC := R^T S_3 R \quad \text{where} \quad R = Q^{-1}B(B^T Q^{-1}B)^{-1}. \tag{5.1}$$

We next study the properties of PC. Let $V_h(\Gamma) \supset \widehat{V} = Range(Q^{-1}B)$. Define $P_Q := Q^{-1}B(B^T Q^{-1}B)^{-1}B^T$. A simple calculation gives $P_Q^T Q P_Q = QP_Q$, or equivalently, $(P_Q\mathrm{v}, P_Q\mathrm{u})_{L^2} = (P_Q\mathrm{v}, \mathrm{u})_{L^2}$. Therefore, $P_Q : V_h(\Gamma) \longmapsto \widehat{V}$ is the L^2-projection $\Pi_{\widehat{V}} : L^2(\Gamma) \longmapsto \widehat{V}$ restricted to $V_h(\Gamma) \subset L^2(\Gamma)$. By definition of $L^2(\Gamma)$ projection we have, for any $v \in L^2(\Gamma)$,

$$\|v - \Pi_{\widehat{V}} v\|_{L^2} = \min_{\hat{\mathrm{u}}\in\widehat{V}} \|v - \hat{\mathrm{u}}\|_{L^2} = \min_{\hat{\mu}\in\widehat{W}(\Gamma)} \|v - \Pi_V \hat{\mu}\|_{L^2}. \tag{5.2}$$

Here $\Pi_V : L^2(\Gamma) \longmapsto V_h(\Gamma)$ is the L^2-projection on $V_h(\Gamma)$.

Note that if $\mathrm{z} \in W\backslash_{\ell^2}\mathbb{R}$ then $S_3^\dagger \mathrm{z} := \mathrm{v} \in V\backslash_{BB^T}\mathbb{R}$. Since $B^T P_Q = B^T$ we have $(BB^T P_Q\mathrm{v}, 1_m)_{\ell^2} = (BB^T\mathrm{v}, 1_m)_{\ell^2} = 0$, hence, $P_Q\mathrm{v} = \Pi_{\widehat{V}}\mathrm{v} \in V\backslash_{BB^T}\mathbb{R}$. By definition $S_3\Pi_{\widehat{V}}\mathrm{v} := \mu \in W\backslash_{\ell^2}\mathbb{R}$, therefore, $(BR^T\mu, 1_m)_{\ell^2} = (P_Q^T\mu, 1_m)_{\ell^2} = (\mu, P_Q 1_m)_{\ell^2} = (\mu, 1_m)_{\ell^2} = 0$. We have used that $P_Q 1 = 1$, see that $Q^{-1}B$ is the L^2-projection of $\widehat{W}_h(\Gamma)$ into $W_h(\Gamma)$, $P_Q^2 = P_Q$ and $1_{\hat{m}} \in \widehat{W}_h(\Gamma) \cap W_h(\Gamma)$.

Let $\widehat{W}'$ be the space with the same dimension of $\widehat{W}\backslash_B\mathbb{R}$, that is $\hat{m}-1$, defined by

$$\widehat{W}' := \left\{\hat{\lambda} \in \widehat{W}_h(\Gamma); \exists\, \mathrm{u} \in V\backslash_{BB^T}\mathbb{R}; \hat{\lambda} = B^T\mathrm{u}\right\} \tag{5.3}$$

endowed with the seminorm

$$\|\hat{\lambda}\|^2_{\widehat{W}'} := |R\hat{\lambda}|^2_{S_3} = (PC\hat{\lambda},\hat{\lambda})_{\ell^2}. \tag{5.4}$$

Remember that S_3 is symmetric semi-positive definite. We next prove that $\|\cdot\|^2_{\widehat{W}'}$ is a norm. To verify this we only need to show that $\|\hat{\lambda}\|_{\widehat{W}'} = 0$ implies $\hat{\lambda} = 0$. For this we need the following lemma.

Lemma 5.1 *For any $\hat{\mu} \in Range(B^T)$, there exists a $\mathrm{u} \in \widehat{V} = Range(P_Q)$ such that $\hat{\mu} = B^T\mathrm{u}$.*

Proof: For each $\hat{\mu} \in Range(B^T)$, there exists $\mathrm{v} \in V_h(\Gamma)$ such that $\hat{\mu} = B^T\mathrm{v}$. Choosing $\mathrm{u} = P_Q\mathrm{v}$, it follows from the definition of P_Q that $B^T\mathrm{u} = B^T P_Q\mathrm{v} = B^T\mathrm{v} = \hat{\mu}$. □

Consider any $\hat{\lambda} \in \widehat{W}'$ such that $\|\hat{\lambda}\|_{\widehat{W}'} = 0$. By Lemma 5.1, $\hat{\lambda} = B^T\mathrm{u}$ for some $\mathrm{u} \in Range(P_Q)$. Since $P_D\mathrm{u} = \mathrm{u}$, we obtain

$$0 = \|\hat{\lambda}\|^2_{\widehat{W}'} = \|B^T\mathrm{u}\|^2_{\widehat{W}'} = \|P_Q\mathrm{u}\|^2_{S_3} = \|\mathrm{u}\|^2_{S_3}.$$

Thus $\mathrm{u} \in ker(S_3) = span\{1_m\}$. Since $1_m \notin V\backslash_{BB^T}\mathbb{R}$, by definition of $\widehat{W}'$ we have $\hat{\lambda} = 0$.

We equip the space $\widehat{W}\backslash_B\mathbb{R}$ with the norm

$$\|\hat{\lambda}\|^2_{\widehat{W}} := \sup_{\hat{\mu}\in\widehat{W}'} \frac{(\hat{\lambda},\hat{\mu})_{\ell^2}}{\|\hat{\mu}\|_{\widehat{W}'}}. \tag{5.5}$$

By a simple calculation we obtain

$$\|\hat{\lambda}\|^2_{\widehat{W}} = (PC^{-1}\hat{\lambda},\hat{\lambda})_{\ell^2}, \quad \text{for all} \quad \hat{\lambda} \in \widehat{W}\backslash_B\mathbb{R} \tag{5.6}$$

Now we prove that PC^{-1} is spectrally equivalent to $\widehat{T}$, or equivalently, PC is an optimal preconditioner for $\widehat{T}$.

Theorem 5.1 *For all $\hat{\lambda} \in \widehat{W}\backslash_B\mathbb{R}$ we have*

$$(PC^{-1}\hat{\lambda},\hat{\lambda})_{\ell^2} \preceq (\widehat{T}\hat{\lambda},\hat{\lambda})_{\ell^2} \preceq (PC^{-1}\hat{\lambda},\hat{\lambda})_{\ell^2}. \tag{5.7}$$

Therefore, the condition of $PC * \widehat{T}$ is bounded independently of h.

Proof: *Lower bound*: It is easy to see that

$$(\widehat{T}\hat{\lambda},\hat{\lambda})_{\ell^2} = \|S_3^{-1/2}B\hat{\lambda}\|^2_{\ell^2} = \sup_{\mathrm{z}\in W\backslash_{\ell^2}\mathbb{R}} \frac{(S_3^{-1/2}B\hat{\lambda},\mathrm{z})^2_{\ell^2}}{(\mathrm{z},\mathrm{z})_{\ell^2}} = \sup_{\mathrm{v}\in V\backslash_{BB^T}\mathbb{R}} \frac{(\lambda, B^T\mathrm{v})^2_{\ell^2}}{(\mathrm{v}, S_3\mathrm{v})_{\ell^2}}. \tag{5.8}$$

By Lemma 5.1, for each $\hat{\mu} = B^T\mathrm{v}$, there exists $\mathrm{u} := P_Q\mathrm{v} \in \widehat{V} \subset W\backslash_{\ell^2}\mathbb{R}$ such that $\hat{\mu} = B^T\mathrm{u}$. Hence,

$$\begin{aligned}
\sup_{\mathrm{v}\in V\backslash_{BB^T}\mathbb{R}} \frac{(\hat{\lambda}, B^T\mathrm{v})^2_{\ell^2}}{(\mathrm{v}, S_3\mathrm{v})_{\ell^2}} &\geq \sup_{\mathrm{v}\in V\backslash_{BB^T}\mathbb{R}} \frac{(\hat{\lambda}, B^T P_Q\mathrm{v})^2_{\ell^2}}{(P_Q\mathrm{v},\, S_3 P_Q\mathrm{v})_{\ell^2}} = \sup_{\mathrm{u}\in V\backslash_{BB^T}\mathbb{R}} \frac{(\hat{\lambda}, B^T\mathrm{u})^2_{\ell^2}}{(RB^T\mathrm{u},\, S_3RB^T\mathrm{u})_{\ell^2}} \\
&= \sup_{\hat{\mu}\in\widehat{W}'} \frac{(\hat{\lambda}, \hat{\mu})^2_{\ell^2}}{(R\hat{\mu},\, S_3R\hat{\mu})_{\ell^2}} = \sup_{\hat{\mu}\in\widehat{W}'} \frac{(\hat{\lambda}, \hat{\mu})^2_{\ell^2}}{||\hat{\mu}||^2_{\widehat{W}'}} \\
&= ||\hat{\lambda}||^2_{\widehat{W}} = (PC^{-1}\hat{\lambda}, \hat{\lambda})_{\ell^2}.
\end{aligned}$$

This gives the lower bound.

Upper bound: By (5.8) and the stability of P_Q in S_3 seminorm (see Lemma 5.5) we get

$$\begin{aligned}
(\widehat{T}\hat{\lambda}, \hat{\lambda})_{\ell^2} &= \sup_{\mathrm{v}\in V\backslash_{BB^T}\mathbb{R}} \frac{(\hat{\lambda}, B^T\mathrm{v})^2_{\ell^2}}{(\mathrm{v}, S_3\mathrm{v})_{\ell^2}} \preceq \sup_{\mathrm{v}\in V\backslash_{BB^T}\mathbb{R}} \frac{(\hat{\lambda}, B^T\mathrm{v})^2_{\ell^2}}{(P_Q\mathrm{v}, S_3P_Q\mathrm{v})_{\ell^2}} \\
&= \sup_{\mathrm{v}\in V\backslash_{BB^T}\mathbb{R}} \frac{(\hat{\lambda}, B^T\mathrm{v})^2_{\ell^2}}{(RB^T\mathrm{v}, S_3RB^T\mathrm{v})_{\ell^2}} = \sup_{\hat{\mu}\in\widehat{W}'} \frac{(\hat{\lambda}, \hat{\mu})^2_{\ell^2}}{(R\hat{\mu}, S_3R\hat{\mu})_{\ell^2}} \\
&= \sup_{\hat{\mu}\in\widehat{W}'} \frac{(\hat{\lambda}, \hat{\mu})^2_{\ell^2}}{||\hat{\mu}||^2_{\widehat{W}'}} = ||\hat{\lambda}||^2_{\widehat{W}} = (PC^{-1}\hat{\lambda}, \hat{\lambda})_{\ell^2}.
\end{aligned}$$

This concludes the proof. □

Remark 5.1 *By the proof of Theorem 5.1 we have*

$$(R^TAR)^\dagger \asymp B^TA^\dagger B. \tag{5.9}$$

We next prove the stability of P_Q in the S_3 norm (see Lemma 5.5 below). This result was used in the proof of the upper bound of the Theorem 5.1. Before proving this result we define an operator and prove three auxiliary lemmas. Let $v \in \hat{H}^s(\Gamma)$, $s > 1/2$. By Sobolev embedding Theorem we have $v \in C^0(\Gamma)$. For all $v \in \Pi_{k=1}^K C^0(\Gamma_k)$, we can define $I_{\widehat{W}} : \Pi_{k=1}^K C^0(\Gamma_k) \longmapsto \widehat{W}_h(\Gamma)$ the natural interpolation operator given by $I_{\widehat{W}}(v) = \sum_{j=1}^{m-K} v(x_j)\psi_j$. The next lemma say about an approximation property of the operator $I_{\widehat{W}}$. This result is important to prove a similar approximation property of the operator $\Pi_{\widehat{V}}$. Next we use this to prove the stability P_Q in some Sobolev norms. Finally we use this result to prove the stability P_Q in the S_3 norm.

Lemma 5.2 *For all* $v \in H^{3/2}_{t,00}(\Gamma)$ *we have*

$$\|v - I_{\widehat{W}} v\|_{L^2(\Gamma)} \preceq h^{3/2} \|v\|_{H^{3/2}_{t,00}(\Gamma)}. \tag{5.10}$$

Proof: The sketch of the proof: it is enough to prove the statement in a dense subset $C^{\infty}_{t,00}([0,1]) \subset H^{3/2}_{t,00}([0,1])$. We define $C^{\infty}_{t,00}([0,1])$ in (5.11) below. First prove that for all $v \in C^{\infty}_{t,00}([0,1])$ we have $\|v - I_{\widehat{W}} v\|_{L^2([0,1])} \preceq h^2 \|v\|_{H^2([0,1])}$ and $\|v - I_{\widehat{W}} v\|_{L^2([0,1])} \preceq h \|v\|_{H^1([0,1])}$. We conclude the proof by interpolation. It is sufficient to prove that for all v belonging to

$$C^{\infty}_{t,00}([0,1]) := \{v \in C^{\infty}([0,1]); v' \in C^{\infty}_0([0,1])\}, \tag{5.11}$$

we have $\|v - I_{\widehat{W}} v\|_{L^2([0,1])} \preceq h^{3/2} \|v\|_{H^{3/2}_{t,00}([0,1])}$, where $[0,1] = \cup_{i=1}^N [x_i, x_{i+1}]$, with $x_1 = 0$, $x_{N+1} = 1$ and $h = \max_i\{|x_{i+1} - x_i|\}$. Next we prove that for all $v \in C^{\infty}_{t,00}([0,1])$ we have $\|v - I_{\widehat{W}} v\|_{L^2([0,1])} \preceq h^2 \|v\|_{H^2([0,1])}$. Let $u = v - I_{\widehat{W}} v$. We consider two cases separately:

1. Since $u(x_i) = 0$ for all $2 \le i \le N-1$, then, by Mean Value Theorem there exits $c_i \in [x_i, x_{i+1}]$, such that $u'(c_i) = 0$.

2. Denoting $c_1 = 0$ and $c_{N+1} = 1$ by hypothesis we have $u'(c_1) = u'(c_{N+1}) = 0$.

Therefore, we have

$$\|u\|^2_{L^2([0,1])} = \sum_{i=1}^{N} \int_{[x_i, x_{i+1}]} \left(\int_{x_i}^{x} \int_{c_i}^{s} u''(t)\, dt ds \right)^2 dx \preceq h^4 \sum_{i=1}^{N} \int_{[x_i, x_{i+1}]} (u''(x))^2\, dx.$$

This implies that $\|u\|_{L^2([0,1])} \preceq h^2 \sum_{i=1}^N |u|_{H^2([x_i, x_{i+1}])} = h^2 \sum_{i=1}^N |v|_{H^2([x_i, x_{i+1}])} = h^2 |v|_{H^2([0,1])}$.

Applying similar arguments we obtain $\|v - I_{\widehat{W}} v\|_{L^2([0,1])} \preceq h \|v\|_{H^1([0,1])}$. Since $H^{3/2}_{t,00}([0,1]) = \left[H^2_{t,00}([0,1]), H^1([0,1])\right]_{1/2}$, by interpolation the result follows. □

The next lemma is used to prove the stability of P_Q in the norms $H^{-1}(\Gamma)$ and T; (see Lemma 5.4).

Lemma 5.3 *For all $v \in H^{3/2}_{t,00}(\Gamma)$ we have*

$$\|v - \Pi_{\widehat{V}} v\|_{L^2(\Gamma)} \preceq h^{3/2} \|v\|_{H^{3/2}_{t,00}(\Gamma)}. \tag{5.12}$$

Proof: By Lemma 5.2 we have $\|v - I_{\widehat{W}} v\|_{L^2(\Gamma)} \preceq h^{3/2} \|v\|_{H^{3/2}_{t,00}(\Gamma)}$, where $I_{\widehat{W}}$ is the interpolant in $\widehat{W}$. Using this, we obtain

$$
\begin{aligned}
\|v - \Pi_{\widehat{V}} v\|_{L^2(\Gamma)} &= \min_{\hat{\mu}\in\widehat{W}(\Gamma)} \|v - \Pi_V \hat{\mu}\|_{L^2(\Gamma)} \\
&\le \min_{\hat{\mu}\in\widehat{W}(\Gamma)} \{\|\Pi_V(v - \hat{\mu})\|_{L^2(\Gamma)} + \|v - \Pi_V v\|_{L^2(\Gamma)}\} \\
&\le \min_{\hat{\mu}\in\widehat{W}(\Gamma)} \{\|v - \hat{\mu}\|_{L^2(\Gamma)}\} + \|v - \Pi_V v\|_{L^2(\Gamma)} \qquad (5.13)\\
&\le h^{3/2}\|v\|_{H^{3/2}_{t,00}(\Gamma)} + \|v - I_{\widehat{W}} v\|_{L^2(\Gamma)} \\
&\le h^{3/2}\|v\|_{H^{3/2}_{t,00}(\Gamma)}.
\end{aligned}
$$

This concludes the proof. □

Note that by (5.13), an inverse inequality and triangular inequality we have also the H^1-stability of P_Q. Now we can prove an important result about the stability of P_Q.

Lemma 5.4 *The projection P_Q is stable in the norms $H^{-1}(\Gamma)$ and T. That means,*

$$
\|P_Q \mathrm{v}\|_{H^{-1}(\Gamma)} \le \|\mathrm{v}\|_{H^{-1}(\Gamma)} \quad \text{and} \quad \|P_Q \mathrm{v}\|_T \le \|\mathrm{v}\|_T \quad \text{for all} \quad \mathrm{v} \in V_h(\Gamma).
$$

Proof: Using properties of projections and Lemma 5.3 we have for all $\mathrm{v} \in V_h(\Gamma)$,

$$
\begin{aligned}
\|\mathrm{v} - P_Q \mathrm{v}\|_{H^{-3/2}_{t,00}(\Gamma)} &= \sup_{\varphi\in H^{3/2}_{t,00}(\Gamma)} \frac{(\mathrm{v} - P_Q\mathrm{v}, \varphi)_{L^2(\Gamma)}}{\|\varphi\|_{H^{3/2}_{t,00}(\Gamma)}} \\
&= \sup_{\varphi\in H^{3/2}_{t,00}(\Gamma)} \frac{(\mathrm{v} - P_Q\mathrm{v}, \varphi - \Pi_{\widehat{V}}\varphi)_{L^2(\Gamma)}}{\|\varphi\|_{H^{3/2}_{t,00}(\Gamma)}} \qquad (5.14)\\
&\le \sup_{\varphi\in H^{3/2}_{t,00}(\Gamma)} \frac{\|\mathrm{v} - P_Q\mathrm{v}\|_{L^2(\Gamma)}\|\varphi - \Pi_{\hat{V}}\varphi\|_{L^2(\Gamma)}}{\|\varphi\|_{H^{3/2}_{t,00}(\Gamma)}} \\
&\le h^{3/2}\|\mathrm{v} - P_Q\mathrm{v}\|_{L^2(\Gamma)} \le h\|\mathrm{v}\|_{H^{-1/2}(\Gamma)}.
\end{aligned}
$$

In the last inequality of (5.14) we have used the inverse inequality that $\|\mathrm{v}\|_{L^2(\Gamma)} \le h^{-1/2}\|\mathrm{v}\|_{H^{-1/2}(\Gamma)}$ for all $\mathrm{v} \in V_h(\Gamma)$. By triangular inequality and (5.14) we obtain

$$
\|P_Q\mathrm{v}\|_{H^{-3/2}_{t,00}(\Gamma)} \le \|\mathrm{v}\|_{H^{-3/2}_{t,00}(\Gamma)} + \|\mathrm{v} - P_Q\mathrm{v}\|_{H^{-3/2}_{t,00}(\Gamma)} \le \|\mathrm{v}\|_{H^{-3/2}_{t,00}(\Gamma)} + h\|\mathrm{v}\|_{H^{-1/2}(\Gamma)}. \qquad (5.15)
$$

In the same way we obtain

$$
\begin{aligned}
\|\mathrm{v} - P_Q \mathrm{v}\|_{H^{-1/2}(\Gamma)} &= \sup_{\varphi\in H^{1/2}(\Gamma)} \frac{(\mathrm{v} - P_Q\mathrm{v}, \varphi)_{L^2(\Gamma)}}{\|\varphi\|_{H^{1/2}(\Gamma)}} \\
&= \sup_{\varphi\in H^{1/2}(\Gamma)} \frac{(\mathrm{v} - P_Q\mathrm{v}, \varphi - \Pi_{\widehat{V}}\varphi)_{L^2(\Gamma)}}{\|\varphi\|_{H^{1/2}(\Gamma)}} \qquad (5.16)\\
&\le \sup_{\varphi\in H^{1/2}(\Gamma)} \frac{\|\mathrm{v} - P_Q\mathrm{v}\|_{L^2(\Gamma)}\|\varphi - \Pi_{\widehat{V}}\varphi\|_{L^2(\Gamma)}}{\|\varphi\|_{H^{1/2}(\Gamma)}} \\
&\le h^{1/2}\|\mathrm{v} - P_Q\mathrm{v}\|_{L^2(\Gamma)} \le \|\mathrm{v}\|_{H^{-1/2}(\Gamma)}.
\end{aligned}
$$

By triangle inequality we get $\|P_Q\mathrm{v}\|_{H^{-1/2}(\Gamma)} \le \|\mathrm{v}\|_{H^{-1/2}(\Gamma)}$. Using this, Theorem 4.1 and

(5.15) we have

$$\|P_Q \mathrm{v}\|_T \asymp \|P_Q \mathrm{v}\|_{H^{-3/2}_{t,00}(\Gamma)} + h\|P_Q \mathrm{v}\|_{H^{-1/2}(\Gamma)} \le \|\mathrm{v}\|_{H^{-3/2}_{t,00}(\Gamma)} + h\|\mathrm{v}\|_{H^{-1/2}(\Gamma)} \asymp \|\mathrm{v}\|_T. \tag{5.17}$$

The proof of stability in $H^{-1}(\Gamma)$ is similar to (5.16). □

Next we prove the stability of P_Q in the S_3 norm.

Lemma 5.5 *For all* $\mathrm{v} \in V\backslash_{BB^T}\mathbb{R}$ *we have* $\|P_Q \mathrm{v}\|_{S_3} \le \|\mathrm{v}\|_{S_3}$.

Proof: It is sufficient to show that

$$(P_Q T^\dagger P_Q^T T\lambda, \lambda)_T \le (\lambda, \lambda)_T, \quad \text{for all} \quad \lambda \in W\backslash_Q \mathbb{R}. \tag{5.18}$$

Indeed, making the change of variables $Q^{-1}S_3\mathrm{v} = \lambda \in W\backslash_Q\mathbb{R}$ we obtain

$$T\lambda = QS_3^\dagger Q Q^{-1} S_3 \mathrm{v} = Q\mathrm{v}.$$

Using this and observing that $QP_QQ^{-1} = P_Q^T$ we obtain

$$(P_Q T^\dagger P_Q^T T\lambda, \lambda)_T \;=\; (P_Q Q^{-1} S_3 Q^{-1} P_Q^T Q\mathrm{v}, Q\mathrm{v})_{\ell^2} \;=\; (P_Q^T S_3 P_Q \mathrm{v}, \mathrm{v})_{\ell^2}. \tag{5.19}$$

From (5.18) and (5.19) we have

$$\begin{aligned}(P_Q^T S_3 P_Q \mathrm{v}, \mathrm{v})_{\ell^2} &= (P_Q T^\dagger P_Q^T T\lambda, \lambda)_T \le (\lambda, \lambda)_T = (\underbrace{Q^{-1}S_3\mathrm{v}}_{\lambda}, T\,\underbrace{Q^{-1}S_3\mathrm{v}}_{\lambda})_{\ell^2} \\ &= (Q^{-1}S_3\mathrm{v}, Q\mathrm{v})_{\ell^2} = (S_3\mathrm{v}, \mathrm{v})_{\ell^2},\end{aligned}$$

which is the desired inequality. For proving (5.18) we do the following:

$$\begin{aligned}(P_Q T^\dagger P_Q^T T\lambda, \lambda)_T &= \|T^{-1/2} P_Q^T T\lambda\|_{\ell^2}^2 &&= \sup_{\mathrm{x}\in W\backslash_Q\mathbb{R}} \frac{(T^{-1/2}P_Q^T T\lambda, \mathrm{x})_{\ell^2}^2}{\|\mathrm{x}\|_{\ell^2}^2} \\ &= \sup_{\mu\in W\backslash_Q\mathbb{R}} \frac{(P_Q^T T\lambda, \mu)_{\ell^2}^2}{\|T^{1/2}\mu\|_{\ell^2}^2} &&= \sup_{\mu\in W\backslash_Q\mathbb{R}} \frac{(T^{1/2}\lambda, T^{1/2}P_Q\mu)_{\ell^2}^2}{\|\mu\|_T^2} \\ &\le \sup_{\mu\in W\backslash_Q\mathbb{R}} \frac{\|T^{1/2}\lambda\|_{\ell^2}^2\,\|T^{1/2}P_Q\mu\|_{\ell^2}^2}{\|\mu\|_T^2} &&= \sup_{\mu\in W\backslash_Q\mathbb{R}} \frac{\|\lambda\|_T^2\,\|P_Q\mu\|_T^2}{\|\mu\|_T^2}.\end{aligned}$$

We now use the Lemma 5.4, that is, $\|P_Q\mu\|_T \le \|\mu\|_T$, to obtain

$$\sup_{\mu\in W\backslash_Q\mathbb{R}} \frac{\|\lambda\|_T^2\,\|P_Q\mu\|_T^2}{\|\mu\|_T^2} \le \sup_{\mu\in W\backslash_Q\mathbb{R}} \frac{\|\lambda\|_T^2\,\|\mu\|_T^2}{\|\mu\|_T^2} \le \|\lambda\|_T^2.$$

Therefore, we conclude the proof of (5.18). □

Next we extend the FETI stabilization, Theorem 5.1, to the norms L^2 and $H^{-1/2}$ restricted to $W\backslash_Q\mathbb{R}$ and $\widehat{W}\backslash_B\mathbb{R}$. We need to define $Q_{Ker} := (I - P_{1_m})\, Q\, (I - P_{1_m})$, where

$$P_{1_m}\lambda := \frac{({1_m}^T Q\lambda)}{({1_m}^T Q 1_m)}\, 1_m \tag{5.20}$$

is the Q-projection onto $span\{1_m\}$. Note that ${1_m}^T Q\,(I - P_{1_m}) = 0$ and if λ satisfies ${1_m}^T Q\lambda = 0$ then $\lambda^T Q\,\lambda = \lambda^T Q_{Ker}\lambda$, that is, if $\lambda \in W\backslash_Q\mathbb{R}$ then $\|\lambda\|^2_{L^2} = \lambda^T Q_{Ker}\lambda$. We define also, $\widehat{Q}_{Ker} = (I - P_{1_{\hat m}})\, \widehat{Q}\, (I - P_{1_{\hat m}})$, where

$$P_{1_{\hat m}}\hat\lambda = \frac{({1_m}^T B\,\hat\lambda)}{({1_{\hat m}}^T B^T 1_m)}\, 1_{\hat m} \tag{5.21}$$

is the B-projection onto $span\{1_{\hat m}\}$. Note that ${1_m}^T B\,(I - P_{1_{\hat m}}) = 0$ and if $\hat\lambda$ satisfies ${1_m}^T B\,\hat\lambda = 0$ then $\hat\lambda^T\, \widehat{Q}\,\hat\lambda = \hat\lambda^T \widehat{Q}_{Ker}\hat\lambda$, that is, if $\hat\lambda \in \widehat{W}\backslash_B\mathbb{R}$ then $\|\hat\lambda\|^2_{L^2} = \hat\lambda^T \widehat{Q}_{Ker}\hat\lambda$. Using these and the fact of the projection P_Q of the Lemma 5.4 is obviously stable in L^2-norm, we have that P_Q is stable in the $H^{-1/2}$-norm by interpolation. Using these statements and Theorems 4.1 and 4.2 we have

$$C_{\rm c} = \alpha Q^T_{ext} A^\dagger Q_{ext} + \beta Q_{ker} + T \asymp (\alpha + h^2)\, \|\cdot\|^2_{H^{-1/2}} + \beta\|\cdot\|^2_{L^2} + \|\cdot\|^2_{H^{-3/2}_{t,00}} \quad \text{in} \quad W\backslash_Q\mathbb{R}$$

and

$$C_{\rm d} = \alpha B^T_{ext} A^\dagger B_{ext} + \beta \widehat{Q}_{ker} + \widehat{T} \asymp (\alpha + h^2)\, \|\cdot\|^2_{H^{-1/2}} + \beta\|\cdot\|^2_{L^2} + \|\cdot\|^2_{H^{-3/2}_{t,00}} \quad \text{in} \quad \widehat{W}\backslash_B\mathbb{R}.$$

Repeating the proof of Theorem 5.1 and using the statements above we can conclude the following important result:

Theorem 5.2 *Theorem 5.1 remains true if we use the matrices $C_{\rm c}$ and $C_{\rm d}$ instead of T and $\widehat{T}$ respectively. In this case we use $((\alpha + h^2)\, S_1^\dagger + \beta Q_{ker} + S_3^\dagger)^\dagger$ instead of S_3 in the FETI preconditioner PC defined in (5.1).*

5.1 Numerical Illustration of Theorem 5.2

In this section we present numerical results which illustrate Theorem 5.2. We consider two domains Ω:

1) Ω is the square $[0, 1]^2$,

2) Ω is the polygon $\Omega = \{(x, y) \in \mathbb{R}^2;\ x \geq 0,\ y \geq 0,\ y \leq 1 \text{ and } y \geq \mathrm{a}x - \mathrm{a}\}$, where a > 0 is a given parameter; see Figure 5.1. In both cases, the triangulation of Ω is constructed as showed in Figure 5.1. We divide each edge of Γ into 2^N parts of equal length, where N is an integer denoting the number of refinements.

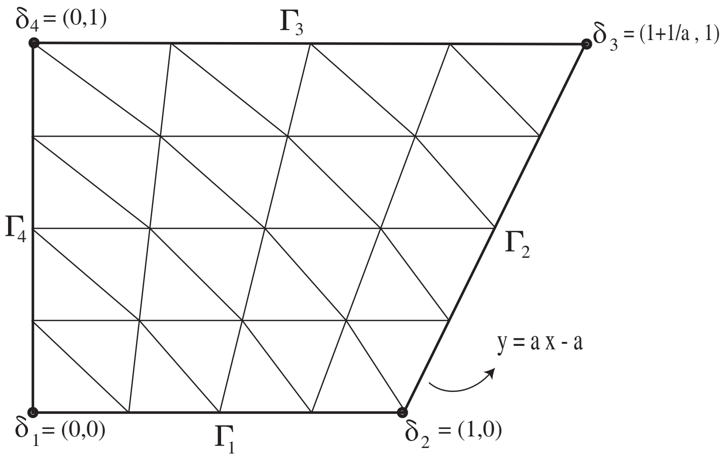


Figure 5.1: Illustration of the discretization used in the numerical tests.

The matrices C_c, C_d and PC depend on the relaxation parameters α and β, the mesh size $h = O(2^{-N})$ and the shape parameter of Ω given by ”a”. We divide the tests in two categories: 1) The tests such $\alpha \in \left\{1, (10)^{-3}, (10)^{-6}, 0\right\}$ and $\beta = 0$; 2) The tests such $\alpha = 0$ and $\beta \in \left\{1, (10)^{-3}, (10)^{-6}\right\}$.

a=∞	$\beta = 1$,	$\alpha = 0$	$\beta = (0.1)^3$,	$\alpha = 0$	$\beta = (0.1)^6$,	$\alpha = 0$	$\beta = 0$,	$\alpha = 0$
N ↓	cond	iter	cond	iter	cond	iter	cond	iter
3	1.28868	1	1.26968	3	3.14023	3	3.25166	3
4	1.34891	2	1.34471	3	2.61393	5	3.2516	3
5	1.34891	2	1.34833	3	1.57169	5	3.25159	3
6	1.34892	2	1.34884	2	1.30637	4	3.25159	3
7	1.34892	2	1.34891	2	1.34121	3	3.25159	3

Table 5.1: Condition number and number of iterations of PCG ($residual < 10^{-6}$) for the matrix C_d and preconditioner PC as in Theorem 5.2. Ω is a square domain.

a=5	$\beta = 1$,	$\alpha = 0$	$\beta = (0.1)^3$,	$\alpha = 0$	$\beta = (0.1)^6$,	$\alpha = 0$	$\beta = 0$,	$\alpha = 0$
N ↓	cond	iter	cond	iter	cond	iter	cond	iter
3	1.28867	4	1.26729	6	3.31176	7	3.41005	5
4	1.44865	4	1.43655	7	2.86571	8	3.4492	5
5	1.45666	4	1.44271	5	1.76559	9	3.4634	5
6	1.4607	4	1.44365	5	1.353	7	3.46895	5
7	1.46272	4	1.44356	4	1.4263	6	3.4713	6

Table 5.2: Condition number and number of iterations of PCG ($residual < 10^{-6}$) for the matrix C_d and preconditioner PC as in Theorem 5.2. Ω is a deformed square, $a = 5$; see Figure 5.1.

a=∞	$\beta = 0$,	$\alpha = 1$	$\beta = 0$,	$\alpha = (0.1)^3$	$\beta = 0$,	$\alpha = (0.1)^6$	$\beta = 0$,	$\alpha = 0$
N ↓	cond	iter	cond	iter	cond	iter	cond	iter
3	1.4283	3	2.3728	3	3.24996	3	3.25166	3
4	1.42616	3	1.81165	3	3.24483	3	3.2516	3
5	1.42585	3	1.53769	3	3.22481	3	3.25159	3
6	1.42578	2	1.45364	3	3.14901	3	3.25159	3
7	1.42576	2	1.43238	3	2.90074	3	3.25159	3

Table 5.3: Condition number and number of iterations of PCG ($residual < 10^{-6}$) for the matrix C_d and preconditioner PC as in Theorem 5.2. Ω is a square domain.

a=5	$\beta = 0,$	$\alpha = 1$	$\beta = 0,$	$\alpha = (0.1)^3$	$\beta = 0,$	$\alpha = (0.1)^6$	$\beta = 0,$	$\alpha = 0$
N ↓	cond	iter	cond	iter	cond	iter	cond	iter
3	1.4919	5	2.53517	5	3.4085	5	3.4101	5
4	1.4921	5	1.92528	5	3.4428	5	3.4492	5
5	1.4909	4	1.5943	4	3.4378	5	3.4634	5
6	1.4911	4	1.4975	4	3.3703	5	3.4689	5
7	1.4912	4	1.4914	4	3.1277	5	3.4713	5

Table 5.4: Condition number and number of iterations of PCG ($residual < 10^{-6}$) for the matrix C_d and preconditioner PC as in Theorem 5.2. Ω is a deformed square, $a = 5$; see Figure 5.1.

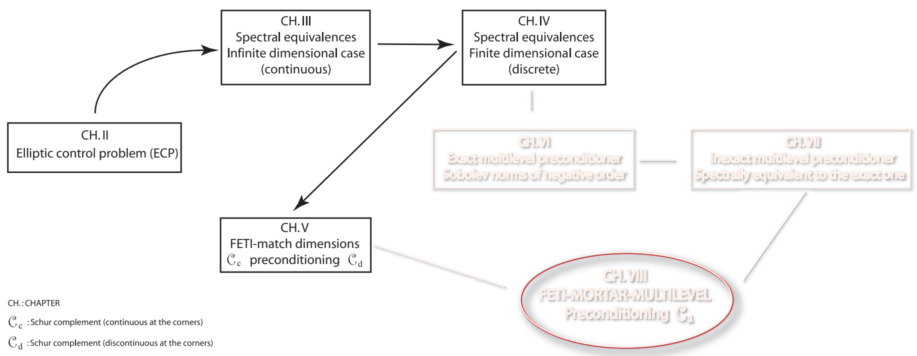
a=∞	$\beta = 1,$	$\alpha = 0$	$\beta = (0.1)^3,$	$\alpha = 0$	$\beta = (0.1)^6,$	$\alpha = 0$	$\beta = 0,$	$\alpha = 0$
N ↓	cond	iter	cond	iter	cond	iter	cond	iter
3	3.85571	6	206.044	7	11462.2	8	12185	8
4	4.91987	10	209.997	18	71575.4	72	109844	46
5	5.0193	11	202.294	18	162089	171	874969	289
6	5.05372	11	196.948	18	188462	191	6.876e+6	>500
7	5.06884	11	193.997	18	190238	185	5.431e+7	>500

Table 5.5: Condition number and number of iterations of CG ($residual < 10^{-6}$) for the matrix C_d as in Theorem 5.2 without preconditioner. Ω is a square domain.

a=∞	$\beta = 0,$	$\alpha = 1$	$\beta = 0,$	$\alpha = (0.1)^3$	$\beta = 0,$	$\alpha = (0.1)^6$
N ↓	cond	iter	cond	iter	cond	iter
3	162.38	6	6237.21	8	12173.2	8
4	354.542	18	22120.7	38	109404	47
5	690.579	31	51157.7	88	860938	288
6	1339.13	44	104675	147	6.453e+6	>500
7	2626.33	51	208860	212	4.303e+7	>500

Table 5.6: Condition number and number of iterations of CG ($residual < 10^{-6}$) for the matrix C_d defined in Theorem 5.2 without preconditioner. Ω is a square domain.

The diagram below illustrates the steps completed



6
An Exact Multilevel Preconditioner

In this chapter, using multilevel based preconditioners, we develop spectral approximations for matrices associated to several Sobolev norms ; see [2, 7, 8, 31, 34] and references therein.

This chapter is organized as follows. In the first section we introduce a multilevel method and we provide some technical tools which will be used in the proof of Theorem 6.1. This theorem provides a multilevel preconditioner for matrices associated to the norm $H^{-3/2}_{t,00}(\Gamma)$. Theorem 6.1 extend a known result on multilevel methods to the critical exponent $-3/2$. In the second section we prove Theorem 6.1, one of the most important results of this thesis.

6.1 Notation and Technical Tools

From now on, we assume that the triangulation $\mathcal{T}_h$ of Γ has a *multilevel* structure. We consider $\mathcal{T}_h$ as the restriction of $\mathcal{T}_h(\Omega)$ to Γ. We assume that the triangulation $\mathcal{T}_h$ is obtained from $(L-1)$ successive refinements of an initial coarse triangulation $\mathcal{T}_0$ with initial grid size h_0. Let $h_\ell = h_{\ell-1}/2$ be the grid size on the ℓ-th triangulation $\mathcal{T}_\ell$ and associate the standard P_1 finite element space $V_\ell(\Gamma)$ generated by continuous and piecewise linear basis functions $\{\varphi_i^\ell\}_{i=1}^{m_\ell}$. Hence we have

$$V_0(\Gamma) \subset V_1(\Gamma) \subset \cdots \subset V_L(\Gamma) := V_h(\Gamma) \subset ... \subset L^2.$$

Let P_ℓ denote the $L^2(\Gamma)$-orthogonal projection onto $V_\ell(\Gamma)$, and let $\Delta P_\ell := (P_\ell - P_{\ell-1})$ be the $L^2(\Gamma)$-orthogonal projection onto $V_\ell(\Gamma) \cap V_{\ell-1}(\Gamma)^\perp$. We have that P_0, $(P_1 - P_0), \ldots, (P_L - P_{L-1})$ restricted to $V_L(\Gamma)$ are mutually L^2-orthogonal projections which satisfy:

$$Id = P_0 + (P_1 - P_0) + \cdots + (P_L - P_{L-1}). \tag{6.1}$$

Note that P_L is the identity matrix denoted as Id.

The matrix form of P_ℓ restricted to $V_L(\Gamma)$ is given by

$$P_\ell = R_\ell^T Q_\ell^{-1} R_\ell Q, \tag{6.2}$$

where R_ℓ is the $m_\ell \times m_L$ restriction matrix. The i-th row of R_ℓ is obtained by interpolating the basis function $\varphi_i^\ell \in V_\ell := V_\ell(\Gamma)$ at the nodes of the finest triangulation $\mathcal{T}_L := \mathcal{T}_h$.

We now explain how to associate the splitting $\sum_{\ell=0}^{L} \mu_\ell \|(P_\ell - P_{\ell-1})\mathrm{v}\|^2_{L^2(\Gamma)}$ to a matrix. Let $\Delta_\ell := (P_\ell - P_{\ell-1})Q^{-1} = R_\ell^T Q_\ell^{-1} R_\ell - R_{\ell-1}^T Q_{\ell-1}^{-1} R_{\ell-1}$. Then we have

$$\Delta_k Q \Delta_\ell = \delta_{k\ell}\Delta_\ell \quad \text{and} \quad \sum_{\ell=0}^{L} \mu_\ell \|(P_\ell - P_{\ell-1})\mathrm{v}\|^2_{L^2(\Gamma)} = \sum_{\ell=0}^{L} \mu_\ell \mathrm{v}^T Q (P_\ell - P_{\ell-1})\mathrm{v}, \tag{6.3}$$

where $P_{-1} = 0$. We observe that $Q(P_\ell - P_{\ell-1}) = Q\Delta_\ell Q$ is symmetric semi-positive definite.

Follows from [7, 31] that for $-3/2 < s < 3/2$

$$\|\mathrm{v}\|^2_{H^s(\Gamma)} \asymp \sum_{\ell=0}^{L} h_\ell^{-2s} \|(P_\ell - P_{\ell-1})\mathrm{v}\|^2_{L^2(\Gamma)} = \Big(\sum_{\ell=0}^{L} h_\ell^{-2s} \Delta_\ell Q\mathrm{v}, Q\mathrm{v}\Big), \quad \text{for all} \quad \mathrm{v} \in V_L. \tag{6.4}$$

This constraint for s comes from the fact that for $V_h(\Gamma) \not\subset H^s(\Gamma)$ if $s \geq 3/2$, therefore, the equivalence deteriorates when s tends to $3/2$. Results for negative norms are obtained by duality. By (6.4), for all $\mathrm{u} \in V_L$ we have

$$\begin{aligned} \|\mathrm{u}\|^2_{H^{1/2}(\Gamma)} &\asymp \big(\textstyle\sum_{\ell=0}^{L} h_\ell^{-1}\, \Delta_\ell Q\mathrm{u}, Q\mathrm{u}\big), \\ \|\mathrm{u}\|^2_{H^{-1/2}(\Gamma)} &\asymp \big(\textstyle\sum_{\ell=0}^{L} h_\ell\, \Delta_\ell Q\mathrm{u}, Q\mathrm{u}\big), \\ \|\mathrm{u}\|^2_{H^{-1}(\Gamma)} &\asymp \big(\textstyle\sum_{\ell=0}^{L} h_\ell^2\, \Delta_\ell Q\mathrm{u}, Q\mathrm{u}\big). \end{aligned} \tag{6.5}$$

Now we explain how to invert a matrix of the form $\sum_{k=0}^{L} \mu_k^{-1} \Delta_k Q$. We assume that $\mu_k > 0$, $0 \leq k \leq L$. From (6.1) and (6.3) we obtain

$$\Big(\sum_{k=0}^{L} \mu_k^{-1} \Delta_k Q\Big)\Big(\sum_{\ell=0}^{L} \mu_\ell \Delta_\ell Q\Big) = Id, \tag{6.6}$$

and since

$$\Big(\sum_{\ell=0}^{L} \mu_\ell\, \Delta_\ell Q\mathrm{u}, Q\mathrm{u}\Big) = \mathrm{u}^T \sum_{\ell=0}^{L} \mu_\ell\, Q\Delta_\ell Q\mathrm{u}, \tag{6.7}$$

we can use (1.2), (6.5) and (6.6) to obtain

$$\begin{cases} S_1 & \asymp \; Q \sum_{\ell=0}^{L} h_\ell^{-1}\, \Delta_\ell Q, \\ Q S_1^\dagger Q & \asymp \; Q \sum_{\ell=0}^{L} h_\ell\, \Delta_\ell Q. \end{cases} \tag{6.8}$$

The above equivalences yield simultaneous approximation for the spectral representations of $G := \beta Q + \alpha Q S_1^\dagger Q$ in terms of the Δ_ℓ and Q. More precisely,

$$G \asymp Q \sum_{\ell=1}^{L} (\beta + \alpha h_\ell)\, \Delta_\ell Q, \tag{6.9}$$

and using (6.6) and (6.9), the following spectral equivalency holds

$$G^{-1} \asymp \sum_{\ell=0}^{L} (\beta + \alpha h_\ell)^{-1} \Delta_\ell. \tag{6.10}$$

Numerically we can see that $\sum_{\ell=0}^{L} (h_\ell^{-3})\Delta_\ell$ is a quasi-optimal preconditioner for $QS_3^\dagger Q$. Before we prove that $\sum_{\ell=0}^{L} (h_\ell^{-3})\Delta_\ell$ is a quasi-optimal preconditioner for $QS_3^\dagger Q$, we first introduce some tools.

Let $\tilde{V}_\ell(\Gamma)$ be a sequence of nested subspaces defined by:

$$\tilde{V}_\ell(\Gamma) := \{\tilde{\mathrm{v}} \in C^1(\Gamma); \tilde{\mathrm{v}}_i := \tilde{\mathrm{v}}|_{[x_i^\ell, x_{i+1}^\ell]} \in P_3([x_i^\ell, x_{i+1}^\ell]), 0 \le i \le m_{\ell-1}\}, \tag{6.11}$$

where $\Gamma = \cup_{i=0}^{m_\ell - 1} [x_i^\ell, x_{i+1}^\ell]$, $x_0^\ell = x_{m_\ell}^\ell$ and P_3 are the cubic polynomials. The basis functions $\tilde{\varphi}_i^1$ and $\tilde{\varphi}_i^2$, $1 \le i \le m_\ell$, of $\tilde{V}_\ell(\Gamma)$ are defined by

$$\begin{aligned}
&\tilde{\varphi}_i^1(x_i) = 1, \tfrac{d}{dx}\tilde{\varphi}_i^2(x_i) = 1,\\
&\tilde{\varphi}_i^1(x_{i-1}) = \tfrac{d}{dx}\tilde{\varphi}_i^1(x_{i-1}) = \tilde{\varphi}_i^1(x_{i+1}) = \tfrac{d}{dx}\tilde{\varphi}_i^1(x_{i+1}) = \tfrac{d}{dx}\tilde{\varphi}_i^1(x_i) = 0,\\
&\tilde{\varphi}_i^2(x_{i-1}) = \tfrac{d}{dx}\tilde{\varphi}_i^2(x_{i-1}) = \tilde{\varphi}_i^2(x_{i+1}) = \tfrac{d}{dx}\tilde{\varphi}_i^2(x_{i+1}) = \tilde{\varphi}_i^2(x_i) = 0.
\end{aligned} \tag{6.12}$$

These basis functions can be obtained from $\hat{\tilde{\varphi}}_1(\hat{x}) = 2\hat{x}^3 - 3\hat{x}^2 + 1$, $\hat{\tilde{\varphi}}_2(\hat{x}) = \hat{x}^3 - 2\hat{x}^2 + \hat{x}$, $\hat{\tilde{\varphi}}_3(\hat{x}) = -2\hat{x}^3 + 3\hat{x}^2$ and $\hat{\tilde{\varphi}}_4(\hat{x}) = \hat{x}^3 - \hat{x}^2$ defined on the reference element $[0,1]$; see Figure 6.1. Indeed, let $\tilde{\varphi}_i^1(x) := \hat{\tilde{\varphi}}_3(\hat{T}_{i-1}(x))$ if $x \in [x_{i-1}, x_i]$, and $\tilde{\varphi}_i^1(x) := \hat{\tilde{\varphi}}_1(\hat{T}_i(x))$ if $x \in [x_i, x_{i+1}]$, where $\hat{T}_i : [x_i, x_{i+1}] \longmapsto [0,1]$ is such that $\hat{T}_i^{-1}(t) := x_i + t(x_{i+1} - x_i)$, $t \in [0,1]$. Also let $\tilde{\varphi}_i^2(x) := (\hat{T}_{i-1}'(x)^{-1})\tilde{\varphi}_4(\hat{T}_{i-1}(x))$ if $x \in [x_{i-1}, x_i]$, and $\tilde{\varphi}_i^2(x) := (\hat{T}_i'(x)^{-1})\tilde{\varphi}_2(\hat{T}_i(x))$ if $x \in [x_i, x_{i+1}]$. Note that $\hat{T}_i'(x)^{-1}$ does not depend on $x \in [x_i, x_{i+1}]$, and $\hat{T}_i'(x)^{-1} = O(h_L)$.

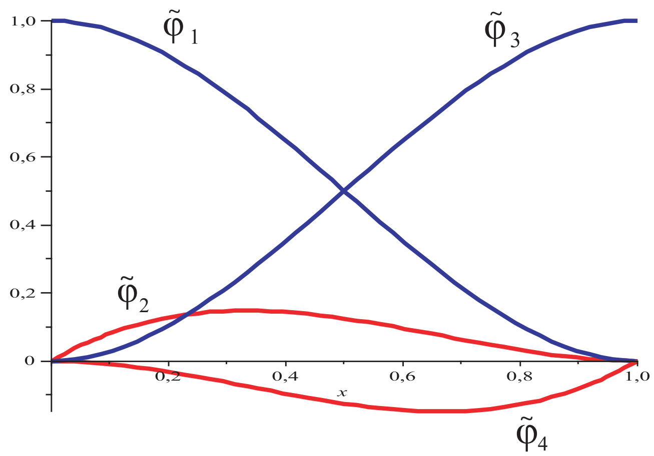


Figure 6.1: Illustration of the basis of $\tilde{V}_L(\Gamma)$.

Define $\tilde{I}_\ell : C^1(\Gamma) \longmapsto \tilde{V}_\ell(\Gamma)$ by

$$\tilde{I}_\ell(\mathrm{v}) := \sum_{i=1}^{m_\ell} \left(\mathrm{v}(x_i)\tilde{\varphi}_i^1(x) + \mathrm{v}'(x_i)\tilde{\varphi}_i^2(x)\right). \tag{6.13}$$

Let $\tilde{P}_\ell : L^2(\Gamma) \longmapsto \tilde{V}_\ell(\Gamma)$ be the L^2 projection onto $\tilde{V}_\ell(\Gamma)$, and let $\Delta\tilde{P}_\ell := \tilde{P}_\ell - \tilde{P}_{\ell-1}$, where $\tilde{P}_{-1} = 0$. For each $\mathrm{v}_L \in V_L := V$ define

$$\|\mathrm{v}_L\|^2_{A^{3/2}} := \sum_{\ell=0}^{L} h_\ell^{-3} \|\Delta P_\ell \mathrm{v}_L\|^2_{L^2}, \tag{6.14}$$

where $\Delta P_\ell := P_\ell - P_{\ell-1}$. Define $\tilde{V}_{\ell,0}(\Gamma) := \left\{\tilde{v}_\ell \in \tilde{V}_\ell(\Gamma);\ \partial\tilde{v}_\ell/\partial\tau(\delta_k) = 0,\ 1 \le k \le K\right\}$ and let $\tilde{P}_{\ell,0} : L^2(\Gamma) \longmapsto \tilde{V}_{\ell,0}(\Gamma)$ be the L^2 projection onto $\tilde{V}_{\ell,0}(\Gamma)$, and denote $\Delta\tilde{P}_{\ell,0} := \tilde{P}_{\ell,0} - \tilde{P}_{\ell-1,0}$. Note that the $dim(\tilde{V}_{\ell,0}(\Gamma)) = dim(\tilde{V}_\ell(\Gamma)) - K$. Note that we have introduced the space $\tilde{V}_{\ell,0}(\Gamma)$ because it belongs to $H^{3/2}_{t,00}(\Gamma_k)$ and due to Theorem 5.1 of [36] we have

$$\|\tilde{\mathrm{v}}_{L,0}\|^2_{H^{3/2}_{t,00}(\Gamma)} \asymp \sum_{\ell=0}^{L} h_\ell^{-3} \|\Delta\tilde{P}_{\ell,0} \mathrm{v}_L\|^2_{L^2(\Gamma)}. \tag{6.15}$$

We need also some definitions and results about interpolation spaces. Let $X \subset Y$ be densely embedded Hilbert spaces and define

$$K(t, v, X, Y)^2 := \inf_{x\in X}\{\|x - v\|^2_Y + t^2\|x\|^2_X\}. \tag{6.16}$$

Some obvious properties of K are: K is nondecreasing function of t and $K(t, v_1 + v_2, X, Y) \le K(t, v_1, X, Y) + K(t, v_2, X, Y)$. For $\theta \in [0, 1]$ the interpolation space and its norm are defined respectively by

$$\begin{aligned} [X, Y]_\theta &:= \left\{ v \in Y; \int_0^\infty t^{-2\theta-1} K(t, v, X, Y)^2 dt < \infty \right\}, \\ \|v\|^2_{[X,Y]_\theta} &:= \|v\|^2_Y + \int_0^\infty t^{-2\theta-1} K(t, v, X, Y)^2 dt. \end{aligned} \tag{6.17}$$

The Sobolev spaces H^s, for $m < s < n$, m and n integers, can alternatively be defined as interpolation spaces: $H^s := [H^n, H^m]_{\frac{s-m}{n-m}}$. For more details, see [1,3,23]. The space $H^{3/2}(\Gamma)$ can be identified with $\left[H^2(\Gamma), L^2(\Gamma)\right]_{3/4}$ endowed with the $H^{3/2}$-norm. For more discussion about interpolation spaces, see [3,22,23].

6.1.1 Technical Tools

In this subsection we provide some technical lemmas, which will be used in the proof of Theorem 6.1.

Lemma 6.1 *Let $X \subset L^2$ be any densely embedded Hilbert space. Denote by $X_L \subset X$ a finite dimensional subspace such that $X_L \neq \{0\}$. Then, for all $\mathrm{v}_L \in X_L$ we have*

$$\|\mathrm{v}_L\|_{X'} := \sup_{0\neq u\in X} \frac{(\mathrm{v}_L, u)_{L^2(\Gamma)}}{\|u\|_X} = \sup_{0\neq \mathrm{u}_L\in X_L} \frac{(\mathrm{v}_L, \mathrm{u}_L)_{L^2(\Gamma)}}{\inf_{u\in X: P_L u = \mathrm{u}_L} \|u\|_X}, \tag{6.18}$$

where P_L is the L^2 projection onto X_L, and X' is the dual of X.

Proof: First, let us prove that $\sup_{0\neq \mathrm{u}_L\in X_L} \frac{(\mathrm{v}_L,\mathrm{u}_L)_{L^2(\Gamma)}}{\inf_{u\in X:P_L u=\mathrm{u}_L}\|u\|_X}$ is well defined. Let $u_L \neq 0$ and assume by contradiction that $\inf_{u\in X:P_L u=\mathrm{u}_L}\|u\|_X = 0$. Then, there exists a sequence $u_n \in X$ such that $P_L u_n = \mathrm{u}_L$ and $\lim\|u_n\|_X = 0$. This implies that $\lim\|u_n\|_{L^2(\Gamma)} = 0$ and since $\|\mathrm{u}_L\|_{L^2(\Gamma)} = \|P_L u_n\|_{L^2(\Gamma)} \leq \|u_n\|_{L^2(\Gamma)} \longrightarrow 0$, we obtain $\mathrm{u}_L = 0$.

Since $(\mathrm{v}_L,u)_{L^2(\Gamma)} = (\mathrm{v}_L,P_L u)_{L^2(\Gamma)}$ for all $u\in X$, we have $\inf_{z\in X:P_L z=P_L u}\|z\|_X \leq \|u\|_X$. Taking $(\mathrm{v}_L,u)_{L^2(\Gamma)} \geq 0$, we have $\frac{(\mathrm{v}_L,u)_{L^2(\Gamma)}}{\|u\|_X} \leq \frac{(\mathrm{v}_L,P_L u)_{L^2(\Gamma)}}{\inf_{z\in X:P_L z=P_L u}\|z\|_X}$, and hence

$$\sup_{0\neq u\in X}\frac{(\mathrm{v}_L,\mathrm{u})_{L^2(\Gamma)}}{\|u\|_X} \leq \sup_{0\neq u\in X}\frac{(\mathrm{v}_L,P_L u)_{L^2(\Gamma)}}{\inf_{z\in X:P_L z=P_L u}\|z\|_X} = \sup_{0\neq \mathrm{u}_L\in X_L}\frac{(\mathrm{v}_L,\mathrm{u}_L)_{L^2(\Gamma)}}{\inf_{z\in X:P_L z=\mathrm{u}_L}\|z\|_X}. \tag{6.19}$$

For proving the other side of the inequality of (6.18), consider $X_{\mathrm{u}_L} := \{u\in X; P_L u = \mathrm{u}_L\}$. Denote by $\overline{X}_{\mathrm{u}_L}$ the closure of X_{u_L} in the X topology. Then there exists $\overline{u}\in\overline{X}_{\mathrm{u}_L}$ such that $\inf_{0\neq z\in X:P_L z=\mathrm{u}_L}\|z\|_X = \|\overline{u}\|_X$. In addition $(\mathrm{v}_L,\mathrm{u}_L)_{L^2} = (\mathrm{v}_L,\overline{u})_{L^2}$. Indeed, let $u_n\in\overline{X}_{\mathrm{u}_L}$ be a sequence such that $\lim u_n = \overline{u}$ in the X topology, then $(\mathrm{v}_L,\mathrm{u}_L)_{L^2} = (\mathrm{v}_L,P_L u_n)_{L^2} = (\mathrm{v}_L,u_n)_{L^2}$ and $\lim(\mathrm{v}_L,u_n)_{L^2(\Gamma)} = (\mathrm{v}_L,\lim u_n)_{L^2(\Gamma)} = (\mathrm{v}_L,\overline{u})_{L^2(\Gamma)}$. Using this we obtain

$$\sup_{0\neq \mathrm{u}_L\in X_L}\frac{(\mathrm{v}_L,\mathrm{u}_L)_{L^2(\Gamma)}}{\inf_{z\in X:P_L z=\mathrm{u}_L}\|z\|_X} = \sup_{0\neq \mathrm{u}_L\in X_L}\frac{(\mathrm{v}_L,\overline{u})_{L^2(\Gamma)}}{\|\overline{u}\|_X} \leq \sup_{0\neq z\in X}\frac{(\mathrm{v}_L,z)_{L^2}}{\|z\|_X},$$

and the proof is complete. □

Lemma 6.2 *Let* $\|\cdot\|^2_{A^{3/2}}$ *be defined as in (6.14). Defining* $\|\mathrm{v}_L\|^2_{A^{-3/2}} := \sup_{0\neq \mathrm{u}_L\in V}\frac{(\mathrm{v}_L,\mathrm{u}_L)^2}{\|\mathrm{u}_L\|^2_{A^{3/2}}}$ *we have*

$$\|\mathrm{v}_L\|^2_{A^{-3/2}} = \sum_{\ell=1}^{L} h_\ell^3\|\Delta P_\ell \mathrm{v}_L\|^2_{L^2}. \tag{6.20}$$

Lemma 6.2 follows straightforwardly from the orthogonality of the projections ΔP_ℓ and its proof can be found in [31].

Peter Oswald in [31] provides semi-orthogonal splittings based on piecewise constants for the Sobolev spaces $\left(H^{1/2}\right)'$ and $\left(H^{1/2}_{00}\right)'$. In the next section we prove a version of this result for piecewise linear and continuous functions for the space $H^{-3/2}_{t,00}$.

6.2 Main Result

Using the notation and the technical tools provided above, we now state and prove Theorem 6.1, which provides a multilevel preconditioner for matrices associated to the norm $H^{-3/2}_{t,00}(\Gamma)$.

Theorem 6.1 *For all* $\mathrm{v}_L\in V_L$*, the following inequalities hold:*

$$\|\mathrm{v}_L\|^2_{H^{-3/2}_{t,00}(\Gamma)} \leq \|\mathrm{v}_L\|^2_{A^{-3/2}} \leq (L+1)^2\|\mathrm{v}_L\|^2_{H^{-3/2}_{t,00}(\Gamma)}. \tag{6.21}$$

Since the proof of Theorem 6.1 is very long, we divide it in two lemmas: Lemma 6.3 refers to the lower bound and Lemma 6.8 refers to the upper bound of (6.21).

Lemma 6.3 (Lower bound) *For all* $\mathrm{v}_L \in V_L$*, the following inequality holds:*

$$\|\mathrm{v}_L\|^2_{H^{-3/2}_{t,00}(\Gamma)} \leq \|\mathrm{v}_L\|^2_{A^{-3/2}}. \tag{6.22}$$

Proof: Since $\|\mathrm{v}_L\|_{H^{-3/2}_{t,00}(\Gamma)} \leq \|\mathrm{v}_L\|_{H^{-3/2}(\Gamma)}$, the lemma follows from the following fact

$$\|\mathrm{u}_L\|^2_{A^{3/2}} := \textstyle\sum_{\ell=0}^{L} h_\ell^{-3}\|\Delta P_\ell \mathrm{u}_L\|^2_{L^2} \leq \inf_{z\in H^{3/2}(\Gamma):P_L z=\mathrm{u}_L} \|z\|^2_{H^{3/2}(\Gamma)}. \tag{6.23}$$

Indeed, if (6.23) holds, then by Lemma 6.1 and Lemma 6.2 we obtain

$$\|\mathrm{v}_L\|^2_{H^{-3/2}(\Gamma)} \leq \sup_{\mathrm{u}_L\in V_L(\Gamma)} \frac{(\mathrm{v}_L,\mathrm{u}_L)^2_{L^2}}{\|\mathrm{u}_L\|^2_{A^{3/2}}} = \|\mathrm{v}_L\|^2_{A^{-3/2}}.$$

We now prove (6.23). Consider $v \in H^{3/2}(\Gamma)$ such that $P_L v = \mathrm{v}_L$. Then

$$\begin{aligned} \|\mathrm{v}_L\|^2_{A^{3/2}} &:= \textstyle\sum_{\ell=0}^{L} h_\ell^{-3}\|\Delta P_\ell \mathrm{v}_L\|^2_{L^2} = \sum_{\ell=0}^{L} h_\ell^{-3}\|\Delta P_\ell v\|^2_{L^2} \\ &\leq \left(\|v\|^2_{L^2(\Gamma)} + \textstyle\sum_{\ell=0}^{L} h_\ell^{-3}\|v - P_\ell v\|^2_{L^2}\right). \end{aligned} \tag{6.24}$$

Since $\|v - P_\ell v\|^2_{L^2(\Gamma)}$ is the best L^2-approximation of v in V_ℓ, we have

$$\begin{aligned} \|v - P_\ell v\|^2_{L^2(\Gamma)} &\leq \|I_\ell u - v\|^2_{L^2(\Gamma)}, \quad \text{for all} \;\; u \in H^2(\Gamma) \\ &\leq \|u - v\|^2_{L^2(\Gamma)} + \|I_\ell u - u\|^2_{L^2(\Gamma)}, \quad \text{for all} \;\; u \in H^2(\Gamma) \\ &\leq \|u - v\|^2_{L^2} + h_\ell^4\|u\|^2_{H^2}, \quad \text{for all} \;\; u \in H^2(\Gamma). \end{aligned} \tag{6.25}$$

Hence $\|v - P_\ell v\|^2_{L^2(\Gamma)} \leq K(h_\ell^2, v, H^2(\Gamma), L^2(\Gamma))^2$, therefore,

$$\|\mathrm{v}_L\|^2_{A^{3/2}} \leq \left(\|v\|^2_{L^2} + \sum_{\ell=0}^{L} h_\ell^{-3} K(h_\ell^2, v, H^2(\Gamma), L^2(\Gamma))^2\right). \tag{6.26}$$

In addition, considering $h_\ell = h_0/2^\ell$ for all ℓ integer, we have

$$\begin{aligned} \int_0^\infty t^{-2\frac{3}{4}-1} K(t, v, H^2(\Gamma), L^2(\Gamma))^2 dt &= \sum_{\ell=-\infty}^{+\infty} \int_{h_\ell^{-2}}^{h_{\ell+1}^{-2}} t^{-\frac{5}{2}} K(t, v, H^2(\Gamma), L^2(\Gamma))^2 dt \\ &\geq \sum_{\ell=-\infty}^{+\infty} \int_{h_\ell^{-2}}^{h_{\ell+1}^{-2}} t^{-\frac{5}{2}} K(h_\ell^{-2}, v, H^2(\Gamma), L^2(\Gamma))^2 dt \\ &= \sum_{\ell=-\infty}^{+\infty} -\frac{2}{3} t^{-\frac{3}{2}}\Big|_{h_\ell^{-2}}^{h_{\ell+1}^{-2}} K(h_\ell^{-2}, v, H^2(\Gamma), L^2(\Gamma))^2 \\ &\geq \sum_{\ell=0}^{L} h_\ell^{-3} K(h_\ell^2, v, H^2(\Gamma), L^2(\Gamma))^2. \end{aligned}$$

Therefore, $\|\mathrm{v}_L\|^2_{A^{3/2}} \preceq \|v\|^2_{H^{3/2}}$ for all $v \in H^{3/2}(\Gamma)$ such that $P_L v = \mathrm{v}_L$, (6.23) holds. □

In order to prove Lemma 6.8, the upper bound of the equivalence (6.21) of Theorem 6.1, we first introduce some tools and results.

We now define an operator $\tilde{\mathbb{I}}_L : V_L(\Gamma) \longmapsto \tilde{V}_{L+1}(\Gamma)$ such that $P_L(\tilde{\mathbb{I}}_L \mathrm{v}_L) = \mathrm{v}_L$. For each arbitrarily fixed $\mathrm{v}_L \in V_L(\Gamma)$ we construct a function $\tilde{\mathrm{v}}_{L+1} \in \tilde{V}_{L+1}(\Gamma)$ such that $P_L \tilde{\mathrm{v}}_{L+1} = \mathrm{v}_L$. We set the nodal values $\tilde{\mathrm{v}}_{L+1}$ and $\frac{d}{dx}\tilde{\mathrm{v}}_{L+1}$ at the grid points of the partition $\mathcal{T}_{L+1}$ as follows: for a grid point $p \in \mathcal{T}_L := \mathcal{T}_h(\Gamma)$ define $\tilde{\mathrm{v}}_{L+1}(p) = \mathrm{v}_L(p)$, and $\frac{d}{dx}\tilde{\mathrm{v}}_L(p)$ equals to the average of the piecewise constant function $\frac{d}{dx}\mathrm{v}_L$ on the two intervals $\tau_i \in \mathcal{T}_L$ attached to p. For the grid point $p \in \mathcal{T}_{L+1}$ interior to $\tau_i := [x_i, x_{i+1}] \in \mathcal{T}_L$, define $\tilde{v}_{L+1}(p)$ and $\frac{d}{dx}\tilde{v}_{L+1}(p)$ such that $\tilde{\mathrm{v}}_{L+1} \in C^1(\Gamma)$ and

$$\int_{\tau_i} (\tilde{\mathrm{v}}_{L+1} - \mathrm{v}_L)(x)\varphi_i(x)dx = 0 \quad \text{and} \quad \int_{\tau_i} (\tilde{\mathrm{v}}_{L+1} - \mathrm{v}_L)(x)\varphi_{i+1}(x)dx = 0, \tag{6.27}$$

where φ_i and φ_{i+1} are the piecewise linear nodal basis functions associated to τ_i. We define $\tilde{\mathbb{I}}_L(\mathrm{v}_L) := \tilde{\mathrm{v}}_{L+1}$; see Figure 6.2. The condition (6.27) gives $P_L \tilde{\mathrm{v}}_{L+1} = \mathrm{v}_L$ as desired.

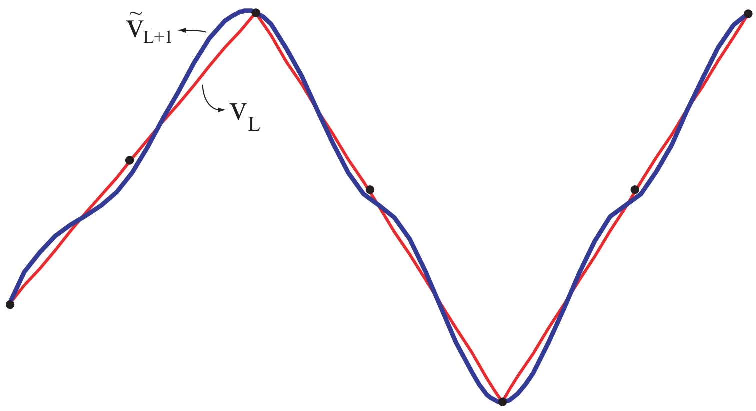


Figure 6.2: Illustration of $\tilde{\mathbb{I}}_L \mathrm{v}_L := \tilde{\mathrm{v}}_{L+1}$.

To calculate the expression of $\tilde{\mathbb{I}}_L$ for an arbitrary refinement, let $a_i = \mathrm{v}_L(x_i)$, $a_{i+1} = \mathrm{v}_L(x_{i+1})$, b_i be the average of the lateral derivatives of v_L at x_i, and b_{i+1} be the average of the lateral derivatives of v_L at x_{i+1}. Also let $x_{i+\theta} := \theta x_{i+1} + (1-\theta)x_i$, $0 < \theta < 1$, be the point of $\mathcal{T}_{L+1}$ belonging to $]x_i, x_{i+1}[$. Tedious calculations gives

$$\begin{aligned} \tilde{\mathrm{v}}_L(x_{i+\theta}) &= q_1^1(\theta)a_i + q_2^1(\theta)a_{i+1} + q_3^1\, h_L\,(\theta)b_i + q_4^1\, h_L\,(\theta)b_{i+1}, \\ \tilde{\mathrm{v}}'_L(x_{i+\theta}) &= h_L^{-1}\, q_1^2(\theta)a_i + h_L^{-1}\, q_2^2(\theta)a_{i+1} + q_3^2(\theta)b_i + q_4^2(\theta)b_{i+1}, \end{aligned} \tag{6.28}$$

where $q_1^1(\theta) = 2\theta^3 - 3\theta^2 + 2/3\theta + 2/3$, $q_2^1(\theta) = -2\theta^3 + 3\theta^2 - 2/3\theta + 1/3$, $q_3^1(\theta) = 1/3\,(3\theta^3 - 2\theta^2)$, $q_4^1(\theta) = 1/3\,(3\theta^3 - 7\theta^2 + 5\theta - 1)$, $q_1^2(\theta) = 6\theta^2 - 6\theta + 2$, $q_2^2(\theta) = -6\theta^2 + 6\theta - 2$,

$q_3^2(\theta) = 3\theta^2$ and $q_4^2(\theta) = 3\theta^2 - 6\theta + 3$. It is easy to check that $\tilde{\mathrm{v}}_{L+1}(x_{i+\theta})$ and $\tilde{\mathrm{v}}'_{L+1}(x_{i+\theta})$ satisfy (6.27).

When the refinement is uniform, that is $\theta = 1/2$, we have a more suitable expression for $\tilde{\mathrm{v}}_{L+1}$:

$$\begin{aligned} \tilde{\mathrm{v}}_{L+1}(x_{i+1/2}) &= \tfrac{1}{2}(a_i + a_{i+1}) + \tfrac{1}{24}h_L(b_{i+1} - b_i), \\ \tilde{\mathrm{v}}'_{L+1}(x_{i+1/2}) &= \tfrac{1}{2h_L}(a_{i+1} - a_i) + \tfrac{3}{4}(b_{i+1} + b_i). \end{aligned} \tag{6.29}$$

Note that $\tilde{\mathrm{v}}_{L+1}(x_{i+1/2})$ is a linear combination of the values $a_{i-1} = \mathrm{v}_L(x_{i-1})$, a_i, a_{i+1} and $a_{i+2} = \mathrm{v}_L(x_{i+2})$, that is, $\tilde{\mathrm{v}}_{L+1}(x_{i+1/2}) = \sum_{j=1}^{4} c_j a_{i-2+j}$, where c_j independ of h_L. Similarly, $\tilde{\mathrm{v}}'_{L+1}(x_{i+1/2}) = \sum_{j=1}^{3} \tilde{c}_j \tilde{b}_{i-2+j}$, where $\tilde{b}_{i-1} = \mathrm{v}'_L(x_{i-1+1/2})$, $\tilde{b}_i = \mathrm{v}'_L(x_{i+1/2})$ and $\tilde{b}_{i+1} = \mathrm{v}'_L(x_{i+1+1/2})$, where $\tilde{c}_j$ independ of h_L. By (6.29) we have also that if v is linear in $\tau_i^{ext} := [x_{i-1}, x_{i+2}]$ then $\tilde{\mathbb{I}}_L \mathrm{v}\big|_{\tau_i} = \mathrm{v}\big|_{\tau_i}$; see Figure 6.3. This implies the $|\cdot|_{H^1}$-stability for $\tilde{\mathbb{I}}_L$ as shown in the next lemma:

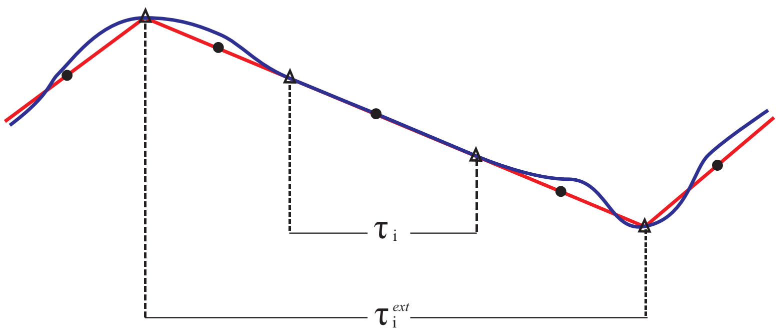


Figure 6.3: Illustration of $\tilde{\mathbb{I}}_L \mathrm{v}$ when v is linear in τ_i^{ext}.

The next lemma says about the stability of $\tilde{\mathbb{I}}_L$ in the norms L^2 and H^1. This result is important in the proof of Lemma 6.5, which is very important in the proof of the Lemma 6.8 (upper bound).

Lemma 6.4 *Let* $\mathrm{v}_L \in V_L(\Gamma)$ *and* $\tilde{\mathbb{I}}_L \mathrm{v}_L = \tilde{\mathrm{v}}_{L+1}$ *as defined above. Then*

$$\|\tilde{\mathrm{v}}_{L+1}\|^2_{L^2(\Gamma)} \lesssim \|\mathrm{v}_L\|^2_{L^2(\Gamma)} \quad \textit{and} \quad |\tilde{\mathrm{v}}_{L+1}|^2_{H^1(\Gamma)} \lesssim |\mathrm{v}_L|^2_{H^1(\Gamma)}. \tag{6.30}$$

Proof: By (6.29) we have that $\|\tilde{\mathrm{v}}_{L+1}\|^2_{L^2(\tau_i)} \lesssim \|\mathrm{v}_L\|^2_{L^2(\tau_i^{ext})}$. Therefore, $\|\tilde{\mathrm{v}}_{L+1}\|^2_{L^2(\Gamma)} \lesssim \|\mathrm{v}_L\|^2_{L^2(\Gamma)}$. Next we prove that $|\tilde{\mathrm{v}}_{L+1}|^2_{H^1(\tau_i)} \lesssim |\mathrm{v}_L|^2_{H^1(\tau_i^{ext})}$. By using an inverse inequality we have that for all constant c

$$|\tilde{\mathbb{I}}_L \mathrm{v}|_{H^1(\tau_i)} = |\tilde{\mathbb{I}}_L(\mathrm{v}_L - c)|_{H^1(\tau_i)} \lesssim h_L^{-1}\, |\tilde{\mathbb{I}}_L(\mathrm{v}_L - c)|_{L^2(\tau_i)} \lesssim h_L^{-1}\, |\mathrm{v}_L - c|_{L^2(\tau_i^{ext})}. \tag{6.31}$$

Taking the infimum over c we have $|\tilde{\mathrm{v}}_{L+1}|^2_{H^1(\tau_i)} \lesssim |\mathrm{v}_L|^2_{H^1(\tau_i^{ext})}$ and which implies that $|\tilde{\mathrm{v}}_{L+1}|^2_{H^1(\Gamma)} \lesssim |\mathrm{v}_L|^2_{H^1(\Gamma)}$. □

Another important result about $\tilde{\mathbb{I}}_L$ is the following:

Lemma 6.5 *If $\ell > k$, then*

$$\left(\tilde{\mathbb{I}}_L \mathrm{w}_\ell - \mathrm{w}_\ell, \tilde{\mathbb{I}}_L \mathrm{w}_k - \mathrm{w}_k\right)_{L^2(\Gamma)} \preceq h_L^3 h_\ell^{-2} h_k^{-1} \|\mathrm{w}_\ell\|_{L^2(\Gamma)} \|\mathrm{w}_k\|_{L^2(\Gamma)}. \tag{6.32}$$

Proof: Let $\tau_i^L = [x_i - h_L, x_i + h_L]$, where the node x_i belongs to the triangulation $\mathcal{T}_k$. Then $(\tilde{\mathbb{I}}_L \mathrm{w}_k - \mathrm{w}_k)(x) = 0$ if $x \in \Gamma \setminus \cup_i \tau_i^L$; see Figure 6.4.

Using Lemma 6.4, Cauchy inequality and Poincaré inequality we have

$$\begin{aligned}
\left(\tilde{\mathbb{I}}_L \mathrm{w}_\ell - \mathrm{w}_\ell, \tilde{\mathbb{I}}_L \mathrm{w}_k - \mathrm{w}_k\right)_{L^2} &\preceq \|\tilde{\mathbb{I}}_L \mathrm{w}_\ell - \mathrm{w}_\ell\|_{L^2(\cup_i \tau_i^L)} \|\tilde{\mathbb{I}}_L \mathrm{w}_k - \mathrm{w}_k\|_{L^2(\cup_i \tau_i^L)} \\
&\preceq h_L^2 |\tilde{\mathbb{I}}_L \mathrm{w}_k - \mathrm{w}_k|_{H^1((\cup_i \tau_i^L))} |\tilde{\mathbb{I}}_L \mathrm{w}_\ell - \mathrm{w}_\ell|_{H^1((\cup_i \tau_i^L))} \\
&\preceq h_L^2 |\mathrm{w}_k|_{H^1((\cup_i \tau_i^L))} |\mathrm{w}_\ell|_{H^1((\cup_i \tau_i^L))}.
\end{aligned}$$

As w_k and w_ℓ are linear functions in τ_i^L we have

$$|\mathrm{w}_\ell|_{H^1((\cup_i \tau_i^L))} \preceq \sqrt{h_L h_\ell^{-1}} |\mathrm{w}_\ell|_{H^1(\Gamma)} \quad \text{and} \quad |\mathrm{w}_k|_{H^1((\cup_i \tau_i^L))} \preceq \sqrt{h_L h_\ell^{-1}} |\mathrm{w}_k|_{H^1(\Gamma)}. \tag{6.33}$$

By inverse inequality the result follows. □

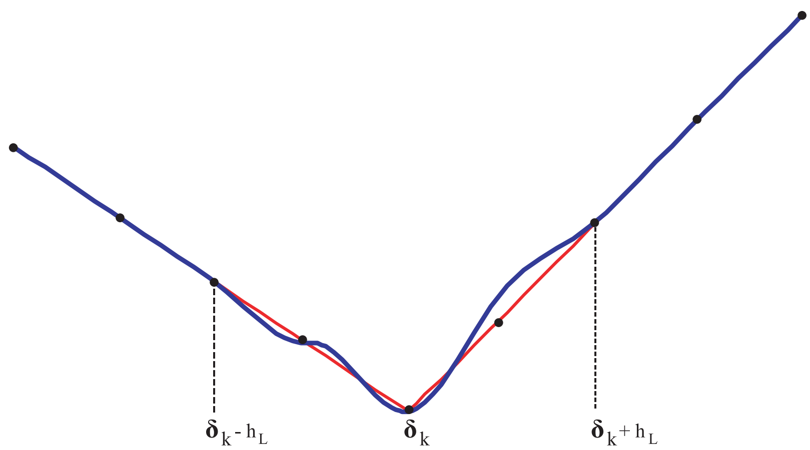


Figure 6.4: Illustration of $\tilde{\mathbb{I}}_L \mathrm{w}_k - \mathrm{w}_k$.

Using the H^1-stability of $\tilde{\mathbb{I}}_h := \tilde{\mathbb{I}}_L$ and standard properties of $\tilde{V}_L(\Gamma)$ we get the following inverse inequality:

Lemma 6.6 *For all* $\mathrm{v} \in V_h(\Gamma)$ *we have* $\|\mathrm{v}\|_{H^{-1}(\Gamma)} \preceq h^{-1/2} \|\mathrm{v}\|_{H^{-3/2}(\Gamma)}$.

Proof: Since $P_h\tilde{\mathbb{I}}_h\varphi = \varphi$ for all $\varphi \in V_h(\Gamma)$, and $\|\tilde{\varphi}\|_{H^{3/2}(\Gamma)} \le \|\tilde{\varphi}\|_{H^{3/2}(\Gamma)} \le h^{-1/2}\|\tilde{\varphi}\|_{H^1(\Gamma)}$ for all $\tilde{\varphi} \in \tilde{V}_h(\Gamma)$, then

$$\begin{aligned}
\|\mathrm{v}\|_{H^{-1}} &= \sup_{\varphi\in V_h(\Gamma)} \frac{(\mathrm{v},\varphi)_{L^2(\Gamma)}}{\|\varphi\|_{H^1(\Gamma)}} \le \sup_{\varphi\in V_h(\Gamma)} \frac{(\mathrm{v},\tilde{\mathbb{I}}_h\varphi)_{L^2(\Gamma)}}{\|\varphi\|_{H^1(\Gamma)}} \le \sup_{\varphi\in V_h(\Gamma)} \frac{(\mathrm{v},\tilde{\mathbb{I}}_h\varphi)_{L^2(\Gamma)}}{\|\tilde{\mathbb{I}}_h\varphi\|_{H^1(\Gamma)}} \\
&\le \sup_{\tilde{\varphi}\in\tilde{V}_h(\Gamma)} \frac{(\mathrm{v},\tilde{\varphi})_{L^2(\Gamma)}}{\|\tilde{\varphi}\|_{H^1(\Gamma)}} \le \sup_{\tilde{\varphi}\in\tilde{V}_h(\Gamma)} \frac{(\mathrm{v},\tilde{\varphi})_{L^2(\Gamma)}}{h^{1/2}\|\tilde{\varphi}\|_{H^{3/2}(\Gamma)}} \le h^{-1/2} \sup_{\tilde{\varphi}\in\tilde{V}_h(\Gamma)} \frac{(\mathrm{v},\tilde{\varphi})_{L^2(\Gamma)}}{\|\tilde{\varphi}\|_{H^{3/2}(\Gamma)}} \\
&\le h^{-1/2} \sup_{\varphi\in H^{3/2}} \frac{(\mathrm{v}_h,\varphi)_{L^2(\Gamma)}}{\|\varphi\|_{H^{3/2}(\Gamma)}} = h^{-1/2}\|\mathrm{v}\|_{H^{-3/2}(\Gamma)},
\end{aligned}$$

as desired. □

We need the next technical lemma in the proof of Lemma 6.8.

Lemma 6.7 *Let* $\{a_i\}_{i=0}^{\infty} \subset \mathbb{R}$ *and* $\varepsilon > 0$. *Then*

$$\left(\sum_{i=0}^{L} a_i\right)^2 \le c_\varepsilon \sum_{i=0}^{L} h_i^{-3\varepsilon} a_i^2, \quad \textit{where} \quad c_\varepsilon = \sum_{i=0}^{\infty} h_i^{3\varepsilon} = h_0 \sum_{i=0}^{\infty} 2^{-3i\varepsilon}. \tag{6.34}$$

Proof: Writing $a_i = h_i^{-3\varepsilon/2} h_i^{3\varepsilon/2} a_i$ and using Cauchy inequality we have

$$\left(\sum_{i=0}^{L} a_i\right)^2 \le \sum_{i=0}^{L} h_i^{3\varepsilon} \sum_{i=0}^{L} h_i^{-3\varepsilon} a_i^2. \tag{6.35}$$

□

Lemma 6.8 (Upper bound) *For all* $\mathrm{v}_L \in V_L$ *the following inequality holds:*

$$\|\mathrm{v}_L\|_{A^{-3/2}}^2 \le (L+1)^2 \|\mathrm{v}_L\|_{H_{t,00}^{-3/2}(\Gamma)}^2. \tag{6.36}$$

Proof: By Lemmas 6.1 and 6.2 we need to prove that

$$\inf_{z\in H_{t,00}^{3/2}(\Gamma):P_L z=\mathrm{u}_L} \|z\|_{H_{t,00}^{3/2}(\Gamma)}^2 \le (L+1)^2 \|\mathrm{u}_L\|_{A^{3/2}}^2. \tag{6.37}$$

Indeed, suppose that (6.37) holds. Then by Lemmas 6.1 and 6.2 we have

$$\|\mathrm{v}_L\|_{H_{t,00}^{-3/2}(\Gamma)}^2 \ge \frac{1}{(L+1)^2} \sup_{\mathrm{u}_L\in V_L} \frac{(\mathrm{v}_L,\mathrm{u}_L)_{L^2}^2}{\|\mathrm{u}_L\|_{A^{3/2}}^2} = \frac{1}{(L+1)^2}\|\mathrm{v}_L\|_{A^{-3/2}}^2.$$

We now introduce the operator $\tilde{\mathbb{I}}_{L,0} : V_L(\Gamma) \longmapsto \tilde{V}_{L+1,0}(\Gamma)$ such that $P_L\tilde{\mathbb{I}}_{L,0}\mathrm{v}_L = \mathrm{v}_L$ and $\tilde{\mathrm{v}}'_L(\delta_k) = 0$ for all $1 \le k \le K$. Define $\tilde{\mathbb{I}}_{L,0}\mathrm{v}_L := \tilde{\mathbb{I}}_L\mathrm{v}_L$ in τ_i such that $\delta_k \notin \tau_i$ for any $1 \le k \le K$ where $\tilde{\mathbb{I}}_L\mathrm{v}_L := \tilde{\mathrm{v}}_L$ is given by (6.29). If $\delta_k \in \tau_i$, and $\delta_k = x_i$, then $\tilde{\mathbb{I}}_{L,0}\mathrm{v}_L$ is defined as (6.29) by making $b_i := \tilde{\mathrm{v}}'_{L,0}(\delta_k) = 0$.

By [29, 36], we have $\|\tilde{v}_{L+1,0}\|^2_{H^{3/2}_{\iota,00}(\Gamma)} \asymp \sum_{\ell=0}^{L+1} h_\ell^{-3}\|\Delta\tilde{P}_{\ell,0}\tilde{v}_{L+1,0}\|^2_{L^2(\Gamma)}$, for all $\tilde{v}_{L+1,0} \in \tilde{V}_{L+1,0}(\Gamma)$. Denoting $\tilde{v}_{L+1,0} := \tilde{\mathbb{I}}_{L+1,0}v_L$ we obtain

$$\begin{aligned}
\inf_{v\in H^{3/2}_{\iota,00}(\Gamma):P_L v=v_L} \|v\|^2_{H^{3/2}_{\iota,00}(\Gamma)} &\leq \|\tilde{v}_{L+1,0}\|^2_{H^{3/2}_{\iota,00}(\Gamma)} \preceq \sum_{\ell=0}^{L+1} h_\ell^{-3}\|\Delta\tilde{P}_{\ell,0}\tilde{v}_{L+1,0}\|^2_{L^2(\Gamma)} \\
&\leq \sum_{\ell=0}^{L+1} h_\ell^{-3}\|\Delta\tilde{P}_{\ell,0}(\tilde{v}_{L+1,0} - v_{\ell-2}) + \Delta\tilde{P}_{\ell,0}v_{\ell-2}\|^2_{L^2(\Gamma)} \\
&\leq \sum_{\ell=0}^{L+1} h_\ell^{-3}\left(\|\tilde{v}_{L+1,0} - v_{\ell-2}\|^2_{L^2(\Gamma)} + \|\Delta\tilde{P}_{\ell,0}v_{\ell-2}\|^2_{L^2(\Gamma)}\right),
\end{aligned} \tag{6.38}$$

where we have denoted $v_{-2} = 0$, and we have used the L^2-stability of $\tilde{P}_{\ell,0}$. Looking separately for each term of the third inequality of (6.38), we have

$$\begin{aligned}
\|\tilde{v}_{L+1,0} - v_{\ell-2}\|^2_{L^2(\Gamma)} &\leq \|\tilde{\mathbb{I}}_{L,0}(v_L - v_{\ell-2})\|^2_{L^2(\Gamma)} + \|\tilde{\mathbb{I}}_{L,0}v_{\ell-2} - v_{\ell-2}\|^2_{L^2(\Gamma)} \\
&\leq \|v_L - v_{\ell-2}\|^2_{L^2(\Gamma)} + \|\tilde{\mathbb{I}}_{L,0}v_{\ell-2} - v_{\ell-2}\|^2_{L^2(\Gamma)}.
\end{aligned} \tag{6.39}$$

We note that in (6.39) we have used the L^2-stability of $\tilde{\mathbb{I}}_{L,0}$. We next estimate the second term of the third inequality of (6.38). Since $\tilde{P}_{L,0}$ is the best L^2 approximation in $V_\ell(\Gamma)$, we have

$$\begin{aligned}
\|\Delta\tilde{P}_{\ell,0}v_{\ell-2}\|^2_{L^2(\Gamma)} &\leq \|\tilde{P}_{\ell,0}v_{\ell-2} - v_{\ell-2}\|^2_{L^2(\Gamma)} + \|\tilde{P}_{\ell-1}v_{\ell-2} - v_{\ell-2}\|^2_{L^2(\Gamma)} \\
&\leq \|\tilde{\mathbb{I}}_{\ell-1,0}v_{\ell-2} - v_{\ell-2}\|^2_{L^2(\Gamma)} + \|\tilde{\mathbb{I}}_{\ell-2,0}v_{\ell-2} - v_{\ell-2}\|^2_{L^2(\Gamma)}.
\end{aligned} \tag{6.40}$$

Inserting (6.39) and (6.40) into (6.38), we obtain

$$\begin{aligned}
\|\tilde{v}_{L+1,0}\|^2_{A^{3/2}} &\leq \sum_{\ell=0}^{L} h_\ell^{-3}(\|v_L - v_{\ell-1}\|^2_{L^2} + \|\tilde{\mathbb{I}}_{L,0}v_\ell - v_\ell\|^2_{L^2}) \\
&+ \sum_{\ell=0}^{L} h_\ell^{-3}(\|\tilde{\mathbb{I}}_{\ell+1,0}v_\ell - v_\ell\|^2_{L^2} + \|\tilde{\mathbb{I}}_{\ell,0}v_\ell - v_\ell\|^2_{L^2}).
\end{aligned} \tag{6.41}$$

We next estimate each term of right-hand side of the inequality in (6.41). By using Lemma 6.7 and that $v_{\ell-1} = \sum_{k=0}^{\ell-1} w_k$, where $w_k = v_k - v_{k-1}$ and $v_{-1} = 0$, we estimate $\sum_{\ell=0}^{L} h_\ell^{-3}\|v_L - v_{\ell-1}\|^2_{L^2}$ as follows:

$$\|v_L - v_{\ell-1}\|^2_{L^2(\Gamma)} = \|\sum_{k=\ell}^{L} w_k\|^2_{L^2(\Gamma)} \preceq (\sum_{k=\ell}^{L} \|w_k\|_{L^2(\Gamma)})^2 \preceq \sum_{k=\ell}^{L} h_k^{-3\varepsilon}h_\ell^{3\varepsilon}\|w_k\|^2_{L^2(\Gamma)}. \tag{6.42}$$

Therefore,

$$\sum_{\ell=0}^{L} h_\ell^{-3} \sum_{k=\ell}^{L} h_k^{-3\varepsilon}h_\ell^{3\varepsilon}\|w_k\|^2_{L^2(\Gamma)} \preceq \sum_{k=0}^{L} h_k^{-3\varepsilon}\|w_k\|^2_{L^2(\Gamma)} \sum_{\ell=0}^{k} h_l^{-3(1-\varepsilon)} \preceq \sum_{k=0}^{L} h_k^{-3}\|w_k\|^2_{L^2(\Gamma)}. \tag{6.43}$$

We next estimate $h_L^{-3}\|\tilde{\mathrm{v}}_{L+1,0} - \mathrm{v}_L\|^2_{L^2}$. For this we do the following:

$$\begin{aligned} \|\tilde{\mathrm{v}}_{L+1,0} - \mathrm{v}_L\|^2_{L^2(\Gamma)} &= \|\tilde{\mathbb{I}}_{L,0}\mathrm{v}_L - \mathrm{v}_L + \tilde{\mathbb{I}}_L\mathrm{v}_L - \tilde{\mathbb{I}}_L\mathrm{v}_L\|^2_{L^2(\Gamma)} \\ &\preceq \|\tilde{\mathbb{I}}_{L,0}\mathrm{v}_L - \tilde{\mathbb{I}}_L\mathrm{v}_L\|^2_{L^2(\Gamma)} + \|\tilde{\mathbb{I}}_L\mathrm{v}_L - \mathrm{v}_L\|^2_{L^2(\Gamma)}. \end{aligned} \tag{6.44}$$

To estimate $\|\tilde{\mathbb{I}}_{L,0}\mathrm{v}_L - \tilde{\mathbb{I}}_L\mathrm{v}_L\|^2_{L^2(\Gamma)}$ consider for each corner δ_k the intervals $\tau^+_{\delta_k} := [x^k_{n_k-1}, \delta_k]$ and $\tau^-_{\delta_k} := [\delta_k, x^{k+1}_2]$. Then $\|\tilde{\mathbb{I}}_{L,0}\mathrm{v}_L - \tilde{\mathbb{I}}_L\mathrm{v}_L\|^2_{L^2(\Gamma)} = \sum_{k=1}^K(\|\tilde{\mathbb{I}}_{L,0}\mathrm{v}_L - \tilde{\mathbb{I}}_L\mathrm{v}_L\|^2_{L^2(\tau\delta_k{}^+)} + \|\tilde{\mathbb{I}}_{L,0}\mathrm{v}_L - \tilde{\mathbb{I}}_L\mathrm{v}_L\|^2_{L^2(\tau^-_{\delta_k})})$. Denote $a^{k+1}_{\delta_k} = \mathrm{v}_L(\delta_k)$, $a^{k+1}_2 = \mathrm{v}_L(x^{k+1}_2)$, $b^{k+1}_{\delta_k+1/2} = \mathrm{v}'_L(x_{\delta_k+1/2})$ where $x_{\delta_k+1/2} := \delta_k + (x^{k+1}_2 - \delta_k)/2$ and $b^k_{\delta_k-1/2} = \mathrm{v}'_L(x_{\delta_k-1/2})$ where $x_{\delta_k-1/2} := \delta_k - (\delta_k - x^k_{n_k-1})/2$ and $b^{k+1}_{3/2} = \mathrm{v}'_L(x^{k+1}_2 + (x^{k+1}_3 - x^{k+1}_2)/2)$. Then by (6.29) we have

$$\begin{aligned} \tilde{\mathbb{I}}_L\mathrm{v}_L(x_{\delta_k+1/2}) &= \tfrac{1}{2}(a^{k+1}_{\delta_k} + a^{k+1}_2) + \tfrac{1}{48}h_L(b^k_{\delta_k-1/2} - b^{k+1}_{3/2}), \\ (\tilde{\mathbb{I}}_L\mathrm{v}_L)'(x_{\delta_k+1/2}) &= \tfrac{1}{2h_L}(a^{k+1}_{\delta_k} - a^{k+1}_2) + \tfrac{3}{8}(b^k_{\delta_k-1/2} + b^{k+1}_{3/2} + 2b^{k+1}_{\delta_k+1/2}), \\ \tilde{\mathbb{I}}_{L,0}\mathrm{v}_L(x_{\delta_k+1/2}) &= \tfrac{1}{2}(a^{k+1}_{\delta_k} + a^{k+1}_2) + \tfrac{1}{48}h_L(b^k_{\delta_k+1/2} + b^{k+1}_{3/2}), \\ (\tilde{\mathbb{I}}_{L,0}\mathrm{v}_L)'(x_{\delta_k+1/2}) &= \tfrac{1}{2h_L}(a^{k+1}_{\delta_k} - a^{k+1}_2) + \tfrac{3}{8}(b^k_{\delta_k+1/2} + b^{k+1}_{3/2}), \end{aligned} \tag{6.45}$$

therefore,

$$\begin{aligned} (\tilde{\mathbb{I}}_{L,0}\mathrm{v}_L - \tilde{\mathbb{I}}_L\mathrm{v}_L)(x_{\delta_k+1/2}) &= \tfrac{1}{48}h_L(b^k_{\delta_k+1/2} - b^k_{\delta_k-1/2} + 2b^{k+1}_{3/2}), \\ (\tilde{\mathbb{I}}_L\mathrm{v}_L)'(x_{\delta_k+1/2}) &= \tfrac{3}{8}(b^k_{\delta_k-1/2} - b^{k+1}_{\delta_k+1/2}). \end{aligned} \tag{6.46}$$

Furthermore $(\tilde{\mathbb{I}}_{L,0}\mathrm{v}_L - \tilde{\mathbb{I}}_L\mathrm{v}_L)(x^{k+1}_2) = 0$, $(\tilde{\mathbb{I}}_{L,0}\mathrm{v}_L - \tilde{\mathbb{I}}_L\mathrm{v}_L)'(x^{k+1}_2) = 0$, $(\tilde{\mathbb{I}}_{L,0}\mathrm{v}_L - \tilde{\mathbb{I}}_L\mathrm{v}_L)(\delta_k) = 0$ and finally $(\tilde{\mathbb{I}}_{L,0}\mathrm{v}_L - \tilde{\mathbb{I}}_L\mathrm{v}_L)'(\delta_k) = \frac{3}{8}(b^k_{\delta_k-1/2} + b^{k+1}_{\delta_k+1/2})$. Since $\|\tilde{\varphi}^1_i\|^2_{L^2} = c_1h_L$ and $\|\tilde{\varphi}^2_i\|^2_{L^2} = c_2h^3_L$ we have

$$\|\tilde{\mathbb{I}}_{L,0}\mathrm{v}_L - \tilde{\mathbb{I}}_L\mathrm{v}_L\|^2_{L^2(\tau^-_{\delta_k})} \preceq h^3_L(|b^k_{\delta_k-1/2}|^2 + |b^k_{\delta_k+1/2}|^2 + |b^{k+1}_{3/2}|^2). \tag{6.47}$$

Each term of the expression in right-hand side of (6.47) can be estimated as follows. Since $\mathrm{v}_L = \sum_{\ell=0}^L \mathrm{w}_\ell$ then using an inverse inequality we obtain

$$\begin{aligned} |b^k_{\delta_k-1/2}|^2 &= \left|\sum_{\ell=0}^L(\mathrm{w}_\ell)'(x_{\delta_k-1/2})\right|^2 &&\leq (L+1)\sum_{\ell=0}^L\left|(\mathrm{w}_\ell)'(x_{\delta_k-1/2})\right|^2 \\ &\preceq (L+1)\sum_{\ell=0}^L h_\ell^{-1}\|(\mathrm{w}_\ell)'\|^2_{L^2} &&\preceq (L+1)\sum_{\ell=0}^L h_\ell^{-3}\|\mathrm{w}_\ell\|^2_{L^2}. \end{aligned} \tag{6.48}$$

Hence, we can conclude that $h_L^{-3}\|\tilde{\mathbb{I}}_{L,0}\mathrm{v}_L - \tilde{\mathbb{I}}_L\mathrm{v}_L\|^2_{L^2(\Gamma)} \preceq (L+1)\sum_{\ell=0}^L h_\ell^{-3}\|\mathrm{w}_\ell\|^2_{L^2}$.

It remains to estimate $\|\tilde{\mathbb{I}}_L\mathrm{v}_L - \mathrm{v}_L\|^2_{L^2(\Gamma)}$. By Lemma 6.5 we have

$$
\begin{aligned}
h_L^{-3}\|\tilde{\mathrm{v}}_{L+1}-\mathrm{v}_L\|^2_{L^2(\Gamma)} &= h_L^{-3}\|\textstyle\sum_{\ell=0}^L(\tilde{\mathbb{I}}_L\mathrm{w}_\ell-\mathrm{w}_\ell)\|^2_{L^2(\Gamma)}\\
&= h_L^{-3}\left(\textstyle\sum_{\ell=0}^L(\tilde{\mathbb{I}}_L\mathrm{w}_\ell-\mathrm{w}_\ell),\sum_{k=0}^L(\tilde{\mathbb{I}}_L\mathrm{w}_k-w_k)\right)\\
&\preceq \textstyle\sum_{k=0}^L\sum_{\ell=k}^L h_\ell^{-2}h_k^{-1}\|\mathrm{w}_\ell\|_{L^2(\Gamma)}\|\mathrm{w}_k\|_{L^2(\Gamma)} \qquad (6.49)\\
&= \textstyle\sum_{k=0}^L\sum_{\ell=k}^L h_k^{-1}h_\ell h_\ell^{-3}\|\mathrm{w}_\ell\|_{L^2(\Gamma)}\|\mathrm{w}_k\|_{L^2(\Gamma)}\\
&\preceq (L+1)\textstyle\sum_{\ell=0}^L h_L^{-3}\|\mathrm{w}_\ell\|^2_{L^2(\Gamma)}.
\end{aligned}
$$

Since we have a sum of $L+1$ terms like $h_L^{-3}\|\tilde{\mathrm{v}}_{L+1}-\mathrm{v}_L\|^2_{L^2}$ in (6.41), we conclude that

$$
\|\tilde{\mathrm{v}}_{L+1,0}\|^2_{H^{3/2}_{t,00}(\Gamma)} \preceq (L+1)^2\sum_{\ell=0}^L 2^{3\ell}\|\mathrm{w}_\ell\|^2_{L^2(\Gamma)} \preceq (L+1)^2\|\mathrm{v}_L\|^2_{A^{3/2}}, \qquad (6.50)
$$

and the proof follows. □

By Theorems 4.1 and 6.1 we conclude that

$$
Q\sum_{\ell=0}^L(\alpha h_\ell+\beta+h_\ell^3)\,\Delta_\ell Q \;\preceq\; C_{\mathrm{c}} \;\preceq\; (L+1)^2 Q\sum_{\ell=0}^L(\alpha h_\ell+\beta+h_\ell^3)\,\Delta_\ell Q. \qquad (6.51)
$$

Using (6.6) we have

$$
(L+1)^{-2}\mathcal{P}C_{\mathrm{c}} \;\preceq\; C_{\mathrm{c}}^{-1} \;\preceq\; \mathcal{P}C_{\mathrm{c}}, \text{ where } \mathcal{P}C_{\mathrm{c}} := \sum_{\ell=0}^L(\alpha h_\ell+\beta+h_\ell^3)^{-1}\,\Delta_\ell. \qquad (6.52)
$$

6.3 Numerical Results

In this section we show numerical results conforming the theory developed in this chapter. We consider two domains Ω:

1) Ω is the square $[0,1]^2$,

2) Ω is the polygon $\Omega=\{(x,y)\in\mathbb{R}^2;\ x\geq 0,\ y\geq 0,\ y\leq 1\ \text{ and }\ y\geq \mathrm{a}x-\mathrm{a}\}$,

where $\mathrm{a}>0$ is a given parameter; see Figure 5.1. In both cases, the triangulation of Ω is constructed as showed in the Figure 5.1. We divide each edge of Γ into 2^N parts of equal length, where N is an integer denoting the number of refinements.

The Reduced Hessian C_{c}, defined as in Theorem 5.2, depend of the relaxation parameters α and β, the mesh size $h=O(2^{-N})$ and the shape parameter of Ω given by "a". We divide the tests in two categories: 1) The tests such $\alpha\in\left\{1,(10)^{-3},(10)^{-6},0\right\}$ and $\beta=0$; 2) The tests such $\alpha=0$ and $\beta\in\left\{1,(10)^{-3},(10)^{-6}\right\}$.

Tables below show the condition number (cond), number of iterations (it), three lowest eigenvalues (eigs nim) and the three greatest eigenvalues (eigs max) for the matrices $\mathcal{P}C_{\mathrm{c}}*C_{\mathrm{c}}$ for different values for the relaxation parameter $\alpha\in\left\{1,(10)^{-3},(10)^{-6},0\right\}$

and $\beta = 0$, for the mesh size $h = 2^{-N}$, and Ω as in **1)**, that is a=∞ (Ω square), where $\mathcal{P}C_c$ is defined in (6.52). We show also numerical results making $\alpha = 0$ and $\beta \in \left\{1, (10)^{-3}, (10)^{-6}, 0\right\}$. When $\beta \neq 0$, by compatibility condition of $A^\dagger$ we have changing the L^2 norm matrix Q by $Q_{Ker} := (I - P_{1_m})\, Q\, (I - P_{1_m})$, where

$$P_{1_m}\lambda := \frac{({1_m}^T Q\lambda)}{({1_m}^T Q 1_m)}\, 1_m \tag{6.53}$$

is the ℓ^2-projection onto $span\{1_m\}$. Note that ${1_m}^T Q\,(I - P_{1_m}) = 0$ and if λ satisfies ${1_m}^T Q\lambda = 0$ then $\lambda^T\, Q\, \lambda = \lambda^T Q_{Ker}\lambda$.

Numerical results show that the greatest eigenvalue of $\left(\sum_{\ell=0}^{L} 1/\beta\, \Delta_\ell\right) * \beta Q_{ker}$ divided by the greatest eigenvalue of $\left(\sum_{\ell=0}^{L} h_\ell^{-3}\, \Delta_\ell\right) * QS_3^\dagger Q$ converges to 36 when h decreases to zero. This jump in the constants of equivalences deteriorates the performance of the preconditioner as shown in Table 6.1. To avoid this, we propose the rescaled preconditioner

$$\mathcal{P}C_c^r := \sum_{\ell=0}^{L} (\alpha h_\ell + C\beta + h_\ell^3)^{-1}\, \Delta_\ell,$$

with $C = 36$, instead of $\mathcal{P}C_c := \sum_{\ell=0}^{L} (\alpha h_\ell + \beta + h_\ell^3)^{-1}\, \Delta_\ell$. This change improve considerably the results (compare Tables 6.1 and 6.3).

a=∞	$\mathcal{P}C_c * C_c$ with $\beta = (0.1)^6$			$\mathcal{P}C_c * C_c$ with $\beta = 0$		
N ↓	cond	eigs min	eigs max	cond	eigs min	eigs max
3	24.5375	0.00499423	0.122546	25.3554	0.00483029	0.122474
		0.00679499	0.118539		0.00652634	0.118467
	it=8	0.00720473	0.0807817	it=8	0.00693006	0.0804496
4	27.7469	0.00486643	0.135028	33.5522	0.00401645	0.134761
		0.00634468	0.134149		0.00522075	0.133872
	it=16	0.0065078	0.119306	it=16	0.00538252	0.118546
5	23.9416	0.00603214	0.144419	41.9737	0.00340741	0.143022
		0.00697228	0.144272		0.00430967	0.142864
	it=22	0.0079075	0.139599	it=25	0.00433603	0.137029
6	36.4422	0.00731919	0.266728	50.5193	0.00293017	0.14803
		0.00769472	0.266713		0.00359212	0.147998
	it=32	0.00893824	0.26669	it=35	0.00362129	0.145917
7	89.5923	0.00782718	0.701255	59.2085	0.00255006	0.150985
		0.00794298	0.701254		0.00301998	0.150978
	it=42	0.00922875	0.701253	it=44	0.00309153	0.150962

Table 6.1: Number of PCG iterations (it), condition number (cond) and extremal eigenvalues for the matrix C_c and preconditioner $\mathcal{P}C_c$ with $\beta = (0.1)^6$, $\alpha = 0$ and $\beta = 0$, $\alpha = 0$. Ω is a square domain.

C=36	$\mathcal{P}C_c^r * C_c$ with $\beta = 1$			$\mathcal{P}C_c^r * C_c$ with $\beta = (0.1)^3$		
N ↓	cond	eigs min	eigs max	cond	eigs min	eigs max
3	1.04295	0.0275481	0.0287312	5.20211	0.0150672	0.0783815
		0.0277439	0.0287113		0.0192738	0.0781236
	it=3	0.0277469	0.0277976	it=7	0.0203725	0.037535
4	1.04237	0.0275684	0.0287364	4.94294	0.0162245	0.0801968
		0.0277505	0.0287312		0.0203398	0.0800384
	it=3	0.0277514	0.0278028	it=7	0.0210076	0.0401337
5	1.04222	0.0275736	0.0287377	4.87258	0.0165515	0.0806486
		0.0277523	0.0287364		0.0207078	0.0805994
	it=2	0.0277525	0.0278044	it=7	0.0211841	0.0409472
6	1.04218	0.0275749	0.028738	4.85515	0.0166339	0.0807604
		0.0277528	0.0287377		0.0208118	0.0807472
	it=2	0.0277528	0.0278048	it=7	0.0212295	0.0411635
7	1.04217	0.0275753	0.0287381	4.85084	0.0166545	0.0807881
		0.0277528	0.028738		0.0208387	0.0807848
	it=2	0.0277528	0.0278049	it=7	0.0212409	0.0412184

Table 6.2: Number of PCG iterations (it), condition number (cond) and extremal eigenvalues for the matrix C_c and preconditioner $\mathcal{P}C_c^r := \sum_{\ell=0}^{L} (\alpha h_\ell + C\beta + h_\ell^3)^{-1} \Delta_\ell$ with $C = 36, \beta = 1, \alpha = 0, \beta = (0.1)^3, \alpha = 0$. Ω is a square domain.

C=36	$\mathcal{P}C_c^r * C_c$ with $\beta = (0.1)^6$			$\mathcal{P}C_c^r * C_c$ with $\beta = 0$		
N ↓	cond	eigs min	eigs max	cond	eigs min	eigs max
3	24.6053	0.00496626	0.122197	25.3554	0.00483029	0.122474
		0.00673302	0.118204		0.00652634	0.118467
	it=8	0.00713975	0.079778	it=8	0.00693006	0.0804496
4	28.1662	0.0047469	0.133702	33.5522	0.00401645	0.134761
		0.00616563	0.132788		0.00522075	0.133872
	it=15	0.00626578	0.115992	it=16	0.00538252	0.118546
5	24.3303	0.00573966	0.139648	41.9737	0.00340741	0.143022
		0.0066376	0.13947		0.00430967	0.142864
	it=20	0.00740123	0.130766	it=25	0.00433603	0.137029
6	20.3042	0.00698434	0.141811	50.5193	0.00293017	0.14803
		0.00735495	0.141772		0.00359212	0.147998
	it=22	0.00844516	0.135325	it=35	0.00362129	0.145917
7	18.9576	0.00751417	0.14245	59.2085	0.00255006	0.150985
		0.00763686	0.142441		0.00301998	0.150978
	it=20	0.00877157	0.136594	it=44	0.00309153	0.150962

Table 6.3: Number of PCG iterations (it), condition number (cond) and extremal eigenvalues for the matrix C_c and preconditioner $\mathcal{P}C_c^r := \sum_{\ell=0}^{L} (\alpha h_\ell + C\beta + h_\ell^3)^{-1} \Delta_\ell$ with $C = 36, \beta = (0.1)^6, \alpha = 0, \beta = 0, \alpha = 0$. Ω is a square domain.

a= ∞	$\mathcal{P}C_c * C_c$ with $\alpha = 1$			$\mathcal{P}C_c * C_c$ with $\alpha = (0.1)^3$		
N ↓	cond	eigs min	eigs max	cond	eigs min	eigs max
3	3.94215	0.119978	0.472971	15.132	0.00836788	0.126623
		0.122913	0.472672		0.0103951	0.122834
	it=7	0.122919	0.441244	it=8	0.0115835	0.0952644
4	4.62312	0.118933	0.549842	13.7601	0.0106986	0.147214
		0.119541	0.540384		0.0115993	0.146781
	it=10	0.119541	0.540357	it=14	0.0135269	0.14579
5	5.12018	0.118259	0.605507	18.3917	0.0125039	0.229969
		0.118369	0.59432		0.0131101	0.22935
	it=10	0.118369	0.594317	it=19	0.0151533	0.229198
6	5.33402	0.11798	0.629308	26.2878	0.0131387	0.345387
		0.117998	0.619555		0.0140697	0.344935
	it=11	0.117998	0.619554	it=22	0.0158111	0.344912
7	5.45327	0.117888	0.642873	35.6393	0.0133125	0.474449
		0.11789	0.637867		0.0144698	0.474354
	it=12	0.11789	0.637867	it=26	0.0160111	0.474353

Table 6.4: Number of PCG iterations (it), condition number (cond) and extremal eigenvalues for the matrix C_c and preconditioner $\mathcal{P}C_c$ with $\beta = 0$, $\alpha = 1$ and $\beta = 0$, $\alpha = (0.1)^3$. Ω is a square domain.

a= ∞	$\mathcal{P}C_c * C_c$ with $\alpha = (0.1)^6$			$\mathcal{P}C_c * C_c$ with $\alpha = 0$		
N ↓	cond	eigs min	eigs max	cond	eigs min	eigs max
3	25.3324	0.00483482	0.122478	25.3554	0.00483029	0.122474
		0.00653318	0.118471		0.00652634	0.118467
	it=8	0.00693503	0.0804647	it=8	0.00693006	0.0804496
4	33.4363	0.00403064	0.13477	33.5522	0.00401645	0.134761
		0.00524073	0.133881		0.00522075	0.133872
	it=16	0.00539447	0.118574	it=16	0.00538252	0.118546
5	41.4318	0.00345247	0.143042	41.9737	0.00340741	0.143022
		0.0043709	0.142885		0.00430967	0.142864
	it=25	0.00437165	0.137072	it=25	0.00433603	0.137029
6	48.1852	0.00307299	0.148073	50.5193	0.00293017	0.14803
		0.00369701	0.148041		0.00359212	0.147998
	it=33	0.00380752	0.145999	it=35	0.00362129	0.145917
7	50.8326	0.00297327	0.151139	59.2085	0.00255006	0.150985
		0.00333204	0.151084		0.00301998	0.150978
	it=43	0.0035952	0.151077	it=44	0.00309153	0.150962

Table 6.5: Number of PCG iterations (it), condition number (cond) and extremal eigenvalues for the matrix C_c and preconditioner $\mathcal{P}C_c$ with $\beta = 0$, $\alpha = (0.1)^6$ and $\beta = 0$, $\alpha = 0$. Ω is a square domain.

7
An Inexact Multilevel Method

The projection P_ℓ defined in (6.2) can be expensive to evaluate because it requires solving a linear system with the mass matrices Q_ℓ. To obtain a more efficient implementation, avoiding these issues we now replace each projection P_ℓ by another operator $\bar{P}_\ell : V_L \longmapsto V_\ell$ and replace $(P_\ell - P_{\ell-1})$ by $(\bar{P}_\ell - \bar{P}_{\ell-1})^*(\bar{P}_\ell - \bar{P}_{\ell-1})$, where $*$ stands for adjoint with respect to the Q norm and $\bar{P}_{-1} = 0$. This approximation is easier to compute than P_ℓ.

We want to answer the following question: under which conditions on $\{\mu_\ell\}$ and $\bar{P}_\ell$ the operators below

$$A_L := \sum_{\ell=0}^{L} \mu_\ell (P_\ell - P_{\ell-1}) \quad \text{and} \quad \bar{A}_L := \sum_{\ell=0}^{L} \mu_\ell (\bar{P}_\ell^* - \bar{P}_{\ell-1}^*)(\bar{P}_\ell - \bar{P}_{\ell-1}) \tag{7.1}$$

are spectrally equivalent in the L^2-norm? In [7] they provides sufficient hypothesis on μ_ℓ and $\bar{P}_\ell$ to obtain the equivalence between the operators A_L and $\bar{A}_L$ defined in (7.1). Furthermore, they construct operators $\{\bar{P}_\ell\}$ satisfying these hypothesis. Unfortunately for negative Sobolev norms, this construction does not work well; see Table 7.2. Next we provide an alternative construction of $\bar{P}_\ell$ which works better for negative norms; see Table 7.1.

Let $\bar{\psi}_i^\ell := -1/2\varphi_{j-1}^{\ell+1} + 3\varphi_j^{\ell+1} - 1/2\varphi_{j+1}^{\ell+1}$, where $x_j^{\ell+1} = x_i^\ell$; see Figure 7.1. For

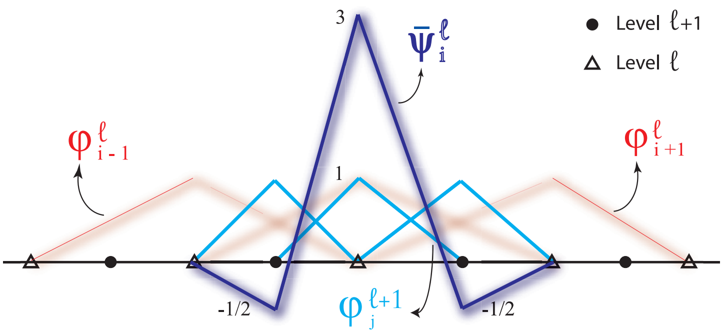


Figure 7.1: Illustration of the basis function $\bar{\psi}_i^\ell$.

$\ell < L$, define $\bar{V}_\ell := span\{\bar{\psi}_i^\ell\}_{i=1}^{m_\ell}$ and $\bar{V}_L := V_L$. Then we have a biorthogonality property between $\bar{V}_\ell$ and V_ℓ for $\ell < L$, that is,

$$(\varphi_{i_1}^\ell, \bar{\psi}_{i_2}^\ell) = 0 \text{ if } i_1 \neq i_2. \tag{7.2}$$

Let $\bar{P}_\ell : L^2 \longmapsto V_\ell$ be defined as follows: for each $v \in L^2$, $\bar{P}_\ell v$ is the unique vector belonging to V_ℓ such that

$$(\bar{P}_\ell v, \bar{\psi}_i^\ell)_{L^2} = (v, \bar{\psi}_i^\ell)_{L^2} \text{ for all } 1 \leq i \leq m_\ell. \tag{7.3}$$

Note that thanks to this biorthogonality, $\bar{P}_\ell$ restricted to V_L can be calculated very fast avoiding solving linear systems with matrices Q_ℓ. In matrix term,

$$\bar{P}_\ell := R_\ell^T \bar{D}_\ell^{-1} \bar{R}_\ell Q, \tag{7.4}$$

where $\bar{R}_\ell$ is defined such that it i-th row is obtained by interpolating the basis function $\bar{\psi}_i^\ell \in V_{\ell+1}$ at the nodes of the finest triangulation $\mathcal{T}_L := \mathcal{T}_h$, and $\bar{D}_\ell := R_\ell Q \bar{R}_\ell^T$. We now state and prove Theorem 7.1, which establishes that A_L and $\bar{A}_L$ defined in (7.1) are spectrally equivalents if $\bar{P}_\ell$ is as in (7.4) and $\mu_\ell = h_\ell^{-2s}$ for all $s > 0$.

Theorem 7.1 *Define* $\|\mathrm{v}\|_{\bar{A}^s}^2 := \sum_{\ell=0}^L h_\ell^{-2s} \|\bar{P}_\ell \mathrm{v} - \bar{P}_{\ell-1}\mathrm{v}\|_{L^2}^2$ *and* $\|\mathrm{v}\|_{A^s}^2 := \sum_{\ell=0}^L h_\ell^{-2s} \|P_\ell \mathrm{v} - P_{\ell-1}\mathrm{v}\|_{L^2}^2$. *Then, for all* $s > 0$ *we have* $\|\mathrm{v}\|_{\bar{A}^s}^2 \asymp \|\mathrm{v}\|_{A^s}^2$.

Proof: To prove that $\|\mathrm{v}\|_{\bar{A}^s}^2 \preceq \|\mathrm{v}\|_{A^s}^2$ it is sufficient to prove that

$$\sum_{\ell=0}^L h_\ell^{-2s} \|\bar{P}_\ell \mathrm{v} - P_\ell \mathrm{v}\|_{L^2}^2 \preceq \sum_{\ell=0}^L h_\ell^{-2s} \|P_\ell \mathrm{v} - P_{\ell-1}\mathrm{v}\|_{L^2}^2. \tag{7.5}$$

In fact, by a triangular inequality and (7.5) we have

$$\begin{aligned}
\|\mathrm{v}\|_{\bar{A}^s}^2 &= \textstyle\sum_{\ell=0}^L h_\ell^{-2s} \|\bar{P}_\ell \mathrm{v} - P_\ell + P_\ell \mathrm{v} - P_{\ell-1}\mathrm{v} + P_{\ell-1}\mathrm{v} - \bar{P}_{\ell-1}\mathrm{v}\|_{L^2}^2 \\
&\leq 3 \textstyle\sum_{\ell=0}^L h_\ell^{-2s} (\|P_\ell \mathrm{v} - \bar{P}_\ell \mathrm{v}\|_{L^2}^2 + \|P_\ell \mathrm{v} - P_{\ell-1}\mathrm{v}\|_{L^2}^2 + \|P_{\ell-1}\mathrm{v} - \bar{P}_{\ell-1}\mathrm{v}\|_{L^2}^2) \\
&\preceq \textstyle\sum_{\ell=0}^L h_\ell^{-2s} \|P_\ell \mathrm{v} - P_{\ell-1}\mathrm{v}\|_{L^2}^2.
\end{aligned}$$

Next we prove (7.5). We observe that $\bar{P}_\ell P_{\ell+1} = \bar{P}_\ell$. Indeed, since $\bar{V}_\ell \subset V_{\ell+1}$ by definition of $\bar{P}_\ell$ and $P_{\ell+1}$ have

$$(\bar{P}_\ell P_{\ell+1}\mathrm{v}, \bar{\psi}_i^\ell)_{L^2} = (P_{\ell+1}\mathrm{v}, \bar{\psi}_i^\ell)_{L^2} = (\mathrm{v}, \bar{\psi}_i^\ell)_{L^2} = (\bar{P}_\ell \mathrm{v}, \bar{\psi}_i^\ell)_{L^2} \tag{7.6}$$

for all $\bar{\psi}_i^\ell \in \bar{V}_\ell$. Using that $\bar{P}_\ell P_{\ell+1} = \bar{P}_\ell$, $\bar{P}_\ell^2 = \bar{P}_\ell$, and the L^2-stability of $\bar{P}_\ell$, we obtain

$$\begin{aligned}
\textstyle\sum_{\ell=0}^{L} h_\ell^{-2s}\|P_\ell \mathrm{v} - \bar{P}_\ell \mathrm{v}\|_{L^2}^2 &= \textstyle\sum_{\ell=0}^{L} h_\ell^{-2s}(P_\ell \mathrm{v} - \bar{P}_\ell \mathrm{v}, P_\ell \mathrm{v} - \bar{P}_\ell \mathrm{v})_{L^2} \\
&= \textstyle\sum_{\ell=0}^{L} h_\ell^{-2s}((P_\ell - \bar{P}_\ell)(P_{\ell+1} - P_\ell)\mathrm{v}, P_\ell \mathrm{v} - \bar{P}_\ell \mathrm{v})_{L^2} \\
&\leq \textstyle\sum_{\ell=0}^{L}(h_\ell^{-s}\|(P_{\ell+1} - P_\ell)\mathrm{v}\|_{L^2})(h_\ell^{-s}\|(P_\ell - \bar{P}_\ell)\mathrm{v}\|_{L^2}) \\
&\leq \tfrac{1}{\sqrt{2}}\left(\textstyle\sum_{\ell=0}^{L} h_\ell^{-2s}\|(P_{\ell+1} - P_\ell)\mathrm{v}\|_{L^2}^2 + \sum_{\ell=0}^{L} h_\ell^{-2s}\|(P_\ell - \bar{P}_\ell)\mathrm{v}\|_{L^2}^2\right).
\end{aligned}$$

Since $1/\sqrt{2} < 1$, (7.5) follows.

It remains to prove that $\|\mathrm{v}\|_{A^s}^2 \leq \|\mathrm{v}\|_{\bar{A}^s}^2$. Let us denote $\bar{P}_{-1} = 0$ and set $\mathrm{v} = \sum_{k=0}^{L}(\bar{P}_k - \bar{P}_{k-1})\mathrm{v}$. Since $(P_\ell - P_{\ell-1})(\bar{P}_k) = 0$ for $k \leq \ell - 1$, we have

$$\begin{aligned}
\|\mathrm{v}\|_{A^s}^2 &= \textstyle\sum_{\ell=0}^{L} h_\ell^{-2s}((P_\ell - P_{\ell-1})\mathrm{v}, (P_\ell - P_{\ell-1})\mathrm{v})_{L^2} \\
&= \textstyle\sum_{\ell=0}^{L} h_\ell^{-2s}((P_\ell - P_{\ell-1})\sum_{k=\ell}^{L}(\bar{P}_k - \bar{P}_{k-1})\mathrm{v}, (P_\ell - P_{\ell-1})\mathrm{v})_{L^2} \\
&\leq \textstyle\sum_{\ell=0}^{L}\sum_{k=\ell}^{L} h_\ell^{-2s}\|(P_\ell - P_{\ell-1})(\bar{P}_k - \bar{P}_{k-1})\mathrm{v}\|_{L^2}\|P_\ell - P_{\ell-1})\mathrm{v}\|_{L^2} \\
&\leq \textstyle\sum_{\ell=0}^{L}\sum_{k=\ell}^{L}(h_\ell^{-s}h_k^{s})(h_k^{-s}\|(\bar{P}_k - \bar{P}_{k-1})\mathrm{v}\|_{L^2})(h_\ell^{-s}\|P_\ell - P_{\ell-1})\mathrm{v}\|_{L^2}).
\end{aligned} \tag{7.7}$$

Consider the triangular matrix $\mathcal{L}$ defined by $\mathcal{L}_{\ell,k} = h_\ell^{-s}h_k^{s}$ for $\ell \leq k$ and $\mathcal{L}_{\ell,k} = 0$ for $\ell \geq k+1$. By hypothesis the refinement is dyadic, that is, $h_k = h_0\,2^{-k}$ and $h_\ell = h_0\,2^{-\ell}$, and also $s > 0$. Hence $\|\mathcal{L}\|_{\ell^2} \leq C_s := h_0\sum_{n=0}^{\infty}((1/2)^{ns}) = h_0(1-(1/2)^s)^{-1}$, and with (7.7) we obtain

$$\|\mathrm{v}\|_{A^s}^2 \leq C_s\left(\sum_{k=0}^{L} h_k^{-2s}\|\bar{P}_k - \bar{P}_{k-1})\mathrm{v}\|_{L^2}^2\right)^{1/2}\left(\sum_{k=0}^{L} h_\ell^{-2s}\|P_\ell - P_{\ell-1})\mathrm{v}\|_{L^2}^2\right)^{1/2}, \tag{7.8}$$

therefore, $\|\mathrm{v}\|_{A^s}^2 \leq \|\mathrm{v}\|_{\bar{A}^s}^2$ and the proof is complete. □

Note that we do not use that $s > 0$ for proving that $\|\mathrm{v}\|_{\bar{A}^s}^2 \leq \|\mathrm{v}\|_{A^s}^2$, so it is true for any s. Unfortunately the constant C_s grows when s tends to zero. In fact we have the following result:

Corollary 7.1 *Let* $\|\cdot\|_{A^0}$ *and* $\|\cdot\|_{\bar{A}^0}$ *are as in Theorem 7.1 for* $s = 0$. *Then for all* $\mathrm{v} \in V_L$ *we have*

$$\|\mathrm{v}\|_{\bar{A}^0}^2 \leq \|\mathrm{v}\|_{A^0}^2 \leq L\,\|\mathrm{v}\|_{\bar{A}^0}^2. \tag{7.9}$$

Proof: It remains to proof that $\|\mathrm{v}\|_{A^0}^2 \leq L\,\|\mathrm{v}\|_{\bar{A}^0}^2$. A simple computation gives

$$\|\mathrm{v}\|_{A^0}^2 = \|\mathrm{v}\|_{L^2}^2 = \Big\|\sum_{\ell=0}^{L}(\bar{P}_\ell - \bar{P}_{\ell-1})\mathrm{v}\Big\|_{L^2}^2 \leq L\sum_{\ell=0}^{L}\|(\bar{P}_\ell - \bar{P}_{\ell-1})\mathrm{v}\|_{L^2}^2 = L\,\|\mathrm{v}\|_{\bar{A}^0}^2. \tag{7.10}$$

This completes the proof. □

Remark 7.1 *Theorem 7.1 provides an optimal preconditioner for negative Sobolev norms with low cost. For positive norms this is no true. However we can provide an optimal preconditioner for positive Sobolev norms using $\bar{P}^*_\ell$, where $*$ stands the adjoint with respect to L^2-norm. Using very similar arguments of the proof of Theorem 7.1 one can prove that for all $s < 0$*

$$\|\mathrm{v}\|^2_{\bar{A}^{*s}} := \sum_{\ell=0}^{L} h_\ell^{-2s} \|(\bar{P}^*_\ell - \bar{P}^*_{\ell-1})\mathrm{v}\|^2_{L^2} \asymp \|\mathrm{v}\|^2_{\bar{A}^s}\,. \tag{7.11}$$

Next we show tables comparing the preconditioner provided in [7] and our preconditioner using the matrix A_L, defined in (7.1). The approximation $\overline{P}_\ell : L^2 \longmapsto V_\ell$ made in [7] is:

$$\overline{P}_\ell v = \sum_{j=1}^{J^\ell} \frac{(v, \varphi_j^\ell)_{L^2}}{(1, \varphi_j^\ell)_{L^2}} \varphi_j^\ell, \tag{7.12}$$

where φ_j^ℓ, $j = 1, ..., J^\ell$, are the nodal basis on V_ℓ. We define $\overline{A}_L := \sum_{\ell=0}^{L} \mu_\ell (\overline{P}^*_\ell - \overline{P}^*_{\ell-1})(\overline{P}_\ell - \overline{P}_{\ell-1})$ and $\bar{A}_L := \sum_{\ell=0}^{L} \mu_\ell (\bar{P}^*_\ell - \bar{P}^*_{\ell-1})(\bar{P}_\ell - \bar{P}_{\ell-1})$.

7.1 Numerical Results

In this section we provide in Tables 7.1, numerical results which illustrate Theorem 7.1 for two values of s, $s = 1$ and $s = 3/2$. Table 7.2 shows the effectiveness of our method compared to the one presented in [7].

In the results presented here we consider a structured triangulation on $\Omega = [0, 1]^2$ with mesh parameter $h = 2^{-N}$, where N is an integer denoting the number of refinements. In that numerical test we generate the matrices $\bar{A}_L$ and $(Q * A_L)^{-1}$ of (7.1), and we use the function *eig* of MATLAB to obtain the eigenvalues of $\bar{A}_L * (Q * A_L)^{-1}$.

Note that the condition number (cond), presented in the second and the fourth columns of Table 7.1, do not depend on the parameter N of the refinement as established in Theorem 7.1. Comparing the Tables 7.1 and 7.2 we conclude that our results are considerably better than that of the method presented in [7] for values of $s > 0$. However, our method is worse for $s = 0$ since it presents a logarithmic growing in the condition number of $\bar{A}_L * (Q * A_L)^{-1}$ while for the other method that does not happen.

We present also numerical tests using the same notation and discretization presented in Section 6.3 just changing $\mathcal{P}C_c$ by $\bar{\mathcal{P}C}_c$ defined by

$$\bar{\mathcal{P}C}_c := \sum_{\ell=0}^{L} (\alpha h_\ell + \beta + h_\ell^3)^{-1}\, \bar{\Delta}_\ell^T Q \bar{\Delta}_\ell, \tag{7.13}$$

where $\bar{\Delta}_\ell = R_\ell^T \bar{D}_\ell \bar{R}_\ell - R_{\ell-1}^T \bar{D}_{\ell-1} \bar{R}_{\ell-1}$; see (7.4). When $\alpha = 0$, using (6.52) and Theorem

7.1 we obtain

$$(L+1)^{-2}\,\bar{\mathcal{P}C}_{\rm c} \;\le\; C_{\rm c}^{-1} \;\le\; L\,\bar{\mathcal{P}C}_{\rm c},\ \text{where}\ \bar{\mathcal{P}C}_{\rm c} := \sum_{\ell=0}^{L}(\beta+h_\ell^3)^{-1}\,\bar{\Delta}_\ell^T Q\bar{\Delta}_\ell. \qquad (7.14)$$

See Section 1.2 for details on $C_{\rm c}$.

N ↓	$\bar{A}_L * (Q * A_L)^{-1};\quad \mu_\ell = h_\ell^{-2}$			$\bar{A}_L * (Q * A_L)^{-1};\quad \mu_\ell = h_\ell^{-3}$		
	cond	eigs min	eigs max	cond	eigs min	eigs max
3	4.72033	0.771932	3.64377	3.63043	0.877897	3.18714
		0.771932	3.31916		0.877897	2.92887
		0.78797	3.31916		0.882766	2.92887
4	4.92431	0.764346	3.76388	3.73629	0.876227	3.27384
		0.764346	3.56677		0.876227	3.11952
		0.771932	3.56677		0.877897	3.11952
5	5.04187	0.760119	3.83242	3.79543	0.875559	3.32312
		0.760119	3.70846		0.875559	3.22707
		0.764346	3.70846		0.876227	3.22707
6	5.11537	0.757533	3.87506	3.83162	0.875269	3.3537
		0.757533	3.79373		0.875269	3.29115
		0.760119	3.79373		0.875559	3.29115
7	5.16423	0.755839	3.90333	3.85533	0.875134	3.37393
		0.755839	3.84777		0.875134	3.33144
		0.757533	3.84777		0.875269	3.33144

Table 7.1: Illustrate the equivalence between $\bar{A}_L$ and A_L of (7.1) for specifics μ_ℓ.

N ↓	$\overline{A}_L * (Q * A_L)^{-1};\quad \mu_\ell = h_\ell^{-2}$			$\overline{A}_L * (Q * A_L)^{-1};\quad \mu_\ell = h_\ell^{-3}$		
	cond	eigs min	eigs max	cond	eigs min	eigs max
3	17.077	0.111111	1.89744	73.4571	0.111111	8.1619
		0.115286	1.4615		0.114798	4.57528
		0.115286	1.22523		0.114798	3.56998
4	19.7906	0.111111	2.19896	110.868	0.111111	12.3186
		0.112149	1.89744		0.111914	8.1619
		0.112149	1.53599		0.111914	6.14152
5	21.9205	0.111111	2.43561	154.663	0.111111	17.1848
		0.11137	2.19896		0.111252	12.3186
		0.11137	1.89744		0.111252	9.3925
6	23.6467	0.111111	2.62741	205.162	0.111111	22.7958
		0.111176	2.43561		0.111116	17.1848
		0.111176	2.19896		0.111116	13.4196
7	25.0746	0.111111	2.78607	262.517	0.111097	29.1649
		0.111127	2.62741		0.111097	22.7958
		0.111127	2.43561		0.111106	18.2292

Table 7.2: Illustrate the equivalence between $\overline{A}_L$ and A_L of (7.1) for specifics μ_ℓ.

a= ∞	$\bar{\mathcal{P}}C_c * C_c$ with $\beta = (0.1)^6$			$\bar{\mathcal{P}}C_c * C_c$ with $\beta = 0$		
N ↓	cond	eigs min	eigs max	cond	eigs min	eigs max
3	22.7676	0.0064887	0.147732	23.3452	0.00632368	0.147627
		0.011134	0.142436		0.0109698	0.142345
	it=15	0.0120026	0.0987038	it=16	0.0116039	0.0984577
4	25.668	0.00622232	0.159714	29.7618	0.00535186	0.159281
		0.00933142	0.158327		0.00861984	0.15789
	it=20	0.0108263	0.129496	it=23	0.00901202	0.128679
5	23.0098	0.00732646	0.168581	35.9693	0.0045967	0.16534
		0.00889362	0.168157		0.0069454	0.164978
	it=23	0.00991208	0.149911	it=27	0.00725875	0.143105
6	64.9098	0.00843315	0.547394	42.1306	0.00400057	0.168546
		0.0089904	0.547388		0.00570964	0.168425
	it=40	0.00932373	0.547315	it=34	0.0060128	0.149558
7	217.391	0.00883855	1.92142	48.3784	0.00351973	0.170279
		0.00905456	1.9213		0.00477599	0.170244
	it=60	0.00914403	1.92128	it=38	0.00508293	0.152831

Table 7.3: Number of PCG iterations (it), condition number (cond) and extremal eigenvalues for the matrix C_c and preconditioner $\bar{\mathcal{P}}C_c$ with $\beta = (0.1)^6$, $\alpha = 0$ and $\beta = 0$, $\alpha = 0$. Ω is a square domain.

a= ∞	$\bar{\mathcal{P}C}_c^r * C_c$ with $\beta = 1$			$\bar{\mathcal{P}C}_c^r * C_c$ with $\beta = (0.1)^3$		
N ↓	cond	eigs min	eigs max	cond	eigs min	eigs max
3	13.0286	0.0118084	0.153848	6.83611	0.0162415	0.111028
		0.0118122	0.140076		0.0162762	0.110798
	it=9	0.0132197	0.140073	it=9	0.0172686	0.107214
4	14.9076	0.0108368	0.161551	9.66956	0.0140581	0.135936
		0.0108377	0.152887		0.0142894	0.134931
	it=12	0.0115837	0.152887	it=13	0.0146748	0.134902
5	16.2195	0.0102258	0.165857	12.1802	0.0124425	0.151553
		0.0102259	0.160315		0.0124902	0.150661
	it=13	0.0106736	0.160315	it=16	0.0127925	0.150657
6	17.1744	0.00981068	0.168493	14.0805	0.0113599	0.159952
		0.00981072	0.164823		0.0113695	0.159349
	it=13	0.0101032	0.164823	it=16	0.0115852	0.159348
7	17.8924	0.00951351	0.170219	15.4996	0.0106309	0.164775
		0.00951352	0.167699		0.0106328	0.164372
	it=15	0.00971625	0.167699	it=19	0.0107838	0.164371

Table 7.4: Number of PCG iterations (it), condition number (cond) and extremal eigenvalues for the matrix C_c and preconditioner $\bar{\mathcal{P}C}_c^r$ with $C = 36$, $\beta = 1$, $\alpha = 0$ and $\beta = (0.1)^3$, $\alpha = 0$. Ω is a square domain.

a= ∞	$\bar{\mathcal{P}C}_c^r * C_c$ with $\beta = (0.1)^6$			$\bar{\mathcal{P}C}_c^r * C_c$ with $\beta = 0$		
N ↓	cond	eigs min	eigs max	cond	eigs min	eigs max
3	22.7567	0.0064683	0.147197	23.3452	0.00632368	0.147627
		0.0110755	0.141956		0.0109698	0.142345
	it=15	0.0119112	0.0978372	it=16	0.0116039	0.0984577
4	25.743	0.006128	0.157753	29.7618	0.00535186	0.159281
		0.00912901	0.156365		0.00861984	0.15789
	it=21	0.0104989	0.125778	it=23	0.00901202	0.128679
5	22.729	0.00708591	0.161055	35.9693	0.0045967	0.16534
		0.00842897	0.160718		0.0069454	0.164978
	it=23	0.00952351	0.1353	it=27	0.00725875	0.143105
6	19.8923	0.00813781	0.16188	42.1306	0.00400057	0.168546
		0.00826507	0.161792		0.00570964	0.168425
	it=21	0.00865989	0.137752	it=34	0.0060128	0.149558
7	19.9685	0.00812474	0.162239	48.3784	0.00351973	0.170279
		0.00824064	0.162217		0.00477599	0.170244
	it=20	0.00854636	0.138703	it=38	0.00508293	0.152831

Table 7.5: Number of PCG iterations (it), condition number (cond) and extremal eigenvalues for the matrix C_c and preconditioner $\bar{\mathcal{P}C}_c^r$ with $C = 36$, $\beta = (0.1)^6$, $\alpha = 0$ and $\beta = 0$, $\alpha = 0$. Ω is a square domain.

a= ∞	$\bar{\mathcal{P}C}_c * C_c$ with $\alpha = 1$			$\bar{\mathcal{P}C}_c * C_c$ with $\alpha = (0.1)^3$		
N ↓	cond	eigs min	eigs max	cond	eigs min	eigs max
3	4.78062	0.187885	0.898209	15.3368	0.00993093	0.152309
		0.188188	0.897257		0.0150829	0.147049
	it=8	0.20272	0.861765	it=15	0.0179356	0.111686
4	5.79026	0.183603	1.06311	14.0124	0.0119613	0.167607
		0.183676	1.05514		0.0151084	0.166313
	it=10	0.19456	1.05512	it=17	0.016217	0.152288
5	6.73812	0.180913	1.21901	18.6371	0.0133857	0.249471
		0.180931	1.20574		0.0153333	0.249405
	it=11	0.190255	1.20573	it=18	0.0156646	0.247646
6	7.41576	0.179247	1.32925	28.6535	0.0139265	0.399044
		0.179251	1.31981		0.015432	0.399029
	it=11	0.187497	1.31981	it=22	0.0155197	0.398816
7	7.8784	0.178157	1.40359	42.5756	0.0140843	0.599645
		0.178158	1.3977		0.0154624	0.599645
	it=12	0.185563	1.3977	it=26	0.0154847	0.599412

Table 7.6: Number of PCG iterations (it), condition number (cond) and extremal eigenvalues for the matrix C_c and preconditioner $\bar{\mathcal{P}C}_c$ with $\beta = 0$, $\alpha = 1$ and $\beta = 0$, $\alpha = (0.1)^3$. Ω is a square domain.

a= ∞	$\bar{\mathcal{P}C}_c * C_c$ with $\alpha = (0.1)^6$			$\bar{\mathcal{P}C}_c * C_c$ with $\alpha = 0$		
N ↓	cond	eigs min	eigs max	cond	eigs min	eigs max
3	23.3277	0.00632862	0.147632	23.3452	0.00632368	0.147627
		0.0109755	0.14235		0.0109698	0.142345
	it=16	0.0116153	0.0984707	it=16	0.0116039	0.0984577
4	29.6748	0.0053678	0.159289	29.7618	0.00535186	0.159281
		0.00863489	0.157897		0.00861984	0.15789
	it=23	0.00904476	0.128699	it=23	0.00901202	0.128679
5	35.5677	0.00464894	0.165352	35.9693	0.0045967	0.16534
		0.00698676	0.16499		0.0069454	0.164978
	it=27	0.00735689	0.143135	it=27	0.00725875	0.143105
6	40.4132	0.00417114	0.168569	42.1306	0.00400057	0.168546
		0.00582634	0.168447		0.00570964	0.168425
	it=31	0.00630793	0.149606	it=34	0.0060128	0.149558
7	42.111	0.00404468	0.170325	48.3784	0.00351973	0.170279
		0.00509294	0.17029		0.00477599	0.170244
	it=36	0.00575649	0.152937	it=38	0.00508293	0.152831

Table 7.7: Number of PCG iterations (it), condition number (cond) and extremal eigenvalues for the matrix C_c and preconditioner $\bar{\mathcal{P}C}_c$ with $\beta = 0$, $\alpha = (0.1)^6$ and $\beta = 0$, $\alpha = 0$. Ω is a square domain.

8
Compendium and Conclusions

In this chapter we apply the theory developed in this thesis to solve the LQECP defined in (1.1). We consider two discretizations for the LQECP: 1) continuous at the corners of the polygonal domain Ω. 2) Discontinuous at the corners of the polygonal domain.

The discrete solution of the LQECP when the continuous at the corners discretization is used is obtained solving the linear system:

$$\begin{bmatrix} M & 0 & A^T \\ 0 & G & Q_{ext}^T \\ A & Q_{ext} & 0 \end{bmatrix} \begin{bmatrix} \mathrm{y} \\ \lambda \\ \mathrm{p} \end{bmatrix} = \begin{bmatrix} \mathrm{f}_1 \\ \mathrm{f}_2 \\ \mathrm{f}_3 \end{bmatrix}, \tag{8.1}$$

where the sub-matrices M, A, Q, Q_{ext} and $G = \alpha\, QS_1^{\dagger}Q + \beta Q$ are defined in Section 1.1.

Eliminating the variables y and p from the equation (8.1), we have:

$$(G + Q_{ext}^T A^{\dagger} M A^{\dagger} Q_{ext})\,\lambda = Q_{ext}^T A^{\dagger} M A^{\dagger} \mathrm{f}_3 - Q_{ext}^T A^{\dagger} \mathrm{f}_1. \tag{8.2}$$

The matrix $C_{\mathrm{c}} = G + Q_{ext}^T A^{\dagger} M A^{\dagger} Q_{ext}$ is known as the *Schur complement* (Reduced Hessian) with respect to the discrete control variable λ. To use the PCG to solve (8.2) we need to precondition the matrix C_{c}. By Theorems 4.1 and 6.1 we conclude that

$$Q \sum_{\ell=0}^{L} (\alpha h_\ell + \beta + h_\ell^3)\, \Delta_\ell Q \;\; \leq \;\; C_{\mathrm{c}} \;\; \leq \;\; (L+1)^2 Q \sum_{\ell=0}^{L} (\alpha h_\ell + \beta + h_\ell^3)\, \Delta_\ell Q, \tag{8.3}$$

therefore,

$$(L+1)^{-2} \mathcal{P}C_{\mathrm{c}} \;\; \leq \;\; C_{\mathrm{c}}^{-1} \;\; \leq \;\; \mathcal{P}C_{\mathrm{c}}, \;\; \text{where } \mathcal{P}C_{\mathrm{c}} := \sum_{\ell=0}^{L} (\alpha h_\ell + \beta + h_\ell^3)^{-1}\, \Delta_\ell. \tag{8.4}$$

When $\beta = 0$, using (8.4) and Theorem 7.1 we obtain

$$(L+1)^{-2}\, \bar{\mathcal{P}}C_{\mathrm{c}} \;\leq\; C_{\mathrm{c}}^{-1} \;\leq\; \bar{\mathcal{P}}C_{\mathrm{c}}, \;\text{where } \bar{\mathcal{P}}C_{\mathrm{c}} := \sum_{\ell=0}^{L} (\alpha h_\ell + h_\ell^3)^{-1}\, \bar{\Delta}_\ell^T Q \bar{\Delta}_\ell, \tag{8.5}$$

and $\bar{\Delta}_\ell = R_\ell^T \bar{D}_\ell \bar{R}_\ell$; see (7.4). When $\alpha = 0$, using (8.4) and Theorem 7.1 we obtain

$$(L+1)^{-2}\,\bar{\mathcal{P}C}_{\mathrm{c}} \;\leq\; C_{\mathrm{c}}^{-1} \;\leq\; L\,\bar{\mathcal{P}C}_{\mathrm{c}}, \text{ where } \bar{\mathcal{P}C}_{\mathrm{c}} := \sum_{\ell=0}^{L} (\beta + h_\ell^3)^{-1}\,\bar{\Delta}_\ell^T Q \bar{\Delta}_\ell. \tag{8.6}$$

The numerical results which illustrate the equivalences (8.5) and (8.6) are presented in Sections 6.3 and 7.1 respectively.

The discrete solution of the LQECP when the discontinuous at the corners discretization is used is obtained solving

$$\begin{bmatrix} M & 0 & A^T \\ 0 & \widehat{G} & B_{ext}^T \\ A & B_{ext} & 0 \end{bmatrix} \begin{bmatrix} \mathrm{y} \\ \hat{\lambda} \\ \mathrm{p} \end{bmatrix} = \begin{bmatrix} \mathrm{f}_1 \\ \mathrm{f}_2 \\ \mathrm{f}_3 \end{bmatrix}, \tag{8.7}$$

where the sub-matrices M, A, B_{ext} and $\widehat{G} = \alpha\, B^T S_1^\dagger B + \beta \widehat{Q}$ are defined in Section 1.1. Eliminating the variables y and p from the equation (8.7), we have:

$$(\widehat{G} + B_{ext}^T A^\dagger M A^\dagger B_{ext})\,\hat{\lambda} = B_{ext}^T A^\dagger M A^\dagger \mathrm{f}_3 - B_{ext}^T A^\dagger \mathrm{f}_1. \tag{8.8}$$

To use the PCG to solve (8.8) we need to precondition the matrix $C_{\mathrm{d}} = \alpha\, B^T S_1^\dagger B + \beta \widehat{Q} + B_{ext}^T A^\dagger M A^\dagger B_{ext}$. To obtain the convergence of PCG when $\beta \neq 0$, by compatibility condition of $A^\dagger$ we change the L^2 norm matrix $\widehat{Q}$ by $\widehat{Q}_{Ker} = (I - P_{1_{m-K}})\,\widehat{Q}\,(I - P_{1_{m-K}})$, where

$$P_{1_{m-K}}\hat{\lambda} = \frac{(1_m{}^T B\,\hat{\lambda})}{(1_m{}^T B 1_{m-K})}\, 1_{m-K}. \tag{8.9}$$

We observe that $P_{1_{m-K}}$ is the B-projection onto $span\{1_{m-K}\} = Ker(A^\dagger)$. Note that $1_m{}^T\, B\,(I - P_{B\mathbf{1}}) = 0$. If $\hat{\lambda}$ satisfies $1_m{}^T B\,\hat{\lambda} = 0$ then $\|\hat{\lambda}\|_{L^2}^2 = \hat{\lambda}^T\,\widehat{Q}\,\hat{\lambda} = \hat{\lambda}^T \widehat{Q}_{Ker}\hat{\lambda}$. From Theorem 5.2 we have

$$C_{\mathrm{d}}^\dagger \asymp D^T Q^{-1}(S_3^\dagger + (\alpha + h) S_1^\dagger + \beta\, Q_{ker})^\dagger Q^{-1} D. \tag{8.10}$$

where $D := B(B^T Q^{-1} B)^{-1}$.

Since $C_{\mathrm{c}}^\dagger = Q^{-1}(S_3^\dagger + (\alpha + h) S_1^\dagger + \beta\, Q_{ker})^\dagger Q^{-1}$, using (8.4) we have

$$(L+1)^{-2} D^T \mathcal{P}C_{\mathrm{c}} D \;\leq\; C_{\mathrm{d}}^\dagger \;\leq\; D^T \mathcal{P}C_{\mathrm{c}} D. \tag{8.11}$$

Inspired in (8.11) we define the following FETI-Mortar-Multilevel preconditioners for the operator C_{d}

$$\mathcal{P}C_{\mathrm{d}} := D^T \sum_{\ell=0}^{L} \mu_\ell\, \Delta_\ell D \;\text{ and }\; \bar{\mathcal{P}C}_{\mathrm{d}} := D^T \sum_{\ell=0}^{L} \mu_\ell\, \bar{\Delta}_\ell^T Q \bar{\Delta}_\ell D. \tag{8.12}$$

When $\beta = 0$, using (8.11) and Theorem 7.1 we obtain

$$(L+1)^{-2}\,\bar{\mathcal{P}C}_{\mathrm{d}} \;\leq\; C_{\mathrm{d}}^{\dagger} \;\leq\; \bar{\mathcal{P}C}_{\mathrm{d}},\ \text{where}\ \bar{\mathcal{P}C}_{\mathrm{d}} = \sum_{\ell=0}^{L}((\alpha+h^2)h_\ell + h_\ell^3)^{-1}\,\bar{\Delta}_\ell^T Q\bar{\Delta}_\ell, \quad (8.13)$$

and $\bar{\Delta}_\ell = R_\ell^T \bar{D}_\ell \bar{R}_\ell$; see (7.4). When $\alpha = 0$, using (8.11) and Theorem 7.1 we obtain

$$(L+1)^{-2}\,\bar{\mathcal{P}C}_{\mathrm{d}} \;\leq\; C_{\mathrm{d}}^{\dagger} \;\leq\; L\,\bar{\mathcal{P}C}_{\mathrm{d}},\ \text{where}\ \bar{\mathcal{P}C}_{\mathrm{d}} = \sum_{\ell=0}^{L}(\beta + h_\ell^3)^{-1}\,\bar{\Delta}_\ell^T Q\bar{\Delta}_\ell. \quad (8.14)$$

8.1 Numerical Results

In this section we provides numerical results to illustrate the equivalences (8.13) and (8.14). We consider the model problem LQECP with $y_*(x_1, x_2) = \sin(\pi x_1)\sin(\pi x_2)$, $f(x_1, x_2) = (2\pi^2 + 1)\sin(\pi x_1)\sin(\pi x_2)$ in two domains Ω:

1) Ω is the square $[0, 1]^2$,

2) Ω is the polygon $\Omega = \{(x, y) \in \mathbb{R}^2;\ x \geq 0,\ y \geq 0,\ y \leq 1 \ \text{and}\ y \geq \mathrm{a}x - \mathrm{a}\}$,

where $\mathrm{a} > 0$ is a given parameter; see Figure 5.1. In both cases, the triangulation of Ω is constructed as showed in Figure 5.1. We divide each edge of Γ into 2^N parts of equal length, where N is an integer denoting the number of refinements.

The reduced Hessian C_{d} depends on the relaxation parameters α and β, the mesh size $h = O(2^{-N})$ and the shape parameter of Ω given by "a". We divide the tests in two categories: 1) The tests such $\alpha \in \left\{1, (10)^{-3}, (10)^{-6}, 0\right\}$ and $\beta = 0$; 2) The tests such $\alpha = 0$ and $\beta \in \left\{1, (10)^{-3}, (10)^{-6}\right\}$.

In Tables 8.1 to 8.4 we show the condition number (cond), number of iterations (it), three lowest eigenvalues (eigs min) and the three greatest eigenvalues (eigs max) for the matrices $\mathcal{P}C_{\mathrm{d}} * C_{\mathrm{d}}$ and $\bar{\mathcal{P}C}_{\mathrm{d}} * C_{\mathrm{d}}$ for different values for the relaxation parameters $\alpha \in \{1, (0.1)^3, (0.1)^6, 0\}$ and $\beta = 0$, for the mesh size $h = 2^{-N}$, and Ω as in **1)**, that is a=∞ (Ω square), where $\mathcal{P}C_{\mathrm{d}}$ and $\bar{\mathcal{P}C}_{\mathrm{d}}$ are defined in (8.12). In Tables 8.5 to 8.7 we use the discretization described above making $\alpha = 0$ and $\beta \in \left\{1, (10)^{-3}, (10)^{-6}\right\}$ and we use the rescaled $\bar{\mathcal{P}C}_{\mathrm{d}}^{r}$ as in Section 7.1.

In Tables 8.8 to 8.11 we repeat the tests presented in Tables 8.1 to 8.4 using a=5, Ω is a deformed square as in **2)**, instead a=∞. We show in Tables 8.12 to 8.15 the effectiveness of our preconditioner, that is, we show the condition number (cond), number of iterations (it), three lowest eigenvalues (eigs min) and the three greatest eigenvalues (eigs max) for the matrices C_{c} and C_{d} without preconditioner for different values for the relaxation parameters $\alpha \in \{1, (0.1)^3, (0.1)^6, 0\}$ and $\beta = 0$, for the mesh size $h = 2^{-N}$, and Ω as in **1)**, that is a=∞ (Ω square).

a=∞	$\mathcal{P}C_d * C_d$, $\alpha = 1$			$\bar{\mathcal{P}}C_d * C_d$, $\alpha = 1$		
N ↓	cond	eigs min	eigs max	cond	eigs min	eigs max
3	3.74286	0.127905	0.478733	4.89369	0.190558	0.932533
		0.132309	0.478077		0.19084	0.930388
	it=6	0.13231	0.441244	it=6	0.20272	0.363332
4	4.56499	0.120448	0.549842	5.76694	0.184276	1.06271
		0.120953	0.540559		0.184347	1.05571
	it=10	0.120953	0.540529	it=9	0.19456	0.73258
5	5.10927	0.118511	0.605507	6.73191	0.181077	1.219
		0.11857	0.594345		0.181095	1.2065
	it=10	0.11857	0.594342	it=11	0.190255	0.874967
6	5.33232	0.118018	0.629308	7.41347	0.179287	1.32914
		0.118025	0.619558		0.179291	1.31983
	it=11	0.118025	0.619557	it=11	0.187497	1.04775
7	5.45303	0.117893	0.642873	7.87786	0.178167	1.40357
		0.117894	0.63787		0.178168	1.3977
	it=11	0.117894	0.63787	it=12	0.185563	1.3977

Table 8.1: Number of PCG iterations (it), condition number (cond) and extremal eigenvalues for the matrix C_d and preconditioners $\mathcal{P}C_d$ and $\bar{\mathcal{P}}C_d$ with $\beta = 0$, $\alpha = 1$. Ω is a square domain.

a= ∞	$\mathcal{P}C_d * C_d$, $\alpha = (0.1)^3$			$\bar{\mathcal{P}}C_d * C_d$, $\alpha = (0.1)^3$		
N ↓	cond	eigs min	eigs max	cond	eigs min	eigs max
3	11.7724	0.0109221	0.12858	12.1309	0.0151495	0.183776
		0.0123442	0.12843		0.0181146	0.165312
	it=10	0.0124745	0.0952644	it=10	0.0197694	0.0531047
4	12.2302	0.0120378	0.147225	11.8899	0.0151684	0.180351
		0.013663	0.146845		0.0162977	0.180056
	it=15	0.0137549	0.14579	it=12	0.0190103	0.13862
5	16.6373	0.0138225	0.229969	18.3699	0.015384	0.282602
		0.0153034	0.229346		0.0157205	0.282565
	it=19	0.01541	0.229211	it=15	0.0189137	0.236477
6	23.4661	0.0147186	0.345387	32.0084	0.0154662	0.495048
		0.015919	0.344965		0.015555	0.495039
	it=24	0.0159663	0.344943	it=22	0.0189146	0.389782
7	32.0013	0.0148259	0.474449	45.7475	0.0154786	0.708108
		0.0160644	0.474355		0.0155011	0.708106
	it=27	0.0160774	0.474353	it=26	0.018829	0.706644

Table 8.2: Number of PCG iterations (it), condition number (cond) and extremal eigenvalues for the matrix C_d and preconditioners $\mathcal{P}C_d$ and $\bar{\mathcal{P}}C_d$ with $\beta = 0$, $\alpha = (10)^{-3}$. Ω is a square domain. .

a= ∞	$\mathcal{P}C_d * C_d, \quad \alpha = (0.1)^6$			$\bar{\mathcal{P}}C_d * C_d, \ \alpha = (0.1)^6$		
N ↓	cond	eigs min	eigs max	cond	eigs min	eigs max
3	16.9087	0.00747638	0.126416	16.3606	0.0110423	0.180658
		0.00809832	0.12573		0.0144977	0.163137
	it=10	0.00809868	0.0826337	it=11	0.0146189	0.0363386
4	24.6192	0.00547759	0.134854	18.4009	0.00866867	0.159511
		0.00615826	0.134127		0.0109549	0.159463
	it=19	0.00617636	0.118574	it=15	0.0110072	0.0793797
5	32.5166	0.00439905	0.143042	23.61	0.00701197	0.165552
		0.00492763	0.142888		0.00857753	0.165542
	it=25	0.0049318	0.137072	it=19	0.00874702	0.113885
6	39.9856	0.00370315	0.148073	28.866	0.00584518	0.168727
		0.00412662	0.148041		0.00694059	0.168655
	it=32	0.00412906	0.145999	it=23	0.00716355	0.13207
7	45.742	0.00330417	0.151139	33.3436	0.00510822	0.170327
		0.00366774	0.151084		0.00585692	0.170298
	it=36	0.00366932	0.151077	it=27	0.00617243	0.152971

Table 8.3: Number of PCG iterations (it), condition number (cond) and extremal eigenvalues for the matrix C_d and preconditioners $\mathcal{P}C_d$ and $\bar{\mathcal{P}}C_d$ with $\beta = 0$, $\alpha = (10)^{-6}$. Ω is a square domain.

a= ∞	$\mathcal{P}C_d * C_d, \quad \alpha = \beta = 0$			$\bar{\mathcal{P}}C_d * C_d, \ \alpha = \beta = 0$		
N ↓	cond	eigs min	eigs max	cond	eigs min	eigs max
3	16.9183	0.00747205	0.126414	16.3687	0.0110366	0.180655
		0.00809237	0.125727		0.0144902	0.163135
	it=10	0.00809285	0.0826221	it=11	0.0146137	0.0363203
4	24.6711	0.00546573	0.134845	18.4324	0.00865352	0.159505
		0.00614412	0.134118		0.0109417	0.15946
	it=19	0.00616222	0.118546	it=15	0.0109873	0.0793046
5	32.7759	0.00436363	0.143022	23.7489	0.00697037	0.165539
		0.00488596	0.142867		0.00854369	0.165525
	it=25	0.00489004	0.137029	it=19	0.00869411	0.113738
6	41.1001	0.00360169	0.14803	29.4534	0.0057278	0.168703
		0.00400859	0.147998		0.00684782	0.168633
	it=32	0.00401088	0.145917	it=24	0.00701507	0.131923
7	50.0385	0.00301738	0.150985	35.5533	0.00478945	0.170281
		0.00333655	0.150978		0.00561942	0.170253
	it=37	0.00333788	0.150962	it=27	0.005776	0.152866

Table 8.4: Number of PCG iterations (it), condition number (cond) and extremal eigenvalues for the matrix C_d and preconditioners $\mathcal{P}C_d$ and $\bar{\mathcal{P}}C_d$ with $\beta = 0$, $\alpha = 0$. Ω is a square domain.

a=∞	$\mathcal{P}C_{\rm d}^r * C_{\rm d}$, $\beta = 1$			$\bar{\mathcal{P}C}_{\rm d}^r * C_{\rm d}$, $\beta = 1$		
N ↓	cond	eigs min	eigs max	cond	eigs min	eigs max
3	1.29889	0.0275581	0.0357951	11.8835	0.012655	0.150385
		0.0277506	0.0357939		0.0126582	0.137029
	it=3	0.027752	0.0357921	it=7	0.0140956	0.137028
4	1.34892	0.0265369	0.0357961	14.6663	0.0109704	0.160895
		0.0276128	0.0357958		0.0111418	0.152284
	it=4	0.027753	0.0357957	it=18	0.0118574	0.151981
5	1.34892	0.0265372	0.0357965	16.117	0.0102742	0.165589
		0.0275971	0.0357964		0.01037	0.160063
	it=4	0.0277528	0.0357964	it=20	0.0108191	0.160035
6	1.34892	0.0265373	0.0357965	17.123	0.00983636	0.168428
		0.0275869	0.0357965		0.00989719	0.164757
	it=4	0.0277528	0.0357965	it=20	0.0101905	0.164734
7	1.34892	0.0265373	0.0357965	17.8579	0.00953025	0.17019
		0.0275812	0.0357965		0.00957144	0.167672
	it=4	0.0277528	0.0357965	it=19	0.00977282	0.167669

Table 8.5: Number of PCG iterations (it), condition number (cond) and extremal eigenvalues for the matrix $C_{\rm d}$ and preconditioners $\mathcal{P}C_{\rm d}^r$ and $\bar{\mathcal{P}C}_{\rm d}^r$ with $C = 36, \beta = 1$, $\alpha = 0$. Ω is a square domain.

a= ∞	$\mathcal{P}C_{\rm d}^r * C_{\rm d}$, $\beta = (0.1)^3$			$\bar{\mathcal{P}C}_{\rm d}^r * C_{\rm d}$, $\beta = (0.1)^3$		
N ↓	cond	eigs min	eigs max	cond	eigs min	eigs max
3	4.21278	0.0187624	0.0790418	6.18644	0.0174769	0.10812
		0.0201405	0.0787854		0.0185057	0.107315
	it=6	0.020939	0.0368369	it=7	0.0191395	0.101466
4	4.70317	0.0170493	0.080186	9.35238	0.0143681	0.134376
		0.0205613	0.080103		0.0147689	0.13348
	it=9	0.0212312	0.040372	it=16	0.0153808	0.133248
5	4.80111	0.0167971	0.0806449	12.0554	0.0125341	0.151103
		0.0207514	0.0806253		0.0128343	0.15023
	it=9	0.0212508	0.0412015	it=18	0.0131638	0.150158
6	4.82297	0.0167462	0.0807664	14.0237	0.0113963	0.159819
		0.0208243	0.0807598		0.0115886	0.159222
	it=9	0.0212452	0.0413345	it=19	0.0117941	0.159196
7	4.8343	0.0167132	0.0807966	15.4674	0.0106503	0.164733
		0.020845	0.0807881		0.0107738	0.164332
	it=8	0.0212445	0.0413186	it=19	0.0109097	0.164324

Table 8.6: Number of PCG iterations (it), condition number (cond) and extremal eigenvalues for the matrix $C_{\rm d}$ and preconditioners $\mathcal{P}C_{\rm d}^r$ and $\bar{\mathcal{P}C}_{\rm d}^r$ with $C = 36$, $\beta = (0.1)^3$, $\alpha = 0$. Ω is a square domain.

a=∞	$\mathcal{P}C_d^r * C_d$, $\beta = (0.1)^6$			$\bar{\mathcal{P}}C_d^r * C_d$, $\beta = (0.1)^6$		
N ↓	cond	eigs min	eigs max	cond	eigs min	eigs max
3	16.5168	0.00762725	0.125977	16.0215	0.011183	0.179168
		0.00833072	0.125386		0.0146643	0.162169
	it=10	0.00833542	0.0817004	it=11	0.0147509	0.140273
4	21.3576	0.00626356	0.133775	16.9268	0.00932856	0.157903
		0.00708126	0.133002		0.0114492	0.157242
	it=19	0.00709804	0.115996	it=14	0.0118683	0.136591
5	20.6957	0.00674764	0.139647	18.4409	0.00873534	0.161087
		0.0076287	0.139471		0.00966535	0.160781
	it=23	0.00764755	0.130766	it=16	0.0107628	0.13531
6	19.0236	0.0074545	0.141811	19.0921	0.00847904	0.161882
		0.00849943	0.141772		0.00874346	0.161798
	it=23	0.00852479	0.135325	it=22	0.0101097	0.137756
7	18.6749	0.00762791	0.14245	19.7251	0.00822503	0.162239
		0.00835184	0.142441		0.00827889	0.162217
	it=25	0.00879678	0.136594	it=24	0.00960455	0.138704

Table 8.7: Number of PCG iterations (it), condition number (cond) and extremal eigenvalues for the matrix C_d and preconditioners $\mathcal{P}C_d^r$ and $\bar{\mathcal{P}}C_d^r$ with $C = 36$, $\beta = (0.1)^6$, $\alpha = 0$. Ω is a square domain.

a=5	$\mathcal{P}C_d * C_d$, $\alpha = 1$			$\bar{\mathcal{P}}C_d * C_d$, $\alpha = 1$		
N ↓	cond	eigs min	eigs max	cond	eigs min	eigs max
3	4.42896	0.127627	0.565253	6.04547	0.184568	1.1158
		0.137335	0.546831		0.210787	1.03939
	it=12	0.143043	0.492379	it=14	0.216061	0.383604
4	5.72514	0.115381	0.66057	7.15471	0.178449	1.27675
		0.125345	0.579617		0.201274	1.13001
	it=13	0.126907	0.568472	it=16	0.207223	0.762533
5	6.42905	0.111827	0.718942	8.30002	0.175254	1.45461
		0.115625	0.677315		0.196143	1.34056
	it=14	0.119736	0.655346	it=17	0.202725	0.95417
6	6.78815	0.109882	0.745896	9.14683	0.173432	1.58635
		0.112162	0.740071		0.192696	1.5056
	it=13	0.114037	0.715908	it=18	0.200141	1.12649
7	7.05013	0.108674	0.766163	9.73636	0.172282	1.67739
		0.11009	0.754356		0.190114	1.62359
	it=13	0.111266	1.55131	it=19	0.19848	1.55131

Table 8.8: Number of PCG iterations (it), condition number (cond) and extremal eigenvalues for the matrix C_d and preconditioners $\mathcal{P}C_d$ and $\bar{\mathcal{P}}C_d$ with $\beta = 0$, $\alpha = 1$. Ω is a deformed square (a=5); see Figure 5.1.

a=5	$\mathcal{P}C_{\mathrm{d}} * C_{\mathrm{d}}$, $\alpha = (0.1)^3$			$\bar{\mathcal{P}}C_{\mathrm{d}} * C_{\mathrm{d}}$, $\alpha = (0.1)^3$		
N ↓	cond	eigs min	eigs max	cond	eigs min	eigs max
3	16.4496	0.0116098	0.190976	16.7072	0.014326	0.239347
		0.0128619	0.153156		0.0209261	0.230376
	it=24	0.0150381	0.132103	it=23	0.023758	0.0551084
4	18.4366	0.0126554	0.233323	18.7832	0.0138906	0.26091
		0.0138553	0.207832		0.0199815	0.249473
	it=25	0.0158826	0.191126	it=25	0.0220555	0.14701
5	22.4622	0.0141778	0.318464	26.8985	0.0138814	0.373389
		0.0149508	0.310243		0.0196429	0.349343
	it=28	0.017453	0.305836	it=29	0.0215711	0.271161
6	31.5032	0.0145042	0.456929	45.8599	0.0138389	0.634648
		0.0156377	0.444911		0.0193818	0.575355
	it=36	0.0186013	0.431691	it=37	0.0213449	0.478777
7	41.6359	0.0143943	0.599319	65.5278	0.0137111	0.898459
		0.015546	0.594012		0.0186415	0.811256
	it=41	0.0177313	0.774638	it=47	0.0205343	0.774638

Table 8.9: Number of PCG iterations (it), condition number (cond) and extremal eigenvalues for the matrix C_{d} and preconditioners $\mathcal{P}C_{\mathrm{d}}$ and $\bar{\mathcal{P}}C_{\mathrm{d}}$ with $\beta = 0$, $\alpha = (0.1)^3$. Ω is a deformed square (a=5); see Figure 5.1.

a=5	$\mathcal{P}C_{\mathrm{d}} * C_{\mathrm{d}}$, $\alpha = (0.1)^6$			$\bar{\mathcal{P}}C_{\mathrm{d}} * C_{\mathrm{d}}$, $\alpha = (0.1)^6$		
N ↓	cond	eigs min	eigs max	cond	eigs min	eigs max
3	23.7783	0.00788681	0.187535	21.9398	0.0108514	0.238078
		0.00852078	0.14075		0.0154241	0.2246
	it=26	0.00938045	0.12989	it=25	0.0166189	0.0374946
4	36.4716	0.0058442	0.213148	29.3186	0.00834733	0.244732
		0.00656699	0.175381		0.0115634	0.202794
	it=34	0.00741765	0.141923	it=29	0.0125719	0.103807
5	50.2821	0.00467457	0.235048	38.5302	0.0066349	0.255644
		0.00523454	0.211279		0.00906559	0.232008
	it=40	0.00580229	0.194676	it=34	0.00966878	0.138101
6	63.0617	0.00391923	0.247153	48.0021	0.00544245	0.261249
		0.00435908	0.241923		0.00736037	0.244842
	it=46	0.00470569	0.224582	it=39	0.00782354	0.168939
7	73.406	0.0034809	0.255519	56.748	0.00466557	0.264762
		0.00384682	0.253884		0.00631742	0.253256
	it=48	0.0041526	0.247796	it=40	0.00677316	0.247796

Table 8.10: Number of PCG iterations (it), condition number (cond) and extremal eigenvalues for the matrix C_{d} and preconditioners $\mathcal{P}C_{\mathrm{d}}$ and $\bar{\mathcal{P}}C_{\mathrm{d}}$ with $\beta = 0$, $\alpha = (0.1)^6$. Ω is a deformed square (a=5); see Figure 5.1.

a=5	$\mathcal{P}C_d * C_d$, $\alpha = \beta = 0$			$\bar{\mathcal{P}}C_d * C_d$, $\alpha = \beta = 0$		
N ↓	cond	eigs min	eigs max	cond	eigs min	eigs max
3	23.7929	0.00788188	0.187532	21.9492	0.0108467	0.238077
		0.00851447	0.140737		0.0154167	0.224594
	it=26	0.00937422	0.129888	it=25	0.0166078	0.0374775
4	36.5466	0.00583175	0.213131	29.3597	0.00833526	0.24472
		0.0065527	0.175358		0.0115431	0.202771
	it=34	0.00740152	0.141873	it=31	0.0125455	0.103769
5	50.6752	0.00463761	0.235012	38.7171	0.00660253	0.255631
		0.00519294	0.211161		0.00901018	0.231945
	it=40	0.00575487	0.194563	it=34	0.00959607	0.138038
6	64.7849	0.00381395	0.247086	48.8	0.00535297	0.261225
		0.0042417	0.241749		0.00720228	0.24476
	it=46	0.00456001	0.224393	it=39	0.00761986	0.168509
7	80.1688	0.0031841	0.255265	59.8254	0.00442462	0.264704
		0.00351787	0.253729		0.00589163	0.253113
	it=50	0.00375978	0.24758	it=45	0.00620172	0.24758

Table 8.11: Number of PCG iterations (it), condition number (cond) and extremal eigenvalues for the matrix C_d and preconditioners $\mathcal{P}C_d$ and $\bar{\mathcal{P}}C_d$ with $\beta = 0$, $\alpha = 0$. Ω is a deformed square (a=5); see Figure 5.1.

a=∞	C_c, $\alpha = 1$			C_d, $\alpha = 1$		
N ↓	cond	eigs min	eigs max	cond	eigs min	eigs max
3	150.14	0.00064562	0.0227813	162.38	0.000765596	0.124317
		0.000672951	0.0227755		0.000800106	0.12414
	it=9	0.000672977	0.0215351	it=6	0.000800109	0.00478773
4	313.443	0.00015655	0.0127932	354.542	0.000161486	0.0572534
		0.000158016	0.0127348		0.000162256	0.0572293
	it=19	0.000158017	0.0127344	it=18	0.000162256	0.00469099
5	637.136	3.86345e-5	0.00669891	690.579	3.88318e-5	0.0268164
		3.87046e-5	0.00657042		3.88527e-5	0.0268136
	it=35	3.87046e-5	0.00657042	it=31	3.88527e-5	0.00274682
6	1281.54	9.6119e-6	0.00338987	1339.13	9.61915e-6	0.0128813
		9.61478e-6	0.0033125		9.61976e-6	0.012881
	it=45	9.61478e-6	0.0033125	it=44	9.61976e-6	0.00141256

Table 8.12: Number of CG iterations (it), condition number (cond) and extremal eigenvalues for the matrices C_c and C_d without preconditioner with $\beta = 0$, $\alpha = 1$. Ω is a square domain.

a= ∞	C_c, $\alpha = (0.1)^3$			C_d, $\alpha = (0.1)^3$		
N ↓	cond	eigs min	eigs max	cond	eigs min	eigs max
3	6154.58	1.27108e-6	0.000344004	6237.21	1.56422e-6	0.00975637
		1.32622e-6	0.000338764		1.67256e-6	0.00960263
	it=13	1.34418e-6	0.000283066	it=8	1.67524e-6	1.57134e-5
4	20247.1	1.94967e-7	0.000194967	22120.7	2.01674e-7	0.00446117
		1.96902e-7	0.000193767		2.02839e-7	0.00444169
	it=42	1.96979e-7	0.000171496	it=38	2.02852e-7	1.99392e-5
5	48250.5	4.09997e-8	0.000100609	51157.7	4.12118e-8	0.0021083
		4.10743e-8	0.000100435		4.12354e-8	0.0021061
	it=105	4.10759e-8	9.00502e-5	it=88	4.1235e-8	1.21187e-5
6	101417	9.75862e-9	5.07035e-5	104675	9.76597e-9	0.00102226
		9.76153e-9	5.06807e-5		9.7666e-9	0.001022
	it=162	9.76156e-9	4.55829e-5	it=147	9.7666e-9	6.18031e-6

Table 8.13: Number of CG iterations (it), condition number (cond) and extremal eigenvalues for the matrices C_c and C_d without preconditioner with $\beta = 0$, $\alpha = (0.1)^3$. Ω is a square domain.

a= ∞	C_c, $\alpha = (0.1)^6$			C_d, $\alpha = (0.1)^6$		
N ↓	cond	eigs min	eigs max	cond	eigs min	eigs max
3	12651.1	6.11349e-7	3.21398e-4	12173.2	7.92128e-7	9.64269e-3
		6.329e-7	3.16136e-4		8.61745e-7	9.48922e-3
	it=15	6.68547e-7	2.61967e-4	it=8	8.73042e-7	1.16887e-5
4	103167	3.78283e-8	1.82326e-4	109404	4.03005e-8	0.00440903
		3.80529e-8	1.81138e-4		4.0699e-8	4.38961e-3
	it=69	3.9033e-8	1.58998e-4	it=47	4.07468e-8	1.50736e-5
5	820829	2.38267e-9	9.40976e-5	860938	2.42047e-9	0.00208388
		2.38466e-9	9.39257e-5		2.42325e-9	2.08168e-3
	it=344	2.40795e-9	8.352e-5	it=288	2.42354e-9	9.13601e-6
6	6.27e+6	1.56024e-10	4.7423e-5	6.45e+6	1.5658e-10	0.00101051
		1.56044e-10	4.74007e-5		1.56601e-10	1.01027e-3
	it>500	1.56465e-10	4.22812e-5	it>500	1.56603e-10	4.65896e-6

Table 8.14: Number of CG iterations (it), condition number (cond) and extremal eigenvalues for the matrices C_c and C_d without preconditioner with $\beta = 0$, $\alpha = (0.1)^6$. Ω is a square domain.

a= ∞	C_c, $\alpha=\beta=0$			C_d, $\alpha=\beta=0$		
N ↓	cond	eigs min	eigs max	cond	eigs min	eigs max
3	12664.9	6.10674e-7	3.21376e-4	12185	7.91349e-7	9.64258e-3
		6.32189e-7	3.16114e-4		8.60923e-7	9.4891e-3
	it=15	6.67868e-7	2.61945e-4	it=8	8.72237e-7	1.16847e-5
4	103609	3.76663e-8	1.82313e-4	109844	4.01387e-8	4.40897e-3
		3.78886e-8	1.81125e-4		4.05363e-8	4.38955e-3
	it=69	3.88747e-8	1.58986e-4	it=47	4.05845e-8	1.50687e-5
5	834693	2.34306e-9	9.40911e-5	874969	2.38163e-9	2.08385e-3
		2.34489e-9	9.39192e-5		2.38439e-9	2.08166e-3
	it=341	2.36923e-9	8.35134e-5	it=288	2.38469e-9	9.13303e-6
6	6.68e+6	1.46258e-10	4.74197e-5	6.87e+6	1.46961e-10	1.0105e-3
		1.46269e-10	4.73974e-5		1.46981e-10	1.01026e-3
	it>500	1.46849e-10	4.22779e-5	it>500	1.46983e-10	4.65744e-6

Table 8.15: Number of CG iterations (it), condition number (cond) and extremal eigenvalues for the matrices C_c and C_d without preconditioner with $\beta = 0$, $\alpha = 0$. Ω is a square domain.

a=∞	C_c, $\beta = 1$			C_d, $\beta = 1$		
N ↓	cond	eigs min	eigs max	cond	eigs min	eigs max
3	3.16284	0.0416673	0.131787	3.85571	0.0465994	0.179674
		0.0424679	0.131717		0.0466	0.179451
	it=7	0.042468	0.124262	it=6	0.0466001	0.174919
4	3.18106	0.0208334	0.0662723	4.91987	0.0213183	0.104883
		0.0209337	0.0662617		0.0213273	0.0882382
	it=9	0.0209337	0.0633891	it=10	0.0213274	0.0873875
5	3.18612	0.0104167	0.0331888	5.0193	0.0104718	0.0525609
		0.0104292	0.0331874		0.0104722	0.0435982
	it=9	0.0104292	0.0318541	it=11	0.0104722	0.0433695
6	3.18745	0.00520833	0.0166013	5.05372	0.0052149	0.0263547
		0.0052099	0.0166011		0.00521493	0.0216483
	it=8	0.0052099	0.0159471	it=11	0.00521493	0.0215888

Table 8.16: Number of CG iterations (it), condition number (cond) and extremal eigenvalues for the matrices C_c and C_d without preconditioner with $\beta = 1$, $\alpha = 0$. Ω is a square domain.

a=∞	C_c, $\beta = (0.1)^3$			C_d, $\beta = (0.1)^3$		
N ↓	cond	eigs min	eigs max	cond	eigs min	eigs max
3	185.804	4.22923e-5	0.00785807	206.044	4.75718e-5	0.00980189
		4.3102e-5	0.00777292		4.75932e-5	0.0096478
	it=10	4.31658e-5	0.0025561	it=7	4.76095e-5	0.00343822
4	189.967	2.08718e-5	0.00396495	209.997	2.13593e-5	0.00448539
		2.09717e-5	0.00395352		2.13686e-5	0.00446221
	it=14	2.09738e-5	0.00135187	it=18	2.13689e-5	0.0016147
5	190.706	1.0419e-5	0.00198697	202.294	1.04741e-5	0.00211886
		1.04316e-5	0.00198552		1.04746e-5	0.00211567
	it=15	1.04316e-5	0.000685543	it=18	1.04746e-5	0.0007543
6	190.852	5.20848e-6	0.00099405	196.948	5.21505e-6	0.0010271
		5.21005e-6	0.000993866		5.21508e-6	0.0010266
	it=14	5.21005e-6	0.00034399	it=18	5.21508e-6	0.0003614

Table 8.17: Number of CG iterations (it), condition number (cond) and extremal eigenvalues for the matrices C_c and C_d without preconditioner with $\beta = (0.1)^3$, $\alpha = 0$. Ω is a square domain.

a= ∞	C_c, $\beta = (0.1)^6$			C_d, $\beta = (0.1)^6$		
N ↓	cond	eigs min	eigs max	cond	eigs min	eigs max
3	11853.9	6.5246e-7	0.0077343	11462.2	8.41265e-7	0.0096427
		6.74671e-7	0.00764909		9.11248e-7	0.0094892
	it=15	7.10673e-7	0.0024344	it=8	9.20991e-7	0.0032712
4	66672.6	5.85345e-8	0.0039026	71575.4	6.16001e-8	0.00440905
		5.88261e-8	0.0038912		6.19658e-8	0.0043896
	it=66	5.98763e-8	0.0012898	it=72	6.19984e-8	0.0015356
5	153202	1.27659e-8	0.0019558	162089	1.28565e-8	0.0020839
		1.2775e-8	0.0019543		1.28584e-8	0.0020817
	it=151	1.28094e-8	0.000654375	it=171	1.28592e-8	0.00071837
6	182709	5.35518e-9	0.000978441	188462	5.3619e-9	0.0010105
		5.35627e-9	0.00097826		5.36194e-9	0.00101027
	it=168	5.35773e-9	0.00032838	it=191	5.36195e-9	0.00034453

Table 8.18: Number of CG iterations (it), condition number (cond) and extremal eigenvalues for the matrices C_c and C_d without preconditioner with $\beta = (0.1)^6$, $\alpha = 0$. Ω is a square domain.

8.2 Conclusions

In this thesis we describe multilevel methods for solving iteratively the saddle point systems (8.1) and (8.7). Both methods are based on CG solution of a Schur complement system and require double iteration of $A^{\dagger}$. In both cases the preconditioners described yield rates of convergence depending weakly (polylogarithmic growing) of the mesh parameter h and independent of regularization parameters α and β. To avoid double iterations of the matrix $A^{\dagger}$, one can use the preconditioned GMRES for (8.1) or (8.7) using a block preconditioner presented in [25]. In this case we also need a good preconditioner for C_{c} or C_{d}, which is developed in this thesis. The preconditioner for C_{c} can be used also to solve the linear system arising from a mixed finite element discretization for the biharmonic equation; see Section 1.4. The main idea here is associate the Schur complement matrices C_{c} and C_{d} to Sobolev norms and then use multilevel methods for preconditioning these norms.

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Bibliography

[1] R. A. Adams. **Sobolev spaces**. Academic Press, New York, 1975.

[2] C. Bacuta and J. H. Bramble. **Regularity estimates for solutions of the equations of linear elasticity in convex plane polygonal domains**. *Z. Angew. Math. Phys.*, 54(5):874–878, 2003. Special issue dedicated to Lawrence E. Payne.

[3] J. Bergh and J. Löfström. **Interpolation spaces. An introduction**. Springer-Verlag, Berlin, 1976.

[4] C. Bernardi, Y. Maday and A. T. Patera. **Domain decomposition by the mortar element method**. In *Asymptotic and numerical methods for partial differential equations with critical parameters (Beaune, 1992)*, volume 384 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 269–286. Kluwer Acad. Publ., Dordrecht, 1993.

[5] C. Bernardi, Y. Maday and A. T. Patera. **A new nonconforming approach to domain decomposition: the mortar element method**. In *Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. XI (Paris, 1989–1991)*, volume 299 of *Pitman Res. Notes Math. Ser.*, pages 13–51. Longman Sci. Tech., Harlow, 1994.

[6] D. Braess. **Finite elements: Theory, fast solvers and applications to solid mechanics**. Cambridge University Press, 1997.

[7] J. H. Bramble, J. E. Pasciak and P. S. Vassilevski. **Computational scales of Sobolev norms with application to preconditioning**. *Math. Comp.*, 69:463–480, 2000.

[8] J. H. Bramble, J. E. Pasciak and J. Xu. **Parallel multilevel preconditioners**. In *Numerical analysis 1989 (Dundee, 1989)*, volume 228 of *Pitman Res. Notes Math. Ser.*, pages 23–39. Longman Sci. Tech., Harlow, 1990.

[9] R. Brown. **The mixed problem for Laplace's equation in a class of Lipschitz domains**. *Comm. Partial Differential Equations*, 19(7-8):1217–1233, 1994.

[10] E. Casas and M. Mateos. **Error estimates for the numerical approximation of Neumann control problems**. *Comput. Optim. Appl.*, 39(3):265–295, 2008.

[11] M. Costabel. **Boundary integral operators on Lipschitz domains: elementary results**. *SIAM J. Math. Anal.*, 19(3), 1988.

[12] R. S. Falk. **Approximation of the biharmonic equation by a mixed finite element method**. *SIAM J. Numer. Anal.*, 15(3):556–567, 1978.

[13] C. Farhat, J. Mandel and F.-X. Roux. **Optimal convergence properties of FETI domain decomposition method**. *Comput. Methods Appl. Mech. Eng.*, (115):367–388, 1994.

[14] C. Farhat and F.-X. Roux. **A method of Finite Element Tearing and Interconnecting and its parallel solution algorithm**. *Int. J. Numer. Meth. Eng.*, (32):1205–1227, 1991.

[15] C. Farhat and F.-X. Roux. **Implicit parallel processing in structural mechanics**. *Computational Mechanics Advances*, 2(1):1–124, 1994.

[16] R. Glowinski and O. Pironneau. **Numerical methods for the first biharmonic equation and the two-dimensional Stokes problem**. *SIAM Rev.*, 21(2):167–212, 1979.

[17] G. H. Golub and C. F. Van Loan. **Matrix computations**. *Johns Hopkins Studies in the Mathematical Sciences*. Johns Hopkins University Press, Baltimore, MD, third edition, 1996.

[18] P. Grisvard. **Elliptic problems in nonsmooth domain**. Pitman, 1985.

[19] P. Grisvard. **Singularities in boundary value problems**. Springer-Verlag, 1992.

[20] D. S. Jerison and C. E. Kenig. **The Neumann problem on Lipschitz domains**. *Bulletin of the American Mathematical Society*, 4(2):203–207, 1981.

[21] A. Klawonn and O. B. Widlund. **FETI and Neumann-Neumann iterative substructuring methods: connections and new results**. *Comm. Pure Appl. Math.*, 54(1):57–90, 2001.

[22] J. L. Lions. **Some methods in the mathematical analysis of systems and their control**. Taylor and Francis, 1981.

[23] J. L. Lions and E. Magenes. **Problèmes aux limites non homogènes et applications**, volume 1. Dunod Paris, 1968.

[24] J. Mandel and R. Tezaur. **Convergence of a substructuring method with Lagrange multipliers**. *Numer. Math.*, 73:473–487, 1996.

[25] T. P. Mathew, M. Sarkis and C. E. Schaerer. **Analysis of block matrix preconditioners for elliptic optimal control problems**. *Numer. Linear Algebra Appl.*, 14(4):257–279, 2007.

[26] I. Mitrea. **Boundary problems for harmonic functions and norm estimates for inverses of singular integrals in two dimensions**. *Numer. Funct. Anal. Optim.*, 26(7-8):851–878, 2005.

[27] I. Mitrea and M. Mitrea. **The Poisson problem with mixed boundary conditions in Sobolev and Besov spaces in non-smooth domains**. *Trans. Amer. Math. Soc.*, 359(9):4143–4182 (electronic), 2007.

[28] J. Necas. **Les méthodes directes en théorie des équations elliptiques**. Masson, Paris, 1967.

[29] P. Oswald. **Hierarchical conforming finite element methods for the biharmonic equation**. *SIAM J. Numer. Anal.*, 29(6):1610–1625, 1992.

[30] P. Oswald. **On discrete norm estimates related to multilevel preconditioners in the finite element method**. In *Constructive Theory of Functions*, *Proc. Int. Conf. Varna*, pages 203–214. Bulg. Acad. Sci., Sofia, 1992.

[31] P. Oswald. **Multilevel norms for $H^{-1/2}$**. *Computing*, 61:235–255, 1998.

[32] R. Tezaur. **Analysis of Lagrange multiplier based domain decomposition**. PhD thesis, *University of Colorado at Denver*, 1998.

[33] A. Toselli and O. Widlund. **Domain decompposition methods - algorithms and theory**, volume 34. Springer Series in Computational Mathematics, 2004.

[34] J. Xu. **An introduction to multigrid convergence theory**. In *Iterative methods in scientific computing (Hong Kong, 1995)*, pages 169–241. Springer, Singapore, 1997.

[35] X. Zhang. **Multilevel additive Schwarz methods**. PhD thesis, *Courant Inst. Math. Sci., Dept. Comp.*, 1991.

[36] X. Zhang. **Multilevel Schwarz methods for the biharmonic Dirichlet problem**. *SIAM J. Sci. Comput.*, 15(3):621–644, 1994.