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Modelos integráveis não-lineares compostos, fluxos de gradação negativa e soluções sólitons.

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Resumo

Nesta tese iremos discutir a estrutura algébrica dos modelos mistos mKdV/sinh-Gordon, AKNS/Lund-Regge, bem como suas correspondentes versões super-simétricas e a hierarquia mKdV par negativa. Ademais, usaremos o método dressing para obter explicitamente soluções sólitons para as hierarquias aqui introduzidas.

Palavras Chaves: Método Dressing; Sólitons; Equação de Curvatura Nula.

Áreas do conhecimento: Teoria de Campo; Sistemas Integráveis

Abstract

In this thesis we discus the algebraic structure of the mixed mKdV/sinh-Gordon, AKNS/Lund-Regge models, its corresponding supersymmetric versions and the negative even mKdV hierarchy. Moreover, we use the dressing method to obtain explicit soliton solutions for the whole hierarchy introduced in this thesis.

Sumário

Introdução

Uma construção sistemática de hierarquias integráveis dentro de um formalismo álgebrico foi proposto em [1, 2, 3], com a interessante característica de permitir associar as equações de evolução graus positivos e negativos da estrutura algébrica subjacente. Em particular a primeira equação de grau negativo é sempre associada ao modelo relativístico, por exemplo, equação sinh-Gordon, equação Lund-Regge, equação super sinh-Gordon, equação super Lund-Regge.

A relação entre os modelos mKdV e sinh-Gordon foi observada por alguns autores [4, 5]. Este fato foi explicado e estendido a outros modelos integráveis em [6, 7] e em [8] a relação entre as soluções sólitons de ambos os modelos foi verificada explicitamente. Em particular, observou-se em [6] que a hierarquia mKdV ímpar consiste de uma série de equações de movimento não-lineares associadas a uma certa estrutura algébrica da álgebra de lie graduada ímpar, tal que, cada equação corresponde a uma evolução temporal de acordo com o tempo $t = t_{2n+1}$. Em [9] observou-se que o mesmo não ocorre para a parte negativa da hierarquia mKdV. Para ordem negativa a estrutura é menos restritiva, o que dá ordem a uma subclasse de equações de movimento descrita por equações de evolução associadas a graus pares. A primeira equação desta sub-hierarquia foi considerada em [10] com o uso de técnicas de recursão.

Em [11] nós estudamos a sub-hierarquia mKdV par negativa e tratamos o modelo mais simples, correspondente a t_{-2}, em detalhes e a observação crucial que a solução constante igual a zero não é admitida levou-nos a extender o método dressing para incorporar soluções de vácuo constantes e diferente de zero. Isto implica em deformar o operador de vértice usual, mas ainda preservando a peculiar propriedade de nilpotência necessária para encontrar as soluções sólitons. Aplicando o formalismo de dressing modificado, soluções multi-sólitons são construídas para toda a hierarquia mKdV negativa par.

Na segunda parte desta tese, iremos considerar o sistema misto,

$$\frac{a_3}{4}\left(\phi_{xxxx} - 6\phi_x^2\phi_{xx}\right) - \phi_{xt_3} + 2\eta \sinh(2\phi) = 0, \tag{1}$$

que tem recebido considerável atenção nos últimos anos [12, 13, 14, 15, 16, 17, 18]. Esta equação não-linear representa a bem conhecida equação mKdV para $\eta = 0$ e a equação sinh-Gordon para $a_3 = 0$. Ela foi introduzido em [12] onde, aplicando o método de espalhamento inverso soluções sólitons foram obtidas. Soluções multi-sólitons foram também consideradas em [15] pelo método de Hirota. Além disso, uma solução 2-breather foi discutida em [17] em conexão com pulsos ópticos em meios transparentes. As soluções obtidas em [12, 15] indicam a integrabilidade do modelo.

Com isto em mente, em [19] nós consideramos o modelo misto mKdV/sinh-Gordon (1) dentro do formalismo de curvatura nula. Nós mostramos que uma sistemática solução para o modelo misto mkdv/sinh-Gordon é obtida pelo método dressing e obtemos suas soluções multi-sólitons. Tal formalismo é estendido ao modelo misto AKNS/Lund-Regge e suas versões supersimétricas.

A apresentação desta tese será como se segue: No capítulo um faremos uma breve introdução ao formalismo de curvatura nula e o método dressing, que será utilizado nos capítulos posteriores.

No capítulo dois introduziremos equações de movimento associadas a gradações negativas pares para a hierarquia mKdV usando o formalismo de curvatura nula e estudaremos suas soluções sóliton através de uma modifição no método dressing, esta modificação é necessária para obtermos soluções sólitons dos modelos introduzidos, pois diferente do capítulo anterior agora tem-se um vácuo diferente de zero.

Nos capítulos três e quatro estudaremos a hierarquia mista mKdV/sinh-Gordon e sua versão supersimétrica através do formalismo de curvatura nula e estudaremos suas soluções sólitons através do método dressing.

Nos capítulos cinco e seis introduziremos a hierarquia mista NLS/Lund Regge e sua versão supersimétrica usando o formalismo de curvatura nula e estudaremos suas soluções sólitons através do método dressing.

Ao capítulo sete está reservado nossas conclusões e perspectivas.

1 Construção algébrica de hierarquias integráveis

Nesta tese iremos considerar modelos não-lineares que podem ser escritos em termos de uma equação de curvatura nula

$$[\partial_x + A_x, \partial_{t_n} + A_{t_n}] = 0. \tag{1.1}$$

Uma das vantagens de tal construção é que a integrabilidade do modelo é automaticamente garantida. Além disso, iremos considerar que os potenciais A_x e A_{t_n} pertencem a uma álgebra de Kac-Moody graduada e a váriavel t_n representa os diversos tempos da hierarquia.
Os modelos não-lineares são determinados através da escolha dos potenciais A_x e A_{t_n}, que podem ser escritos como,

$$A_x = E^{(1)} + A_0, \tag{1.2}$$

$$A_{t_n} = D_n^{(n)} + D_n^{(n-1)} + \cdots + D_n^{(0)}, \tag{1.3}$$

onde o elemento E é chamado de semi-simples* e permite decompor a álgebra em núcleo, $\mathcal{K}(E)$,

$$\mathcal{K}(E) = \left\{ X \in \hat{\mathcal{G}} | [E, X] = 0 \right\}, \tag{1.4}$$

e seu complemento chamado imagem, $\mathcal{M}(E)$. Assumiremos também a seguinte estrutura

$$[\mathcal{K}, \mathcal{K}] \subset \mathcal{K}, \quad [\mathcal{M}, \mathcal{M}] \subset \mathcal{K}, \quad [\mathcal{M}, \mathcal{K}] \subset \mathcal{M}. \tag{1.5}$$

O operador A_0 parametriza os campos da teoria, possui grau zero e pertence a imagem da álgebra.

*Assumiremos, por simplicidade, que o elemento semi-simples E possui grau 1, isto é, $E \equiv E^{(1)}$.

Substituindo as equações (1.2) e (1.3) em (1.1) obtemos,

$$\left[\partial_x + E^{(1)} + A_0, \partial_{t_n} + D_n^{(n)} + D_n^{(n-1)} + \cdots + D_n^{(0)}\right] = 0. \tag{1.6}$$

Separando a equação (1.6) grau a grau resulta,

$$\begin{aligned}
\left[E, D_n^{(n)}\right] &= 0, \\
\partial_x D_n^{(n)} + \left[A_0, D_n^{(n)}\right] + \left[E, D_n^{(n-1)}\right] &= 0, \\
\partial_x D_n^{(n-1)} + \left[A_0, D_n^{(n-1)}\right] + \left[E, D_n^{(n-2)}\right] &= 0, \\
\vdots &= \vdots \\
\partial_x D_n^{(1)} + \left[A_0, D_n^{(1)}\right] + \left[E, D_n^{(0)}\right] &= 0, \\
\partial_x D_n^{(0)} - \partial_{t_n} A_0 + \left[A_0, D_n^{(0)}\right] &= 0.
\end{aligned} \tag{1.7}$$

Cada uma das equações (1.7) pode ser decomposta em suas componentes $\mathcal{K}(E)$ e $\mathcal{M}(E)$. Observa-se que uma solução local para $D_n^{(i)}$, $i = 0, 1, \cdots, n-1$ pode ser encontrada em termos do campo A_0 iniciando recursivamente a partir da equação de grau mais alto em (1.7) até chegar a última. Em particular, a projeção de grau zero em $\mathcal{M}(E)$ resulta a equação de movimento

$$\partial_x D_{\mathcal{M}}^{(0)} - \partial_{t_n} A_0 + \left[A_0, D_{\mathcal{K}}^{(0)}\right] = 0. \tag{1.8}$$

A equação (1.8) representa uma série de equações de evolução não-lineares associadas com o tempo t_n.

1.1 Hierarquia negativa

Nesta seção, estenderemos o formalismo anterior para o caso do número n ser negativo e em particular, veremos que o caso t_{-1} corresponde a um modelo relativístico.
Para este fim definiremos o operador $A_{t_{-n}}$ como

$$A_{t_{-n}} = D_{-n}^{(-n)} + D_{-n}^{(-n+1)} + \cdots + D_{-n}^{(-1)}, \tag{1.9}$$

onde $n > 0$. Portanto a equação de curvatura nula (1.1) pode ser escrita como,

$$\left[\partial_x + E^{(1)} + A_0, \partial_{t_{-n}} + D_{-n}^{(-n)} + D_{-n}^{(-n+1)} + \cdots + D_{-n}^{(-1)}\right] = 0. \tag{1.10}$$

Decompondo a equação (1.10) grau a grau, temos

$$\begin{aligned} \partial_x D_{-n}^{(-n)} + \left[A_0, D_{-n}^{(-n)}\right] &= 0, \\ \partial_x D_{-n}^{(-n+1)} + \left[A_0, D_{-n}^{(-n+1)}\right] + \left[E, D_{-n}^{(-n)}\right] &= 0, \\ &\vdots \\ \left[E, D_{-n}^{(-1)}\right] - \partial_{t_{-n}} A_0 &= 0. \end{aligned} \tag{1.11}$$

A equação (1.11) pode ser resolvida recursivamente, entretanto, diferente do caso anterior, em geral, os operadores $D_{-n}^{(-i)}$ são funcionais não locais dos campos contidos em A_0. Para o caso particular $n = 1$ iremos obter uma solução local fechada.
Considerando $n = 1$ na equação (1.10)

$$\left[\partial_x + E^{(1)} + A_0, \partial_{t_{-1}} + D_{-1}^{(-1)}\right] = 0, \tag{1.12}$$

e a escolha,

$$A_0 = -\partial_x B\, B^{-1}, \tag{1.13}$$

$$D_{t_{-1}}^{(-1)} = B\epsilon_- B^{-1}, \tag{1.14}$$

a equação de curvatura nula (1.12) corresponde a equação de Leznov-Saveliev [23]

$$\bar{\partial}(B^{-1}\partial B) + [\epsilon_-, B^{-1}\epsilon_+ B] = 0, \tag{1.15}$$

onde o espaço-tempo é representado pelas coordenadas no cone de luz $z = t + x$, $\bar{z} = t - x$. Portanto, podemos concluir que a equação de Leznov-Saveliev pode ser inserida dentro da construção geral de hierarquias integráveis, sendo associada a evolução temporal para graus negativos.

1.2 Integração dos modelos

O método dressing é baseado na existência de duas transformações de gauge $\Theta_\pm$, que mapeiam soluções triviais, vácuo, em soluções não triviais.
Considere o operador de Lax

$$\mathcal{L}_x = \partial_x + E^{(1)} + A_0. \tag{1.16}$$

O qual define uma hierarquia integrável associada ao problema linear [20]

$$\mathcal{L}_x T = (\partial_x + \mathcal{A}_x)\, T = \left(\partial_x + E^{(1)} + A_0\right) T = 0. \tag{1.17}$$

Na configuração de vácuo os campos físicos vão a zero, de modo que,

$$\left(\partial_x + E^{(1)}\right) T_{vac} = 0. \tag{1.18}$$

As transformações de gauge são geradas por,

$$T = \Theta_- T_{vac}, \tag{1.19}$$
$$T = \Theta_+ T_{vac} g^{-1}, \tag{1.20}$$

onde g é um elemento constante do grupo de Lie G e iremos supor que $\Theta_\pm$ são elementos de grupo com a forma

$$\Theta_- = e^{t(-1)+t(-2)+t(-3)+\cdots}, \tag{1.21}$$
$$\Theta_+ = e^{v(0)} e^{v(1)+v(2)+v(3)+\cdots}, \tag{1.22}$$

onde $t(-i)$ e $v(i)$ são combinações lineares dos geradores de grau negativo ($-i$) e positivos (i) respectivamente, $i = 0, 1, 2, \cdots$. Ademais, B é um elemento de grupo não constante e possui grau zero. Reescrevendo a equação (1.19) como

$$T_{vac} = \Theta_-^{-1} T, \tag{1.23}$$

e substituindo em (1.18) obtemos

$$\left(\Theta_-^{-1}\partial_x + \partial_x \Theta_-^{-1} + E^{(1)}\Theta_-^{-1}\right) T = 0. \tag{1.24}$$

Multiplicando (1.24) por Θ_- pela esquerda e comparando com a expressão (1.17) obtemos

$$E^{(1)} + A_0 = \Theta_- E^{(1)} \Theta_-^{-1} + \Theta_- \partial_x \Theta_-^{-1}. \tag{1.25}$$

Portanto, a transformação de gauge Θ_- transforma a configuração de vácuo numa configuração não trivial. Raciocínio análogo vale para Θ_+, isto é,

$$A_x = \Theta_\pm A_{x,\text{vac}} \Theta_\pm^{-1} + \Theta_\pm \partial_x \Theta_\pm^{-1}, \tag{1.26}$$
$$A_{t_n} = \Theta_\pm A_{t_n,\text{vac}} \Theta_\pm^{-1} + \Theta_\pm \partial_{t_n} \Theta_\pm^{-1}. \tag{1.27}$$

Igualando as equações (1.19) e (1.20) obtemos

$$\left(\Theta_-\right)^{-1} \Theta_+ = T_{vac} g T_{vac}^{-1}, \tag{1.28}$$

isto é, T_{vac} transforma o elemento de grupo g constante num elemento de grupo $\left(\Theta_-\right)^{-1}\Theta_+$ não constante. Considerando Θ_+, a componente de grau zero da equação (1.26) admite a solução

$$e^{v(0)} = B e^{\nu \hat{c}}, \tag{1.29}$$

onde usamos $A_x = E^1 - \partial_x B B^{-1} - \partial_x \nu \hat{c}$ ($\hat{c}$ é o elemento central da álgebra, ver apêndice A). Da equação (1.28) nós obtemos

$$\cdots e^{\cdots - t(-2) - t(-1)} B e^{\nu \hat{c}} e^{v(1) + v(2) \cdots} = T_{vac} g T_{vac}^{-1}. \tag{1.30}$$

Tomando o valor esperado da relação (1.30) na representação de peso mais alto da álgebra obtemos as funções τ, isto é,

$$\tau_i =< \lambda_i | (\Theta_-)^{-1} \Theta_+ | \lambda_i >= \left\langle \lambda_i | T_{vac} g T_{vac}^{-1} | \lambda_i \right\rangle. \tag{1.31}$$

A partir da escolha do elemento de grupo g obteremos as funções τ e conseqüentemente as soluções n-sólitons do modelo.
O elemento de grupo g possui a forma geral

$$g = e^{V_1(\gamma_1)} e^{V_2(\gamma_2)} \cdots e^{V_n(\gamma_n)}, \tag{1.32}$$

onde $V_n(\gamma_n)$ é o operador de vértice, com a propriedade

$$\left[E^{(n)}, V_n(\gamma_n) \right] = f_n^{\pm}(\gamma_n) V_n(\gamma_n). \tag{1.33}$$

Operadores de vértice V em geral possuem a propriedade de nilpotência, isto é, $V^n = 0$. Neste contexto, isto explica o ansatz feito por Hirota para a determinação das funções tau [6, 21, 22]. O elemento T_{vac} é obtido a partir dos operadores de Lax na configuração de vácuo:

$$\left(\partial_x + E^{(1)} \right) T_{vac} = 0 \quad \longrightarrow \quad \partial_x T_{vac} = -E^{(1)} T_{vac}, \tag{1.34}$$

$$\left(\partial_{t_n} + E^{(n)} \right) T_{vac} = 0 \quad \longrightarrow \quad \partial_{t_n} T_{vac} = -E^{(n)} T_{vac}. \tag{1.35}$$

Integrando as equações (1.34) e (1.35) obtemos

$$T_{vac} = e^{-E^{(1)} x - E^{(n)} t_n}. \tag{1.36}$$

1.3 Exemplo

Um exemplo bem conhecido do formalismo apresentado é a hierarquia mKdV, neste modelo usamos a seguinte estrutura algébrica:

$$\hat{\mathcal{G}} = \hat{sl}(2),$$

$$\mathcal{Q} = \frac{1}{2} H^{(0)} + 2d, \tag{1.37}$$

$$E^{(1)} = E_{\alpha}^{(0)} + E_{-\alpha}^{(1)}. \tag{1.38}$$

O operador de graduação (1.37) e as relações de comutação da álgebra implicam,

$$\begin{aligned} \left[\mathcal{Q}, E^{(n)}_{\pm\alpha}\right] &= (2n \pm 1)\, E^{(n)}_{\pm\alpha}, \\ \left[\mathcal{Q}, H^{(n)}\right] &= 2nH^{(n)}. \end{aligned} \tag{1.39}$$

De acordo com as relações (1.39) a álgebra $\hat{sl}(2)$ decompõe-se nos seguintes subespaços

$$\begin{aligned} \hat{\mathcal{G}}^{(2n)} &= \left\{H^{(n)}\right\}, \\ \hat{\mathcal{G}}^{(2n+1)} &= \left\{E^{(n)}_{\alpha}, E^{(n+1)}_{-\alpha}\right\}, \end{aligned} \tag{1.40}$$

isto é, os operadores $E^{(n)}_{\pm\alpha}$ possuem grau ímpar e os operadores $H^{(n)}$ possuem grau par. O elemento semi-simples gera a decomposição $\mathcal{G} = \mathcal{K}\,(E) \oplus \mathcal{M}\,(E)$ com

$$\begin{aligned} \mathcal{K}\,(E) &= \left\{E^{(n)}_{\alpha} + E^{(n+1)}_{-\alpha}\right\}, \\ \mathcal{M}\,(E) &= \left\{E^{(n)}_{\alpha} - E^{(n+1)}_{-\alpha}, H^{(n)}\right\}. \end{aligned} \tag{1.41}$$

O operador A_0 possui grau zero e pertence a $\mathcal{M}\,(E)$, logo, das relações (1.40) e (1.41) obtemos,

$$A_0 = v(x, t_n)H^{(0)}, \tag{1.42}$$

onde $v(x, t_n)$ é o campo físico do modelo e o índice n fixa cada equação da hierarquia.

As equações de movimento são obtidas a partir da equação de curvatura nula

$$\left[\partial_x + E^{(1)} + A_0, \partial_{t_n} + D^{(n)}_n + D^{(n-1)}_n + \cdots + D^{(0)}_n\right] = 0, \tag{1.43}$$

tomando-se valores de n específicos. Considere $n = 3$, os operadores $D^{(i)}_3$ podem ser escritos em suas componentes núcleo e imagem como

$$\begin{aligned} D^{(3)}_3 &= a_3\left(E^{(1)}_{\alpha} + E^{(2)}_{-\alpha}\right) + b_3\left(E^{(1)}_{\alpha} - E^{(2)}_{-\alpha}\right), \\ D^{(2)}_3 &= c_2 H^{(1)}, \\ D^{(1)}_3 &= a_1\left(E^{(0)}_{\alpha} + E^{(1)}_{-\alpha}\right) + b_1\left(E^{(0)}_{\alpha} - E^{(1)}_{-\alpha}\right), \\ D^{(0)}_3 &= c_0 H^{(0)}. \end{aligned} \tag{1.44}$$

Substituindo (1.44) em (1.7) resulta o sistema de equações

$$\begin{aligned} b_3 &= 0, \\ a_3 &= \text{constante}, \\ c_2 &= a_3 v, \\ b_1 &= \frac{1}{2}\partial_x c_2 \\ \partial_x a_1 + 2vb_1 &= 0, \\ \partial_x b_1 + 2va_1 - 2c_0 &= 0, \end{aligned} \tag{1.45}$$

cujo resultado é dado por,

$$\begin{aligned} b_1 &= \frac{a_3}{2} v_x, \\ a_1 &= -\frac{a_3}{2} v^2, \\ c_0 &= \frac{a_3}{4}\left(v_{xx} - 2v^3\right), \end{aligned} \tag{1.46}$$

onde $v = v\,(x, t_3)$. Da equação (1.8) obtemos a equação de movimento

$$\partial_x c_0 - \partial_{t_3} v = 0. \tag{1.47}$$

Fazendo $a_3 = 1$ por simplicidade e usando as relações (1.46), a equação de movimento é escrita como

$$4v_{t_3} + 6v^2 v_x - v_{xxx} = 0. \tag{1.48}$$

A equação (1.48) é chamada modified Korteweg-de Vries (mKdV) e corresponde a $n = 3$ na hierarquia mKdV.

Tomando operadores graduados de ordem mais alta, obteremos outros modelos da hierarquia. Considerando $n = 5$ temos,

$$\begin{aligned} D_5^{(5)} &= a_5\left(E_\alpha^{(2)} + E_{-\alpha}^{(3)}\right) + b_5\left(E_\alpha^{(2)} - E_{-\alpha}^{(3)}\right), \\ D_5^{(4)} &= c_4 H^{(2)}, \\ D_5^{(3)} &= a_3\left(E_\alpha^{(1)} + E_{-\alpha}^{(2)}\right) + b_3\left(E_\alpha^{(1)} - E_{-\alpha}^{(2)}\right), \\ D_5^{(2)} &= c_2 H^{(1)}, \\ D_5^{(1)} &= a_1\left(E_\alpha^{(0)} + E_{-\alpha}^{(1)}\right) + b_1\left(E_\alpha^{(0)} - E_{-\alpha}^{(1)}\right), \\ D_5^{(0)} &= c_0 H^{(0)}. \end{aligned} \tag{1.49}$$

Substituindo (1.49) em (1.7) resulta o sistema de equações

$$\begin{aligned}
a_5 &= \text{constante}, \\
b_5 &= 0, \\
c_4 &= a_5 v, \\
\partial_x c_4 - 2b_3 &= 0, \\
\partial_x a_3 + 2vb_3 &= 0, \\
\partial_x b_3 + 2va_3 - 2c_2 &= 0, \\
\partial_x c_2 - 2b_1 &= 0, \\
\partial_x a_1 + 2vb_1 &= 0, \\
\partial_x b_1 + 2va_1 - 2c_0 &= 0,
\end{aligned} \tag{1.50}$$

cujo resultado é dado por,

$$\begin{aligned}
b_3 &= \frac{a_5}{2} v_x, \\
a_3 &= -\frac{a_5}{2} v^2, \\
c_2 &= \frac{a_5}{4}\left(v_{xx} - 2v^3\right), \\
b_1 &= \frac{a_5}{8}\left(v_{xxx} - 6v^2 v_x\right), \\
a_1 &= \frac{a_5}{8}\left(v_x^2 + 3v^4 - 2vv_{xx}\right), \\
c_0 &= \frac{a_5}{16}\left(v_{xxxx} - 10v^2 v_{xx} - 10vv_x^2 + 6v^5\right).
\end{aligned} \tag{1.51}$$

Novamente, a equação (1.8) determina a equação que descreve a dinâmica do campo físico:

$$\partial_x c_0 - \partial_{t_5} v = 0. \tag{1.52}$$

onde $v = v(x, t_5)$. Fazendo $a_5 = 1$ por simplicidade e usando as relações (1.51), a equação de movimento de quinta ordem da hierarquia mKdV é escrita como

$$16v_{t_5} + 10v^2 v_{xxx} + 40vv_x v_{xx} + 10v_x^3 - 30v^4 v_x - v_{xxxxx} = 0. \tag{1.53}$$

Considere agora a equação de curvatura nula em termos de operadores de graus negativos,

$$\left[\partial_x + E^{(1)} + A_0, \partial_{t_{-n}} + D_{-n}^{(-n)} + D_{-n}^{(-n+1)} + \cdots + D_{-n}^{(-1)}\right] = 0. \tag{1.54}$$

Fazendo $n = 1$ em (1.54) e usando o fato que os operadores $D_{-1}^{(-i)}$ podem ser escritos em suas componentes núcleo e imagem temos:

$$D_{-1}^{(-1)} = a_{-1}\left(E_{\alpha}^{(-1)} + E_{-\alpha}^{(0)}\right) + b_{-1}\left(E_{\alpha}^{(-1)} - E_{-\alpha}^{(0)}\right). \tag{1.55}$$

Substituindo (1.55) em (1.11) obtemos o sistema de equações,

$$\begin{aligned} \partial_x a_{-1} + 2vb_{-1} &= 0, \\ \partial_x b_{-1} + 2va_{-1} &= 0, \end{aligned} \tag{1.56}$$

cujo resultado é dado por,

$$\begin{aligned} a_{-1} &= \eta\cosh(2\phi), \\ b_{-1} &= -\eta\sinh(2\phi), \end{aligned} \tag{1.57}$$

onde introduzimos a coordenada relativística do modelo, ϕ, e a relação $v = -\phi_x$, com $v = v(x, t_{-1})$. Desta forma, a equação de movimento relativística da hierarquia mKdV é o modelo sinh-Gordon dado por,

$$\phi_{xt} = 2\sinh(2\phi). \tag{1.58}$$

Considerando agora $n = 3$ em (1.54) e usando o fato que os operadores $D_{-3}^{(-i)}$ podem ser escritos em suas componentes núcleo e imagem temos:

$$\begin{aligned} D_{-3}^{(-3)} &= a_{-3}\left(E_{\alpha}^{(-2)} + E_{-\alpha}^{(-1)}\right) + b_{-3}\left(E_{\alpha}^{(-2)} - E_{-\alpha}^{(-1)}\right), \\ D_{-3}^{(-1)} &= a_{-1}\left(E_{\alpha}^{(-1)} + E_{-\alpha}^{(0)}\right) + b_{-1}\left(E_{\alpha}^{(-1)} - E_{-\alpha}^{(0)}\right). \end{aligned} \tag{1.59}$$

Substituindo (1.59) em (1.11) resulta o sistema de equações,

$$\begin{aligned} \partial_x a_{-3} + 2vb_{-3} &= 0, \\ \partial_x b_{-3} + 2va_{-3} &= 0, \\ \partial_x c_{-2} - 2b_{-3} &= 0, \\ \partial_x a_{-1} + 2vb_{-1} &= 0, \\ \partial_x b_{-1} + 2va_{-1} - 2c_{-2} &= 0, \end{aligned} \tag{1.60}$$

cujo resultado é dado por,

$$
\begin{aligned}
a_{-3} &= \frac{1}{2}\left(e^{-2\int vdx} + e^{2\int vdx}\right), \\
b_{-3} &= \frac{1}{2}\left(e^{-2\int vdx} - e^{2\int vdx}\right), \\
c_{-2} &= \int \left(e^{-2\int vdx} - e^{2\int vdx}\right) dx, \\
a_{-1} &= e^{-2\int vdx}\int e^{2\int vdx} c_{-2}dx - e^{2\int vdx}\int e^{-2\int vdx} c_{-2}dx, \\
b_{-1} &= e^{-2\int vdx}\int e^{2\int vdx} c_{-2}dx + e^{2\int vdx}\int e^{-2\int vdx} c_{-2}dx, \qquad (1.61)
\end{aligned}
$$

onde $v = v(x, t_{-3})$. Portanto a equação de movimento correspondente ao tempo t_{-3} da hierarquia mKdV é dada por,

$$
\begin{aligned}
v_{t_{-3}} &= 2e^{-2\int vdx}\int e^{2\int vdx}dx \int \left(e^{-2\int vdx} - e^{2\int vdx}\right) dxdx \\
&+ 2e^{2\int vdx}\int e^{-2\int vdx}dx \int \left(e^{-2\int vdx} - e^{2\int vdx}\right) dxdx. \qquad (1.62)
\end{aligned}
$$

1.3.1 Sólitons

O operador de vértice é definido por [24, 25]

$$V(\gamma) = \sum_{n\in\mathbb{Z}} \left\{ \left(H^{(n)} - \frac{1}{2}\hat{c}\delta_{n,0}\right)\gamma^{-2n} + \left(E_{\alpha}^{(n)} - E_{-\alpha}^{(n+1)}\right)\gamma^{-2n-1} \right\}, \qquad (1.63)$$

e obedece a equação de auto-valores

$$\left[E^{(2n+1)}, V(\gamma)\right] = \left[E_{\alpha}^{(n)} + E_{-\alpha}^{(n+1)}, V(\gamma)\right] = -2\gamma^{2n+1}V(\gamma). \qquad (1.64)$$

As funções τ são dadas por,

$$\tau_i =< \lambda_i|(\Theta_-)^{-1}\Theta_+|\lambda_i >= \left\langle \lambda_i|T_{vac}gT_{vac}^{-1}|\lambda_i\right\rangle. \qquad (1.65)$$

Os operadores de grau zero permitem escrever $B = e^{\phi H^{(0)} + \nu\hat{c}}$, desta maneira as funções τ podem ser reescritas como

$$
\begin{aligned}
\tau_0 &= e^{\nu} = \left\langle \lambda_0|T_{vac}gT_{vac}^{-1}|\lambda_0\right\rangle, \qquad (1.66) \\
\tau_1 &= e^{\nu+\phi} = \left\langle \lambda_1|T_{vac}gT_{vac}^{-1}|\lambda_1\right\rangle. \qquad (1.67)
\end{aligned}
$$

Para obtermos a solução 1-sóliton usaremos apenas um operador de vértice, isto é:

$$g = e^{aV_1(\gamma_1)}, \qquad (1.68)$$

portanto,

$$T_{vac}gT_{vac}^{-1} = e^{-E^{(1)}x-E^{(2n+1)}t_{2n+1}}\left(1+aV_1(\gamma_1)\right)e^{E^{(1)}x+E^{(2n+1)}t_{2n+1}}, \quad (1.69)$$

$$= 1+a\rho_1 V_1(\gamma_1), \quad (1.70)$$

onde

$$\rho_1 = e^{2\gamma_1 x+2\gamma_1^{2n+1}t_{2n+1}}. \quad (1.71)$$

Assim, as funções τ são dadas por:

$$\tau_0 = e^{\nu} = 1+\frac{a}{2}\rho_1, \quad (1.72)$$

$$\tau_1 = e^{\nu+\phi} = 1-\frac{a}{2}\rho_1. \quad (1.73)$$

Obtidas as funções τ a solução 1-sóliton dos modelos associados a $n=-1,-2,1,2$ da hierarquia mKdV são dados respectivamente por:

$$\phi(x,t_{-1}) = \ln\frac{\tau_1}{\tau_0} = \ln\left[\frac{1-\frac{a}{2}e^{2\gamma_1 x+\frac{2}{\gamma_1}t_{-1}}}{1+\frac{a}{2}e^{2\gamma_1 x+\frac{2}{\gamma_1}t_{-1}}}\right], \quad (1.74)$$

$$v(x,t_{-3}) = -\left[\ln\left(\frac{1-\frac{a}{2}e^{2\gamma_1 x+\frac{2}{\gamma_1^3}t_{-3}}}{1+\frac{a}{2}e^{2\gamma_1 x+\frac{2}{\gamma_1^3}t_{-3}}}\right)\right]_x, \quad (1.75)$$

$$v(x,t_3) = -\left[\ln\left(\frac{1-\frac{a}{2}e^{2\gamma_1 x+2\gamma_1^3 t_3}}{1+\frac{a}{2}e^{2\gamma_1 x+2\gamma_1^3 t_3}}\right)\right]_x, \quad (1.76)$$

$$v(x,t_5) = -\left[\ln\left(\frac{1-\frac{a}{2}e^{2\gamma_1 x+2\gamma_1^5 t_5}}{1+\frac{a}{2}e^{2\gamma_1 x+2\gamma_1^5 t_5}}\right)\right]_x. \quad (1.77)$$

Para obtermos a solução 2-sólitons usaremos o produto de dois operadores de vértice, isto é:

$$g = e^{aV_1(\gamma_1)}e^{bV_2(\gamma_2)}, \quad (1.78)$$

desta forma,

$$T_{vac}gT_{vac}^{-1} = e^{-E^{(1)}x-E^{(2n+1)}t_{2n+1}}\left(1+aV_1(\gamma_1)\right)\left(1+bV_2(\gamma_2)\right)e^{E^{(1)}x+E^{(2n+1)}t_{2n+1}},$$

$$= 1+a\rho_1V_1(\gamma_1)+b\rho_2V_2(\gamma_2)+ab\rho_1\rho_2V_1(\gamma_1)V_2(\gamma_2), \quad (1.79)$$

onde

$$\rho_i = e^{2\gamma_i x+2\gamma_i^{2n+1}t_{2n+1}}. \quad (1.80)$$

As funções τ são dadas por:

$$\tau_0 = e^{\nu} = 1 + a\frac{1}{2}\rho_1 + b\frac{1}{2}\rho_2 + ab\frac{1}{4}\left(\frac{\gamma_1 - \gamma_2}{\gamma_1 + \gamma_2}\right)^2 \rho_1\rho_2, \quad (1.81)$$

$$\tau_1 = e^{\nu+\phi} = 1 - a\frac{1}{2}\rho_1 - b\frac{1}{2}\rho_2 + ab\frac{1}{4}\left(\frac{\gamma_1 - \gamma_2}{\gamma_1 + \gamma_2}\right)^2 \rho_1\rho_2, \quad (1.82)$$

Portanto, a solução 2-sólitons dos modelos associados a $n = -1, -2, 1, 2$ da hierarquia mKdV são dados respectivamente por:

$$\begin{aligned}\phi(x, t_{-1}) &= \ln\frac{\tau_1}{\tau_0}, \\ &= \ln\left[\frac{1 - a\frac{1}{2}e^{2\gamma_1 x + \frac{2}{\gamma_1}t_{-1}} - b\frac{1}{2}e^{2\gamma_2 x + \frac{2}{\gamma_2}t_{-1}} + ab\frac{1}{4}\left(\frac{\gamma_1-\gamma_2}{\gamma_1+\gamma_2}\right)^2 e^{2(\gamma_1+\gamma_2)x + 2\left(\frac{1}{\gamma_1}+\frac{1}{\gamma_2}\right)t_{-1}}}{1 + a\frac{1}{2}e^{2\gamma_1 x + \frac{2}{\gamma_1}t_{-1}} + b\frac{1}{2}e^{2\gamma_2 x + \frac{2}{\gamma_2}t_{-1}} + ab\frac{1}{4}\left(\frac{\gamma_1-\gamma_2}{\gamma_1+\gamma_2}\right)^2 e^{2(\gamma_1+\gamma_2)x + 2\left(\frac{1}{\gamma_1}+\frac{1}{\gamma_2}\right)t_{-1}}}\right], \quad (1.83)\end{aligned}$$

$$v(x, t_{-3}) = -\left[\ln\left(\frac{1 - a\frac{1}{2}e^{2\gamma_1 x + \frac{2}{\gamma_1^3}t_{-3}} - b\frac{1}{2}e^{2\gamma_2 x + \frac{2}{\gamma_2^3}t_{-3}} + ab\frac{1}{4}\left(\frac{\gamma_1-\gamma_2}{\gamma_1+\gamma_2}\right)^2 e^{2(\gamma_1+\gamma_2)x + 2\left(\frac{1}{\gamma_1^3}+\frac{1}{\gamma_2^3}\right)t_{-3}}}{1 + a\frac{1}{2}e^{2\gamma_1 x + \frac{2}{\gamma_1^3}t_{-3}} + b\frac{1}{2}e^{2\gamma_2 x + \frac{2}{\gamma_2^3}t_{-3}} + ab\frac{1}{4}\left(\frac{\gamma_1-\gamma_2}{\gamma_1+\gamma_2}\right)^2 e^{2(\gamma_1+\gamma_2)x + 2\left(\frac{1}{\gamma_1^3}+\frac{1}{\gamma_2^3}\right)t_{-3}}}\right)\right]_x, \quad (1.84)$$

$$v(x, t_3) = -\left[\ln\left(\frac{1 - a\frac{1}{2}e^{2\gamma_1 x + 2\gamma_1^3 t_3} - b\frac{1}{2}e^{2\gamma_2 x + 2\gamma_2^3 t_3} + ab\frac{1}{4}\left(\frac{\gamma_1-\gamma_2}{\gamma_1+\gamma_2}\right)^2 e^{2(\gamma_1+\gamma_2)x + 2\left(\gamma_1^3+\gamma_2^3\right)t_3}}{1 + a\frac{1}{2}e^{2\gamma_1 x + 2\gamma_1^3 t_3} + b\frac{1}{2}e^{2\gamma_2 x + 2\gamma_2^3 t_3} + ab\frac{1}{4}\left(\frac{\gamma_1-\gamma_2}{\gamma_1+\gamma_2}\right)^2 e^{2(\gamma_1+\gamma_2)x + 2\left(\gamma_1^3+\gamma_2^3\right)t_3}}\right)\right]_x, \quad (1.85)$$

$$v(x, t_5) = -\left[\ln\left(\frac{1 - a\frac{1}{2}e^{2\gamma_1 x + 2\gamma_1^5 t_5} - b\frac{1}{2}e^{2\gamma_2 x + 2\gamma_2^5 t_5} + ab\frac{1}{4}\left(\frac{\gamma_1-\gamma_2}{\gamma_1+\gamma_2}\right)^2 e^{2(\gamma_1+\gamma_2)x + 2\left(\gamma_1^5+\gamma_2^5\right)t_5}}{1 + a\frac{1}{2}e^{2\gamma_1 x + 2\gamma_1^5 t_5} + b\frac{1}{2}e^{2\gamma_2 x + 2\gamma_2^5 t_5} + ab\frac{1}{4}\left(\frac{\gamma_1-\gamma_2}{\gamma_1+\gamma_2}\right)^2 e^{2(\gamma_1+\gamma_2)x + 2\left(\gamma_1^5+\gamma_2^5\right)t_5}}\right)\right]_x. \quad (1.86)$$

2 Gradação negativa par para a hierarquia mKdV

No capítulo anterior introduzimos a equação de curvatura nula e assumimos que os potenciais A_x e A_{t_n} estão contidos numa álgebra de Kac-Moody graduada. Isto nos permitiu, após algumas manipulações, encontrar a equação

$$\partial_x D^{(0)}_{\mathcal{M}} - \partial_{t_n} A_0 + \left[A_0, D^{(0)}_{\mathcal{K}}\right] = 0, \tag{2.1}$$

que representa uma série de equações de evolução não-lineares associadas com o tempo t_n. Ainda neste capítulo introduzimos o método dressing para obtermos as soluções sólitons dos modelos encontrados com o uso da equação de curvatura nula e como exemplo estudamos a hierarquia mKdV [11] usando a estrutura algébrica:

$$\begin{aligned} \hat{\mathcal{G}} &= \hat{sl}(2), \\ \mathcal{Q} &= \frac{1}{2}H^{(0)} + 2d, \\ E &= E^{(0)}_{\alpha} + E^{(1)}_{-\alpha}. \end{aligned} \tag{2.2}$$

Neste exemplo, observou-se que ao tempo t_3 esta associada a equação mKdV e vimos que ao decompormos a equação de curvatura nula associada ao modelo, para o mais alto grau obtemos:

$$\left[E, D^{(3)}_3\right] = 0, \tag{2.3}$$

o que implica que $D^{(3)}_3$ pertence ao núcleo, que possui apenas grau ímpar, de acordo com as equações (1.39) e (1.41). Entretanto, notou-se em [9, 11] que isto não acontece para a hierarquia mKdV negativa, isto é, para t_{-2n}, com $n > 0$. Desta maneira, considere a equação de curvatura nula,

$$\left[\partial_x + A_x, \partial_{t_{-2n}} + A_{t_{-2n}}\right] = 0, \tag{2.4}$$

com os potenciais A_x e $A_{t_{-2n}}$ escritos como,

$$\begin{aligned} A_x &= E + A_0, \\ A_{t_{-2n}} &= D_{-2n}^{(-2n)} + D_{-2n}^{-(2n-1)} + D_{-2n}^{-(2n-2)} + \cdots + D_{-2n}^{(-1)}. \end{aligned} \tag{2.5}$$

Substituindo (2.5) em (2.4) e separando grau a grau obtemos

$$\begin{aligned} \partial_x D_{-2n}^{(-2n)} + \left[A_0, D_{-2n}^{(-2n)}\right] &= 0, \\ \partial_x D_{-2n}^{-(2n-1)} + \left[A_0, D_{-2n}^{-(2n-1)}\right] + \left[E, D_{-2n}^{(-2n)}\right] &= 0, \\ \vdots &= \vdots \\ \partial_x D_{-2n}^{(-1)} + \left[A_0, D_{-2n}^{(-1)}\right] + \left[E, D_{-2n}^{(-2)}\right] &= 0, \\ -\partial_{t_{-2n}} A_0 + \left[E, D_{-2n}^{(-1)}\right] &= 0. \end{aligned} \tag{2.6}$$

Considere agora $n = 1$ na relação (2.5). De acordo com as propriedades algébricas da álgebra escolhida os operadores $D_{-2}^{(i)}$ podem ser escritos em suas componentes núcleo e imagem como,

$$\begin{aligned} D_{-2}^{(-2)} &= c_{-2} H^{(-1)}, \\ D_{-2}^{(-1)} &= a_{-1}\left(E_{\alpha}^{(-1)} + E_{-\alpha}^{(0)}\right) + b_{-1}\left(E_{\alpha}^{(-1)} - E_{-\alpha}^{(0)}\right). \end{aligned} \tag{2.7}$$

Substituindo (2.7) em (2.6) obtemos um conjunto de equações,

$$c_{-2} = \text{constante}, \tag{2.8}$$

$$\partial_x a_{-1} + 2v b_{-1} = 0, \tag{2.9}$$

$$\partial_x b_{-1} + 2v a_{-1} - 2c_{-2} = 0, \tag{2.10}$$

cuja solução é dada por

$$\begin{aligned} a_{-1} &= c_{-2}\left(e^{-2\int v dx}\int e^{2\int v dx} dx - e^{2\int v dx}\int e^{-2\int v dx} dx\right), \\ b_{-1} &= c_{-2}\left(e^{-2\int v dx}\int e^{2\int v dx} dx + e^{2\int v dx}\int e^{-2\int v dx} dx\right), \end{aligned} \tag{2.11}$$

onde $v = v(x, t_{-2})$ e $v = -\phi_x$. Logo, a equação de movimento é dada por,

$$v_{xxt_{-2}} - 4v^2 v_{t_{-2}} - v^{-1} v_x v_{xt_{-2}} - 4v^{-1} v_x = 0. \tag{2.12}$$

onde fizemos $c_{-2} = 1$ por conveniência. A equação (2.12) foi obtida em [10] usando operador de recursão.

Tomando operadores graduados de ordem mais baixa, obtemos outros modelos da hierarquia. Como exemplo considere $n = 2$ na relação (2.5). Os operadores

$D_{-4}^{(i)}$ são escritos em suas componentes núcleo e imagem como:

$$\begin{aligned}
D_{-4}^{(-4)} &= c_{-4}H^{(-2)}, \\
D_{-4}^{(-3)} &= a_{-3}\left(E_{\alpha}^{(-2)} + E_{-\alpha}^{(-1)}\right) + b_{-3}\left(E_{\alpha}^{(-2)} - E_{-\alpha}^{(-1)}\right), \\
D_{-4}^{(-2)} &= c_{-2}H^{(-1)}, \\
D_{-4}^{(-1)} &= a_{-1}\left(E_{\alpha}^{(-1)} + E_{-\alpha}^{(0)}\right) + b_{-1}\left(E_{\alpha}^{(-1)} - E_{-\alpha}^{(0)}\right).
\end{aligned} \tag{2.13}$$

Substituindo (2.13) em (2.6) obtemos um conjunto de equações,

$$\begin{aligned}
c_{-4} &= \text{constante}, \\
\partial_x a_{-3} + 2vb_{-3} &= 0, \\
\partial_x b_{-3} + 2va_{-3} - 2c_{-4} &= 0, \\
\partial_x c_{-2} - 2b_{-3} &= 0, \\
\partial_x a_{-1} + 2vb_{-1} &= 0, \\
\partial_x b_{-1} + 2va_{-1} - 2c_{-2} &= 0,
\end{aligned} \tag{2.14}$$

cuja solução é dada por,

$$\begin{aligned}
a_{-3} &= c_{-4}\left(e^{-2\int vdx}\int e^{2\int vdx}dx + e^{2\int vdx}\int e^{-2\int vdx}dx\right), \\
b_{-3} &= c_{-4}\left(e^{-2\int vdx}\int e^{2\int vdx}dx + e^{2\int vdx}\int e^{-2\int vdx}dx\right), \\
c_{-2} &= \int b_{-3}dx, \\
a_{-1} &= e^{-2\int vdx}\int e^{2\int vdx}c_{-2}dx - e^{2\int vdx}\int e^{-2\int vdx}c_{-2}dx, \\
b_{-1} &= e^{-2\int vdx}\int e^{2\int vdx}c_{-2}dx + e^{2\int vdx}\int e^{-2\int vdx}c_{-2}dx.
\end{aligned} \tag{2.15}$$

onde $v = v\,(x, t_{-4})$ e $v = -\phi_x$. Portanto a equação de movimento pode ser escrita como [11]

$$\phi_{xt_{-4}} + 2e^{-2\int vdx}\int e^{2\int vdx}c_{-2}dx + e^{2\int vdx}\int e^{-2\int vdx}c_{-2}dx = 0. \tag{2.16}$$

2.1 Sólitons

Neste ponto, para obtermos as soluções sóliton das equações (2.12) e (2.16) devemos modificar o método dressing introduzido no capítulo anterior pois, $v = 0$, não é solução destas equações. Portanto, vamos supor a configuração de vácuo:

$$A_{x,\text{vac}} = E_\alpha^{(0)} + E_{-\alpha}^{(1)} + v_0 H^{(0)} - \frac{1}{v_0} t_{-2}\hat{c}, \tag{2.17}$$

$$A_{t_{-2},\text{vac}} = \frac{1}{v_0}\left(E_\alpha^{(-1)} + E_{-\alpha}^{(0)}\right) + H^{(-1)}, \tag{2.18}$$

com $v_0 = \text{constante} \neq 0$. Esta configuração de vácuo não-trivial implica a seguinte modificação da equação (1.36),

$$T_{\text{vac}} = e^{x\left(E_\alpha^{(0)} + E_{-\alpha}^{(1)} + v_0 H^{(0)}\right)} e^{\frac{t_{-2m}}{v_0}\left(E_\alpha^{(-m)} + E_{-\alpha}^{(-m+1)} + v_0 H^{(-m)}\right)}, \tag{2.19}$$

onde temos agora apenas graus negativos pares. No intuito de encontrarmos as novas funções tau devemos encontrar a equação análoga à equação (1.29). Para isto considere o operador de Lax:

$$A_x = E^{(1)} + A_0 + \nu\hat{c}, \tag{2.20}$$

e usando a transformação de gauge,

$$A_x = \Theta_+ A_{x,\text{vac}} \Theta_+^{-1} + \Theta_+ \partial_x \Theta_+^{-1} \tag{2.21}$$

onde

$$\Theta_+ = e^{v(0)} e^{v(1) + v(2) + \cdots}, \tag{2.22}$$

obtemos,

$$E^{(1)} + A_0 + \nu\hat{c} = \Theta_+\left(E^{(1)} + v_0 H^{(0)} - \frac{1}{v_0} t_{-2}\hat{c}\right)\Theta_+^{-1} - \sum_{n=0}^{\infty} \partial_x v(n) + \cdots . \tag{2.23}$$

Tomando a componente de grau zero de (2.23) obtemos,

$$A_0 + \nu\hat{c} = v_0 H^{(0)} - \frac{1}{v_0} t_{-2}\hat{c} - \partial_x v(0), \tag{2.24}$$

e após algumas manipulações encontramos,

$$e^{v(0)} = B e^{\eta\hat{c}} e^{v_0 x H^{(0)}}. \tag{2.25}$$

onde $\eta = -\partial_x \nu$ e fizemos a constante de integração igual a zero.

Usando a representação de peso mais alto as funções tau podem ser escritas como,

$$\tau_0 = e^{\eta} = \left\langle 0 | T_{\text{vac}} g T_{\text{vac}}^{-1} | 0 \right\rangle, \tag{2.26}$$

$$\tau_1 = e^{\eta + \phi + v_0 x} = \left\langle 1 | T_{\text{vac}} g T_{\text{vac}}^{-1} | 1 \right\rangle, \tag{2.27}$$

onde agora temos

$$T_{\text{vac}} = e^{x\left(E_{\alpha}^{(0)}+E_{-\alpha}^{(1)}+v_0H^{(0)}\right)} e^{\frac{t_{-2m}}{v_0}\left(E_{\alpha}^{(-m)}+E_{-\alpha}^{(-m+1)}+v_0H^{(-m)}\right)}, \tag{2.28}$$

o que resulta

$$v(x, t_{-2}) = v_0 - \partial_x \ln\left(\frac{\tau_1}{\tau_0}\right). \tag{2.29}$$

Para a construção das soluções sólitons, o novo operador de vértice é definido por [11],

$$V(k, v_0) = \sum_{n=-\infty}^{\infty} \left(k^2 - v_0^2\right)^{-n} \left[H^{(n)} + \frac{v_0 - k}{2k}\delta_{n,0}\hat{c} + E_{\alpha}^{(n)}\left(k + v_0\right)^{-1} - E_{-\alpha}^{(n+1)}\left(k - v_0\right)^{-1}\right],$$

e satisfaz a relação,

$$\begin{aligned} [b_1, V(k, v_0)] &= -2kV(k, v_0), && (2.30)\\ [b_{-2m}, V(k, v_0)] &= -2k\left(k^2 - v_0^2\right)^{-m} V(k, v_0), && (2.31) \end{aligned}$$

onde

$$\begin{aligned} b_1 &= E_{\alpha}^{(0)} + E_{-\alpha}^{(1)} + v_0H^{(0)}, && (2.32)\\ b_{-2m} &= E_{\alpha}^{(-m)} + E_{-\alpha}^{(-m+1)} + v_0H^{(-m)}. && (2.33) \end{aligned}$$

A solução 1-sóliton será obtida com o uso de um operador de vértice, isto é:

$$g = e^{V_1(\gamma_1)}, \tag{2.34}$$

logo,

$$\begin{aligned} T_{vac}gT_{vac}^{-1} &= e^{x\left(E_{\alpha}^{(0)}+E_{-\alpha}^{(1)}+v_0H^{(0)}\right)} e^{\frac{t_{-2m}}{v_0}\left(E_{\alpha}^{(-m)}+E_{-\alpha}^{(-m+1)}+v_0H^{(-m)}\right)} \left(1 + V_1(\gamma_1)\right)\\ &\quad e^{-x\left(E_{\alpha}^{(0)}+E_{-\alpha}^{(1)}+v_0H^{(0)}\right)} e^{-\frac{t_{-2m}}{v_0}\left(E_{\alpha}^{(-m)}+E_{-\alpha}^{(-m+1)}+v_0H^{(-m)}\right)},\\ &= 1 + \rho_1 V_1(\gamma_1), && (2.35) \end{aligned}$$

com

$$\rho_1 = e^{2k_1x + \frac{2k_1t_{-2m}}{v_0\left(k_1^2 - v_0^2\right)^m}}. \tag{2.36}$$

Desta forma,

$$\begin{aligned} \tau_0 &= e^{\eta} = 1 + \left(v_0 + k_1\right)\rho_1, && (2.37)\\ \tau_1 &= e^{\eta+\phi+v_0x} = 1 + \left(v_0 - k_1\right)\rho_1. && (2.38) \end{aligned}$$

A solução 1-sóliton dos modelos associados a $n = 1, 2$ da hierarquia mKdV par negativa são dados respectivamente por:

$$v(x, t_{-2}) = v_0 - \partial_x \ln \left(\frac{1 + (v_0 - k_1)\, e^{2k_1 x + \frac{2k_1 t_{-2}}{v_0 \left(k_1^2 - v_0^2\right)}}}{1 + (v_0 + k_1)\, e^{2k_1 x + \frac{2k_1 t_{-2}}{v_0 \left(k_1^2 - v_0^2\right)}}} \right). \tag{2.39}$$

$$v(x, t_{-4}) = v_0 - \partial_x \ln \left(\frac{1 + (v_0 - k_1)\, e^{2k_1 x + \frac{2k_1 t_{-4}}{v_0 \left(k_1^2 - v_0^2\right)^2}}}{1 + (v_0 + k_1)\, e^{2k_1 x + \frac{2k_1 t_{-4}}{v_0 \left(k_1^2 - v_0^2\right)^2}}} \right). \tag{2.40}$$

Para a solução 2-sólitons usaremos o produto de dois operadores de vértice, isto é:

$$g = e^{V_1(\gamma_1)} e^{V_2(\gamma_2)}, \tag{2.41}$$

logo,

$$\begin{aligned} T_{vac} g T_{vac}^{-1} &= e^{x\left(E_\alpha^{(0)} + E_{-\alpha}^{(1)} + v_0 H^{(0)}\right)} e^{\frac{t_{-2m}}{v_0}\left(E_\alpha^{(-m)} + E_{-\alpha}^{(-m+1)} + v_0 H^{(-m)}\right)} \left(1 + V_1(\gamma_1)\right)\left(1 + V_2(\gamma_2)\right) \\ & \quad e^{-x\left(E_\alpha^{(0)} + E_{-\alpha}^{(1)} + v_0 H^{(0)}\right)} e^{-\frac{t_{-2m}}{v_0}\left(E_\alpha^{(-m)} + E_{-\alpha}^{(-m+1)} + v_0 H^{(-m)}\right)}, \\ &= 1 + \rho_1 V_1(\gamma_1) + \rho_2 V_2(\gamma_2) + \rho_1 \rho_2 V_1(\gamma_1) V_2(\gamma_2), \end{aligned} \tag{2.42}$$

com

$$\rho_i = e^{2k_i x + \frac{2k_i t_{-2m}}{v_0 \left(k_i^2 - v_0^2\right)^m}}. \tag{2.43}$$

Desta forma,

$$\begin{aligned} \tau_1 &= e^{\eta + \phi + v_0 x} = 1 + \eta_1 \rho_1 + \eta_2 \rho_2 + \eta_3 \rho_1 \rho_2, \\ \tau_0 &= e^{\eta} = 1 + \delta_1 \rho_1 + \delta_2 \rho_2 + \delta_3 \rho_1 \rho_2, \end{aligned}$$

onde

$$\eta_1 = \alpha - k_1, \tag{2.44}$$

$$\eta_2 = \alpha - k_2, \tag{2.45}$$

$$\eta_3 = \eta_1 \eta_2 \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}, \tag{2.46}$$

$$\delta_1 = \alpha + k_1, \tag{2.47}$$

$$\delta_2 = \alpha + k_2, \tag{2.48}$$

$$\delta_3 = \delta_1 \delta_2 \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}. \tag{2.49}$$

A solução 2-sólitons dos modelos associados a $n = 1, 2$ da hierarquia mKdV negativa par são dados respectivamente por:

$$v(x,t_{-2}) = v_0 - \partial_x \ln\left[\frac{1+\eta_1 e^{2k_1x+\frac{2k_1t_{-2}}{v_0\left(k_1^2-v_0^2\right)}}+\eta_2 e^{2k_2x+\frac{2k_2t_{-2}}{v_0\left(k_2^2-v_0^2\right)}}+\eta_3 e^{2(k_1+k_2)x+\frac{2k_1t_{-2}}{v_0\left(k_1^2-v_0^2\right)}+\frac{2k_2t_{-2}}{v_0\left(k_2^2-v_0^2\right)}}}{1+\delta_1 e^{2k_1x+\frac{2k_1t_{-2}}{v_0\left(k_1^2-v_0^2\right)}}+\delta_2 e^{2k_2x+\frac{2k_2t_{-2}}{v_0\left(k_2^2-v_0^2\right)}}+\delta_3 e^{2(k_1+k_2)x+\frac{2k_1t_{-2}}{v_0\left(k_1^2-v_0^2\right)}+\frac{2k_2t_{-2}}{v_0\left(k_2^2-v_0^2\right)}}}\right].$$

$$v(x,t_{-4}) = v_0 - \partial_x \ln\left[\frac{1+\eta_1 e^{2k_1x+\frac{2k_1t_{-4}}{v_0\left(k_1^2-v_0^2\right)^2}}+\eta_2 e^{2k_2x+\frac{2k_2t_{-4}}{v_0\left(k_2^2-v_0^2\right)^2}}+\eta_3 e^{2(k_1+k_2)x+\frac{2k_1t_{-4}}{v_0\left(k_1^2-v_0^2\right)^2}+\frac{2k_2t_{-4}}{v_0\left(k_2^2-v_0^2\right)^2}}}{1+\delta_1 e^{2k_1x+\frac{2k_1t_{-4}}{v_0\left(k_1^2-v_0^2\right)^2}}+\delta_2 e^{2k_2x+\frac{2k_2t_{-4}}{v_0\left(k_2^2-v_0^2\right)^2}}+\delta_3 e^{2(k_1+k_2)x+\frac{2k_1t_{-4}}{v_0\left(k_1^2-v_0^2\right)^2}+\frac{2k_2t_{-4}}{v_0\left(k_2^2-v_0^2\right)^2}}}\right].$$

3 Hierarquia mKdV/sinh-Gordon

Considere a equação de curvatura nula,

$$[\partial_x + A_x, \partial_{t_n} + A_{t_n}] = 0, \tag{3.1}$$

com os potenciais A_x e A_{t_n} escritos como,

$$\begin{aligned} A_x &= E^{(1)} + A_0, \\ A_{t_n} &= D_n^{(n)} + D_n^{(n-1)} + \cdots + D_n^{(0)} + D_n^{(-1)}. \end{aligned} \tag{3.2}$$

Substituindo (3.2) em (3.1) e separando grau a grau obtemos

$$\begin{aligned} \left[E, D_n^{(n)}\right] &= 0, \\ \partial_x D_n^{(n)} + \left[A_0, D_n^{(n)}\right] + \left[E, D_n^{(n-1)}\right] &= 0, \\ \vdots &= \vdots \\ \partial_x D_n^{(1)} + \left[A_0, D_n^{(1)}\right] + \left[E, D_n^{(0)}\right] &= 0, \\ \partial_x D_n^{(0)} + \left[A_0, D_n^{(0)}\right] + \left[E, D_n^{(-1)}\right] - \partial_{t_n} A_0 &= 0, \\ \partial_x D_n^{(-1)} + \left[A_0, D_n^{(-1)}\right] &= 0. \end{aligned} \tag{3.3}$$

Para os modelos tratado neste capítulo usaremos a álgebra de Kac-Moody $\hat{\mathcal{G}} = \hat{sl}(2)$ com a graduação $\mathcal{Q} = \frac{1}{2}H^{(0)} + 2d$, e o elemento semi-simples $E^{(1)} = E_\alpha^{(0)} + E_{-\alpha}^{(1)}$, que permite decompor a álgebra como

$$\begin{aligned} \mathcal{K}(E) &= \left\{E_\alpha^{(n)} + E_{-\alpha}^{(n+1)}\right\}, \\ \mathcal{M}(E) &= \left\{E_\alpha^{(n)} - E_{-\alpha}^{(n+1)}, H^{(n)}\right\}. \end{aligned} \tag{3.4}$$

Para construirmos a equação mKdV/sinh-Gordon faremos $n = 3$ na relação (3.2). De acordo com a equação (3.4), os operadores $D_3^{(i)}$ podem ser escritos em suas componentes núcleo e imagem como

$$\begin{aligned}
D_3^{(3)} &= a_3\left(E_\alpha^{(1)} + E_{-\alpha}^{(2)}\right) + b_3\left(E_\alpha^{(1)} - E_{-\alpha}^{(2)}\right),\\
D_3^{(2)} &= c_2 H^{(1)},\\
D_3^{(1)} &= a_1\left(E_\alpha^{(0)} + E_{-\alpha}^{(1)}\right) + b_1\left(E_\alpha^{(0)} - E_{-\alpha}^{(1)}\right),\\
D_3^{(0)} &= c_0 H^{(0)},\\
D_3^{(-1)} &= a_{-1}\left(E_\alpha^{(-1)} + E_{-\alpha}^{(0)}\right) + b_{-1}\left(E_\alpha^{(-1)} - E_{-\alpha}^{(0)}\right).
\end{aligned} \tag{3.5}$$

Substituindo (3.5) em (3.3) obtemos um conjunto de equações,

$$\begin{aligned}
b_3 &= 0, & a_3 &= \text{constante},\\
c_2 &= a_3 v, & b_1 &= \frac{1}{2}\partial_x c_2,\\
\partial_x a_1 + 2vb_1 &= 0, & \partial_x b_1 + 2va_1 - 2c_0 &= 0,\\
\partial_x a_{-1} + 2vb_{-1} &= 0, & \partial_x b_{-1} + 2va_{-1} &= 0,
\end{aligned} \tag{3.6}$$

cuja solução é dada por

$$\begin{aligned}
a_3 &= \text{constante},\\
b_1 &= \frac{a_3}{2} v_x,\\
a_1 &= -\frac{a_3}{2} v^2,\\
c_0 &= \frac{a_3}{4}\left(v_{xx} - 2v^3\right),\\
a_{-1} &= \eta \cosh(2\phi),\\
b_{-1} &= -\eta \sinh(2\phi),
\end{aligned} \tag{3.7}$$

onde $v = v(x, t_3)$, $v = \phi_x$ e η constante. Logo, a equação de movimento é dada por,

$$\frac{a_3}{4}\left(\phi_{xxxx} - 6\phi_x^2\phi_{xx}\right) - \phi_{xt_3} + 2\eta \sinh(2\phi) = 0. \tag{3.8}$$

A equação (3.8) é chamada mKdV/sinh-Gordon [12, 15, 17, 19]. Fazendo $a_3 = 0$ obtemos a equação sinh-Gordon e fazendo $\eta = 0$ obtemos a equação mKdV.

Tomando operadores graduados de ordem mais alta, obtemos outros modelos da hierarquia. Como exemplo considere $n = 5$ na relação (3.2). Os operadores

$D_5^{(i)}$ são escritos em suas componentes núcleo e imagem como:

$$\begin{aligned}
D_5^{(5)} &= a_5\left(E_\alpha^{(2)}+E_{-\alpha}^{(3)}\right)+b_5\left(E_\alpha^{(2)}-E_{-\alpha}^{(3)}\right),\\
D_5^{(4)} &= c_4H^{(2)},\\
D_5^{(3)} &= a_3\left(E_\alpha^{(1)}+E_{-\alpha}^{(2)}\right)+b_3\left(E_\alpha^{(1)}-E_{-\alpha}^{(2)}\right),\\
D_5^{(2)} &= c_2H^{(1)},\\
D_5^{(1)} &= a_1\left(E_\alpha^{(0)}+E_{-\alpha}^{(1)}\right)+b_1\left(E_\alpha^{(0)}-E_{-\alpha}^{(1)}\right),\\
D_5^{(0)} &= c_0H^{(0)},\\
D_5^{(-1)} &= a_{-1}\left(E_\alpha^{(-1)}+E_{-\alpha}^{(0)}\right)+b_{-1}\left(E_\alpha^{(-1)}-E_{-\alpha}^{(0)}\right).
\end{aligned} \tag{3.9}$$

Substituindo (3.9) em (3.3) obtemos um conjunto de equações,

$$\begin{aligned}
b_5 &= 0, & a_5 &= \text{constante},\\
\partial_x a_3+2vb_3 &= 0, & \partial_x b_3+2va_3-2c_2 &= 0,\\
\partial_x c_4-2b_3 &= 0, & \partial_x c_2-2b_1 &= 0,\\
\partial_x a_1+2vb_1 &= 0, & \partial_x b_1+2va_1-2c_0 &= 0,\\
\partial_x a_{-1}+2vb_{-1} &= 0, & \partial_x b_{-1}+2va_{-1} &= 0,
\end{aligned} \tag{3.10}$$

cuja solução é dada por

$$\begin{aligned}
a_5 &= \text{constante},\\
b_5 &= 0,\\
c_4 &= a_5v,\\
b_3 &= \frac{a_5}{2}v_x,\\
a_3 &= -\frac{a_5}{2}v^2,\\
c_2 &= \frac{a_5}{4}\left(v_{xx}-2v^3\right),\\
b_1 &= \frac{a_5}{8}\left(v_{xxx}-6v^2v_x\right),\\
a_1 &= \frac{a_5}{8}\left(v_x^2+3v^4-2vv_{xx}\right),\\
c_0 &= \frac{a_5}{16}\left(v_{xxxx}-10v^2v_{xx}-10vv_x^2+6v^5\right),\\
a_{-1} &= \eta\cosh\left(2\phi\right),\\
b_{-1} &= -\eta\sinh\left(2\phi\right),
\end{aligned}$$

onde $v=v\left(x,t_3\right)$, $v=\phi_x$ e η constante. Portanto a equação de movimento pode ser escrita como [16, 19]

$$\frac{a_5}{16}\left(v_{xxxxx}-10v^2v_{xxx}-40vv_xv_{xx}-10v_x^3+30v^4v_x\right)-\phi_{xt_5}+2\eta\sinh\left(2\phi\right)=0, \tag{3.11}$$

que é o acoplamento da equação (1.53) com o modelo sinh-Gordon.

3.1 Sólitons

O operador de vértice é definido por,

$$V(\gamma) = \sum_{x\in Z} \left\{ \left(H^{(n)} - \frac{1}{2}\hat{c}\delta_{n,0} \right) \gamma^{-2n} + \left(E_{\alpha}^{(n)} - E_{-\alpha}^{(n+1)} \right) \gamma^{-2n-1} \right\}, \tag{3.12}$$

e obedece a equação de auto-valor:

$$\left[E^{(1)}, V(\gamma) \right] = 2\gamma V(\gamma), \tag{3.13}$$

$$\left[E^{(n)}, V(\gamma) \right] = -2 \left\{ a_{2n+1}\gamma^{2n+1} + \frac{\eta}{\gamma} \right\} V(\gamma). \tag{3.14}$$

onde

$$E^{(1)} = E_{\alpha}^{(0)} + E_{-\alpha}^{(1)}, \tag{3.15}$$

$$E^{(n)} = a_{2n+1} \left(E_{\alpha}^{(n)} + E_{-\alpha}^{(n+1)} \right) + \eta \left(E_{\alpha}^{(-1)} + E_{-\alpha}^{(0)} \right). \tag{3.16}$$

As funções τ são dadas por,

$$\tau_i =< \lambda_i|(\Theta_-)^{-1}\Theta_+|\lambda_i >= \left\langle \lambda_i | T_{vac} g T_{vac}^{-1} | \lambda_i \right\rangle. \tag{3.17}$$

Escrevendo $B = e^{\phi H^{(0)} + \nu\hat{c}}$ podemos reescrever as funções τ como,

$$\tau_0 = e^{\nu} = \left\langle \lambda_0 | T_{vac} g T_{vac}^{-1} | \lambda_0 \right\rangle, \tag{3.18}$$

$$\tau_1 = e^{\nu+\phi} = \left\langle \lambda_1 | T_{vac} g T_{vac}^{-1} | \lambda_1 \right\rangle. \tag{3.19}$$

A solução 1-sóliton será obtida com o uso de um operador de vértice, isto é:

$$g = e^{aV_1(\gamma_1)}, \tag{3.20}$$

logo,

$$\begin{aligned} T_{vac} g T_{vac}^{-1} &= e^{-E^{(1)}x - \left(a_{2n+1}E^{(2n+1)} + \eta E^{(-1)}\right)t_{2n+1}} \left(1 + aV_1(\gamma_1)\right) e^{E^{(1)}x + \left(a_{2n+1}E^{(2n+1)} + \eta E^{(-1)}\right)t_n}, \\ &= 1 + a\rho_1 V_1(\gamma_1), \end{aligned} \tag{3.21}$$

com

$$\rho_1 = e^{2\gamma_1 x + 2\left(a_{2n+1}\gamma_1^{2n+1} + \frac{\eta}{\gamma_1}\right)t_{2n+1}}. \tag{3.22}$$

Desta forma,

$$\begin{aligned} \tau_0 &= e^{\nu} = 1 + \frac{a}{2}\rho_1, & (3.23) \\ \tau_1 &= e^{\nu+\phi} = 1 - \frac{a}{2}\rho_1. & (3.24) \end{aligned}$$

A solução 1-sóliton dos modelos associados a $n = 1, 2$ da hierarquia mKdV/sinh-Gordon são dados respectivamente por:

$$\phi(x, t_3) = \ln\frac{\tau_1}{\tau_0} = \ln\left(\frac{1 - \frac{a}{2}e^{2\gamma_1 x + 2\left(a_3\gamma_1^3 + \frac{\eta}{\gamma_1}\right)t_3}}{1 + \frac{a}{2}e^{2\gamma_1 x + 2\left(a_3\gamma_1^3 + \frac{\eta}{\gamma_1}\right)t_3}}\right), \tag{3.25}$$

$$\phi(x, t_5) = \ln\left(\frac{1 - \frac{a}{2}e^{2\gamma_1 x + 2\left(a_5\gamma_1^5 + \frac{\eta}{\gamma_1}\right)t_5}}{1 + \frac{a}{2}e^{2\gamma_1 x + 2\left(a_5\gamma_1^5 + \frac{\eta}{\gamma_1}\right)t_5}}\right). \tag{3.26}$$

Para a solução 2-sólitons usaremos o produto de dois operadores de vértice, isto é:

$$g = e^{aV_1(\gamma_1)}e^{bV_2(\gamma_2)}, \tag{3.27}$$

logo,

$$\begin{aligned} T_{vac}gT_{vac}^{-1} &= e^{-E^{(1)}x - \left(a_{2n+1}E^{(2n+1)} + \eta E^{(-1)}\right)t_{2n+1}}\left(1 + aV_1(\gamma_1)\right)\left(1 + bV_2(\gamma_2)\right)e^{E^{(1)}x + \left(a_{2n+1}E^{(2n+1)} + \eta E^{(-1)}\right)t_{2n+1}}, \\ &= 1 + a\rho_1 V_1(\gamma_1) + b\rho_2 V_2(\gamma_2) + ab\rho_1\rho_2 V_1(\gamma_1)V_2(\gamma_2), \end{aligned} \tag{3.28}$$

com

$$\rho_i = e^{2\gamma_i x + 2\left(a_{2n+1}\gamma_i^{2n+1} + \frac{\eta}{\gamma_i}\right)t_{2n+1}}. \tag{3.29}$$

Desta forma,

$$\begin{aligned} \tau_0 &= e^{\nu} = 1 + \frac{a}{2}\rho_1 + \frac{b}{2}\rho_2 + ab\frac{1}{4}\left(\frac{\gamma_1 - \gamma_2}{\gamma_1 + \gamma_2}\right)^2\rho_1\rho_2, & (3.30) \\ \tau_1 &= e^{\nu+\phi} = 1 - \frac{a}{2}\rho_1 - \frac{b}{2}\rho_2 + ab\frac{1}{4}\left(\frac{\gamma_1 - \gamma_2}{\gamma_1 + \gamma_2}\right)^2\rho_1\rho_2. & (3.31) \end{aligned}$$

A solução 2-sólitons dos modelos associados a $n = 1, 2$ da hierarquia mKdV/sinh-Gordon são dados respectivamente por:

$$\phi(x, t_3) = \ln\left(\frac{1 - \frac{a}{2}e^{2\gamma_1 x + 2\left(a_3\gamma_1^3 + \frac{\eta}{\gamma_1}\right)t_3} - \frac{b}{2}e^{2\gamma_2 x + 2\left(a_3\gamma_2^3 + \frac{\eta}{\gamma_2}\right)t_3} + ab\frac{1}{4}\left(\frac{\gamma_1 - \gamma_2}{\gamma_1 + \gamma_2}\right)^2 e^{2(\gamma_1 + \gamma_2)x + 2\left(a_3\gamma_1^3 + a_3\gamma_2^3 + \frac{\eta}{\gamma_1} + \frac{\eta}{\gamma_2}\right)t_3}}{1 + \frac{a}{2}e^{2\gamma_1 x + 2a_3\left(\gamma_1^3 + \frac{\eta}{\gamma_1}\right)t_3} + \frac{b}{2}e^{2\gamma_2 x + 2\left(a_3\gamma_2^3 + \frac{\eta}{\gamma_2}\right)t_3} + ab\frac{1}{4}\left(\frac{\gamma_1 - \gamma_2}{\gamma_1 + \gamma_2}\right)^2 e^{2(\gamma_1 + \gamma_2)x + 2\left(a_3\gamma_1^3 + a_3\gamma_2^3 + \frac{\eta}{\gamma_1} + \frac{\eta}{\gamma_2}\right)t_3}}\right),$$

$$\phi(x, t_5) = \ln\left(\frac{1 - \frac{a}{2}e^{2\gamma_1 x + 2\left(a_5\gamma_1^5 + \frac{\eta}{\gamma_1}\right)t_5} - \frac{b}{2}e^{2\gamma_2 x + 2\left(a_5\gamma_2^5 + \frac{\eta}{\gamma_2}\right)t_5} + ab\frac{1}{4}\left(\frac{\gamma_1 - \gamma_2}{\gamma_1 + \gamma_2}\right)^2 e^{2(\gamma_1 + \gamma_2)x + 2\left(a_5\gamma_1^5 + a_5\gamma_2^5 + \frac{\eta}{\gamma_1} + \frac{\eta}{\gamma_2}\right)t_5}}{1 + \frac{a}{2}e^{2\gamma_1 x + 2\left(a_5\gamma_1^5 + \frac{\eta}{\gamma_1}\right)t_5} + \frac{b}{2}e^{2\gamma_2 x + 2\left(a_5\gamma_2^5 + \frac{\eta}{\gamma_2}\right)t_5} + ab\frac{1}{4}\left(\frac{\gamma_1 - \gamma_2}{\gamma_1 + \gamma_2}\right)^2 e^{2(\gamma_1 + \gamma_2)x + 2\left(a_5\gamma_1^5 + a_5\gamma_2^5 + \frac{\eta}{\gamma_1} + \frac{\eta}{\gamma_2}\right)t_5}}\right).$$

4 Hierarquia super mKdV / sinh-Gordon

Considere a superálgebra de Lie afim

$$\hat{\mathcal{G}} = sl(2,1), \tag{4.1}$$

com geradores

$$h_1 = \alpha_1 \cdot H, \qquad h_2 = \alpha_2 \cdot H, \qquad E_{\pm\alpha_1}, \qquad E_{\pm\alpha_2}, \qquad E_{\pm(\alpha_1+\alpha_2)}, \tag{4.2}$$

onde α_1 e α_2, $\alpha_1 + \alpha_2$ são respectivamente raízes bosônicas e fermiônicas (conforme apêndice B).
A hierarquia integrável é definida pela escolha do operador de graduação [1]

$$\mathcal{Q} = \frac{1}{2} h_1^{(0)} + 2d. \tag{4.3}$$

Os modelos deste capítulo serão tratados usando-se uma sub-álgebra $\tilde{sl}(2,1)$. Logo, conforme a graduação (4.3) temos:

$$\begin{aligned}
F_1^{(2n+3/2)} &= (E_{\alpha_1+\alpha_2}^{(n+1/2)} - E_{\alpha_2}^{(n+1)}) + (E_{-\alpha_1-\alpha_2}^{(n+1)} - E_{-\alpha_2}^{(n+1/2)}), \\
F_2^{(2n+1/2)} &= -(E_{\alpha_1+\alpha_2}^{(n)} - E_{\alpha_2}^{(n+1/2)}) + (E_{-\alpha_1-\alpha_2}^{(n+1/2)} - E_{-\alpha_2}^{(n)}), \\
G_1^{(2n+1/2)} &= (E_{\alpha_1+\alpha_2}^{(n)} + E_{\alpha_2}^{(n+1/2)}) + (E_{-\alpha_1-\alpha_2}^{(n+1/2)} + E_{-\alpha_2}^{(n)}), \\
G_2^{(2n+3/2)} &= -(E_{\alpha_1+\alpha_2}^{(n+1/2)} + E_{\alpha_2}^{(n+1)}) + (E_{-\alpha_1-\alpha_2}^{(n+1)} + E_{-\alpha_2}^{(n+1/2)}), \\
K_1^{(2n+1)} &= -E_{-\alpha_1}^{(n+1)} - E_{\alpha_1}^{(n)}, \\
K_2^{(2n+1)} &= h_+^{(n+1/2)} + h_2^{(n+1/2)}, \\
M_1^{(2n+1)} &= E_{-\alpha_1}^{(n+1)} - E_{\alpha_1}^{(n)}, \\
M_2^{(2n)} &= h_1^{(n)}.
\end{aligned} \tag{4.4}$$

Ver no apêndice C as relações de comutação e anti-comutação para os geradores da álgebra $\tilde{sl}(2,1)$ dados acima.
O elemento semi-simples é escrito como

$$E = E^{(1)} = K_1^{(1)} + K_2^{(1)}, \tag{4.5}$$

e induz a decomposição da sub-álgebra $\tilde{sl}(2,1)$ em núcleo

$$\mathcal{K}(E) = \{x \in \tilde{sl}(2,1), [E^{(1)}, x] = 0\},$$

e seu complemento chamado imagem, $\mathcal{M}\,(E)$. Assim,

$$\tilde{sl}(2,1) = \mathcal{K}(E) \oplus \mathcal{M}(E), \tag{4.6}$$

onde

$$\begin{aligned} \mathcal{M}_{Bose} &= \{M_1^{(2n+1)}, M_2^{(2n)}\}, \qquad \mathcal{M}_{Fermi} = \{G_1^{(2n+1/2)}, G_2^{(2n+3/2)}\}, \\ \mathcal{K}_{Bose} &= \{K_1^{(2n+1)}, K_2^{(2n+1)}\}, \qquad \mathcal{K}_{Fermi} = \{F_1^{(2n+3/2)}, F_2^{(2n+1/2)}\}. \end{aligned}$$

A equação de curvatura nula, para o modelo supersimétrico mKdV/sinh-Gordon pode ser escrita como

$$\left[\partial_x + E^{(1)} + A_0 + A_{1/2}, \partial_{t_3} + A_{t_3}\right] = 0, \tag{4.7}$$

onde

$$A_{t_3} = D_3^{(3)} + D_3^{(5/2)} + D_3^{(2)} + D_3^{(3/2)} + D_3^{(1)} + D_3^{(1/2)} + D_3^{(0)} + D_3^{(-1/2)} + D_3^{(-1)}, \tag{4.8}$$

$$A_0 = u\, M_2^{(0)}, \tag{4.9}$$

$$A_{1/2} = \bar{\psi}\, G_1^{(1/2)}, \tag{4.10}$$

e $u = u(x, t_3)$ é um campo bosônico e $\bar{\psi} = \bar{\psi}(x, t_3)$ é um campo fermiônico.
A equação de curvatura nula se decompõe grau a grau da forma:

$$
\begin{aligned}
\left[E, D_3^{(3)}\right] &= 0, \\
\left[E, D_3^{(5/2)}\right] + \left[A_{1/2}, D_3^{(3)}\right] &= 0, \\
\partial_x D_3^{(3)} + \left[A_0, D_3^{(3)}\right] + \left[E, D_3^{(2)}\right] + \left[A_{1/2}, D_3^{(5/2)}\right] &= 0, \\
\partial_x D_3^{(5/2)} + \left[A_0, D_3^{(5/2)}\right] + \left[E, D_3^{(3/2)}\right] + \left[A_{1/2}, D_3^{(2)}\right] &= 0, \\
\partial_x D_3^{(2)} + \left[A_0, D_3^{(2)}\right] + \left[E, D_3^{(1)}\right] + \left[A_{1/2}, D_3^{(3/2)}\right] &= 0, \\
\partial_x D_3^{(3/2)} + \left[A_0, D_3^{(3/2)}\right] + \left[E, D_3^{(1/2)}\right] + \left[A_{1/2}, D_3^{(1)}\right] &= 0, \\
\partial_x D_3^{(1)} + \left[A_0, D_3^{(1)}\right] + \left[E, D_3^{(0)}\right] + \left[A_{1/2}, D_3^{(1/2)}\right] &= 0, \\
\partial_x D_3^{(1/2)} + \left[A_0, D_3^{(1/2)}\right] + \left[E, D_{-1}^{(-1/2)}\right] + \left[A_{1/2}, D_3^{(0)}\right] - \partial_{t_3} A_{1/2} &= 0, \\
\partial_x D_3^{(0)} + \left[A_0, D_3^{(0)}\right] + \left[E, D_{-1}^{(-1)}\right] + \left[A_{1/2}, D_{-1}^{(-1/2)}\right] - \partial_{t_3} A_0 &= 0, \\
\partial_x D_{-1}^{(-1/2)} + \left[A_0, D_{-1}^{(-1/2)}\right] + \left[A_{1/2}, D_{-1}^{(-1)}\right] &= 0, \\
\partial_x D_{-1}^{(-1)} + \left[A_0, D_{-1}^{(-1)}\right] &= 0,
\end{aligned}
\tag{4.11}
$$

com

$$
\begin{aligned}
D_3^{(0)} &= a_1 M_2^{(0)}, \\
D_3^{(1/2)} &= b_1 G_1^{(1/2)} + b_2 F_2^{(1/2)}, \\
D_3^{(1)} &= c_1 M_1^{(1)} + c_2 K_1^{(1)} + c_3 K_2^{(1)}, \\
D_3^{(3/2)} &= d_1 G_2^{(3/2)} + d_2 F_1^{(3/2)}, \\
D_3^{(2)} &= e_1 M_2^{(2)}, \\
D_3^{(5/2)} &= f_1 G_1^{(5/2)} + f_2 F_2^{(5/2)}, \\
D_{-1}^{(-1/2)} &= b_{-1} G_2^{(-1/2)} + b_{-2} F_1^{(-1/2)}, \\
D_{-1}^{(-1)} &= c_{-1} M_1^{(-1)} + c_{-2} K_1^{(-1)} + c_{-3} K_2^{(-1)}.
\end{aligned}
\tag{4.12}
$$

Substituindo as relações (4.12) nas equações (4.11) obtemos um conjunto de equações

$$\begin{array}{rclc}
2f_1 - 2\bar{\psi} & = & 0, & 2u - 2e_1 + 2\bar{\psi} f_2 = 0, \\
2\bar{\psi} f_1 & = & 0, & \partial_x f_1 + 2d_1 - u f_2 = 0, \\
\partial_x f_2 - u f_1 + \bar{\psi} e_1 & = & 0, & \partial_x e_1 - 2c_1 + 2\bar{\psi} d_2 = 0, \\
\partial_x d_1 + 2b_1 - u d_2 - \bar{\psi} c_2 - \bar{\psi} c_3 & = & 0, & \partial_x d_2 - u d_1 + \bar{\psi} c_1 = 0, \\
\partial_x c_1 - 2a_1 + 2u c_2 + 2\bar{\psi} b_2 & = & 0, & \partial_x c_2 + 2u c_1 - 2\bar{\psi} b_1 = 0, \\
\partial_x c_3 + 2\bar{\psi} b_1 & = & 0, & \partial_x b_1 - \partial_{t_3} \bar{\psi} - u b_2 = 0, \\
\partial_x b_2 - u b_1 + \bar{\psi} a_1 & = & 0, & \partial_x a_1 - \partial_{t_3} u = 0.
\end{array}$$

cuja solução é dada por

$$\begin{array}{rcl}
a_1 & = & (\frac{1}{4}\partial_x^2 u + \frac{3}{4} u\bar{\psi}\partial_x\bar{\psi} - \frac{1}{2}u^3), \\
b_1 & = & (\frac{1}{4}\partial_x^2 \bar{\psi} - \frac{1}{2}u^2\bar{\psi}), \\
b_2 & = & \frac{1}{4}(u\partial_x\bar{\psi} - \bar{\psi}\partial_x u), \\
c_1 & = & \frac{1}{2}\partial_x u, \\
c_2 & = & \frac{1}{2}(\bar{\psi}\partial_x\bar{\psi} - u^2), \\
c_3 & = & -\frac{1}{2}\bar{\psi}\partial\bar{\psi}, \\
d_1 & = & -\frac{1}{2}\partial_x\bar{\psi}, \\
d_2 & = & -\frac{1}{2}u\bar{\psi}, \\
e_1 & = & u, \\
f_1 & = & \bar{\psi}, \\
f_2 & = & 0, \\
b_{-1} & = & c_{-3}\psi\cosh(\phi), \\
b_{-2} & = & -c_{-3}\psi\sinh(\phi), \\
c_{-1} & = & c_{-3}\sinh(2\phi), \\
c_{-2} & = & c_{-3}\cosh(2\phi),
\end{array} \tag{4.13}$$

o que implica a equação de movimento

$$\begin{array}{rcl}
-\bar{\psi}_{t_3} + \frac{a_3}{4}\left[\partial_x^3\bar{\psi} - 3u\partial_x(u\bar{\psi})\right] + 2c_{-3}\psi\cosh(\phi) & = & 0, \\
-u_{t_3} + \frac{a_3}{4}\left[\partial_x^3 u - 6u^2\partial_x u + 3\bar{\psi}\partial_x(u\partial_x\bar{\psi})\right] - 2c_{-3}\left[\sinh(2\phi) + \bar{\psi}\psi\sinh(\phi)\right] & = & 0, \\
\psi_x - 2\bar{\psi}\cosh(\phi) & = & 0,
\end{array} \tag{4.14}$$

onde $u = -\partial_x \phi$. A equação (4.14) é a versão supersimétrica da equação mKdV/sinh -Gordon. Fazendo $a_3 = 0$ obtemos a equação super sinh-Gordon e fazendo $c_{-3} = 0$ obtemos a equação super mKdV [1].

Para obtermos o modelo supersimétrico associado ao tempo t_5 da hierarquia supersimétrica mKdV/sinh-Gordon consideraremos o operador graduado A_{t_5} como:

$$\begin{aligned} A_{t_5} &= D_5^{(5)} + D_5^{(9/2)} + D_5^{(4)} + D_5^{(7/2)} + D_5^{(3)} + D_5^{(5/2)} + D_5^{(2)} + D_5^{(3/2)} + D_5^{(1)} + D_5^{(1/2)} \\ &+ D_5^{(0)} + D_5^{(-1/2)} + D_5^{(-1)}, \end{aligned}$$

onde,

$$\begin{aligned} A_0 &= u\, M_2^{(0)}, \\ A_{1/2} &= \bar{\psi}\, G_1^{(1/2)}, \end{aligned} \tag{4.15}$$

e $u = u(x, t_5)$ é um campo bosônico e $\bar{\psi} = \bar{\psi}(x, t_5)$ é um campo fermiônico.

A equação de curvatura nula se decompõe grau a grau da forma:

$$\begin{aligned} \left[E, D_5^{(5)}\right] &= 0, \\ \left[E, D_5^{(9/2)}\right] + \left[A_{1/2}, D_5^{(5)}\right] &= 0, \\ \partial_x D_5^{(5)} + \left[A_0, D_5^{(5)}\right] + \left[E, D_5^{(4)}\right] + \left[A_{1/2}, D_5^{(9/2)}\right] &= 0, \\ \partial_x D_5^{(9/2)} + \left[A_0, D_5^{(9/2)}\right] + \left[E, D_5^{(7/2)}\right] + \left[A_{1/2}, D_5^{(4)}\right] &= 0, \\ \partial_x D_5^{(4)} + \left[A_0, D_5^{(4)}\right] + \left[E, D_5^{(3)}\right] + \left[A_{1/2}, D_5^{(7/2)}\right] &= 0, \\ \partial_x D_5^{(7/2)} + \left[A_0, D_5^{(7/2)}\right] + \left[E, D_5^{(5/2)}\right] + \left[A_{1/2}, D_5^{(3)}\right] &= 0, \\ \partial_x D_5^{(3)} + \left[A_0, D_5^{(3)}\right] + \left[E, D_5^{(2)}\right] + \left[A_{1/2}, D_5^{(5/2)}\right] &= 0, \\ \partial_x D_5^{(5/2)} + \left[A_0, D_5^{(5/2)}\right] + \left[E, D_5^{(3/2)}\right] + \left[A_{1/2}, D_5^{(2)}\right] &= 0, \\ \partial_x D_5^{(2)} + \left[A_0, D_5^{(2)}\right] + \left[E, D_5^{(1)}\right] + \left[A_{1/2}, D_5^{(3/2)}\right] &= 0, \\ \partial_x D_5^{(3/2)} + \left[A_0, D_5^{(3/2)}\right] + \left[E, D_5^{(1/2)}\right] + \left[A_{1/2}, D_5^{(1)}\right] &= 0, \\ \partial_x D_5^{(1)} + \left[A_0, D_5^{(1)}\right] + \left[E, D_5^{(0)}\right] + \left[A_{1/2}, D_5^{(1/2)}\right] &= 0, \\ \partial_x D_5^{(1/2)} + \left[A_0, D_5^{(1/2)}\right] + \left[E, D_5^{(-1/2)}\right] + \left[A_{1/2}, D_5^{(0)}\right] - \partial_{t_5} A_{1/2} &= 0, \\ \partial_x D_5^{(0)} + \left[A_0, D_5^{(0)}\right] + \left[E, D_5^{(-1)}\right] + \left[A_{1/2}, D_5^{(-1/2)}\right] - \partial_{t_5} A_0 &= 0, \\ \partial_x D_5^{(-1/2)} + \left[A_0, D_5^{(-1/2)}\right] + \left[A_{1/2}, D_5^{(-1)}\right] &= 0, \\ \partial_x D_5^{(-1)} + \left[A_0, D_5^{(-1)}\right] &= 0. \end{aligned} \tag{4.16}$$

com

$$\begin{aligned}
D_5^{(5)} &= a_5\left(K_1^{(5)}+K_2^{(5)}\right),\\
D_5^{(9/2)} &= e_{9/2}G_1^{(9/2)}+f_{9/2}F_2^{(9/2)},\\
D_5^{(4)} &= c_4M_2^{(4)},\\
D_5^{(7/2)} &= e_{7/2}G_2^{(7/2)}+f_{7/2}F_1^{(7/2)},\\
D_5^{(3)} &= a_3K_1^{(3)}+b_3K_2^{(3)}+d_3M_1^{(3)},\\
D_5^{(5/2)} &= e_{5/2}G_1^{(5/2)}+f_{5/2}F_2^{(5/2)},\\
D_5^{(2)} &= c_2M_2^{(2)},\\
D_5^{(3/2)} &= e_{3/2}G_2^{(3/2)}+f_{3/2}F_1^{(3/2)},\\
D_5^{(1)} &= a_1K_1^{(1)}+b_1K_2^{(1)}+d_1M_1^{(1)},\\
D_5^{(1/2)} &= e_{1/2}G_1^{(1/2)}+f_{1/2}F_2^{(1/2)},\\
D_5^{(0)} &= c_0M_2^{(0)},\\
D_5^{(-1/2)} &= e_{-1/2}G_2^{(-1/2)}+f_{-1/2}F_1^{(-1/2)},\\
D_5^{(-1)} &= a_{-1}K_1^{(-1)}+b_{-1}K_2^{(-1)}+d_{-1}M_1^{(-1)}. \qquad (4.17)
\end{aligned}$$

Substituindo as relações (4.17) nas equações (4.16) obtemos um conjunto de equações

$$\begin{aligned}
2e_{9/2}-2\bar\psi a_5 &= 0, \quad 2a_5u-2c_4+2\bar\psi f_{9/2}=0,\\
\bar\psi e_{9/2} &= 0, \quad \partial_x e_{9/2}+2e_{7/2}-uf_{9/2}=0,\\
\partial_x f_{9/2}-ue_{9/2}+\bar\psi c_4 &= 0, \quad \partial_x c_4-2d_3+2\bar\psi f_{7/2}=0,\\
\partial_x e_{7/2}+2e_{5/2}-uf_{7/2}-\bar\psi a_3-\bar\psi b_3 &= 0, \quad \partial_x f_{7/2}-ue_{7/2}+\bar\psi d_3=0,\\
\partial_x a_3+2ud_3-2\bar\psi e_{5/2} &= 0, \quad \partial_x d_3+2ua_3-2c_2+2\bar\psi f_{5/2}=0,\\
\partial_x b_3+2\bar\psi e_{5/2} &= 0, \quad \partial_x e_{5/2}-uf_{5/2}+2e_{3/2}=0,\\
\partial_x f_{5/2}-ue_{5/2}+\bar\psi c_2 &= 0, \quad \partial_x c_2-2d_1+2\bar\psi f_{3/2}=0,\\
\partial_x e_{3/2}+2e_{1/2}-uf_{3/2}-\bar\psi a_1-\bar\psi b_1 &= 0, \quad \partial_x f_{3/2}-ue_{3/2}+\bar\psi d_1=0,\\
\partial_x a_1+2ud_1-2\bar\psi e_{1/2} &= 0, \quad \partial_x d_1+2ua_1-2c_0+2\bar\psi f_{1/2}=0,\\
\partial_x b_1+2\bar\psi e_{1/2} &= 0, \quad \partial_x f_{1/2}-ue_{1/2}+\bar\psi c_0=0,\\
\partial_x e_{1/2}-uf_{1/2}+2e_{-1/2}-\partial_{t_5}\bar\psi &= 0, \quad \partial_x c_0-2d_{-1}+2\bar\psi f_{-1/2}-\partial_{t_5}u=0.
\end{aligned}$$

cuja solução é dada por

$$
\begin{aligned}
a_5 &= \text{constante},\\
c_4 &= a_5 u,\\
e_{9/2} &= a_5\bar{\psi},\\
f_{9/2} &= 0,\\
d_3 &= \frac{a_5}{2}\partial_x u,\\
a_3 &= \frac{a_5}{2}(\bar{\psi}\partial_x\bar{\psi} - u^2),\\
e_{7/2} &= -\frac{a_5}{2}\partial_x\bar{\psi},\\
f_{7/2} &= -\frac{a_5}{2}u\bar{\psi},\\
e_{5/2} &= -\frac{a_5}{4}(-\partial_x^2\bar{\psi} + 2u^2\bar{\psi}),\\
f_{5/2} &= \frac{a_5}{4}(u\partial_x\bar{\psi} - \bar{\psi}\partial_x u),\\
b_3 &= -\frac{a_5}{2}\bar{\psi}\partial\bar{\psi},\\
c_2 &= \frac{a_5}{4}(u_{xx} + 3u\bar{\psi}\bar{\psi}_x - 2u^3),\\
e_{3/2} &= \frac{a_5}{8}\left(3uu_x\bar{\psi} + 3u^2\bar{\psi}_x - \bar{\psi}_{xxx}\right),\\
f_{3/2} &= \frac{a_5}{8}\left(3u^3\bar{\psi} - u\bar{\psi}_{xx} - \bar{\psi}u_{xx} + u_x\bar{\psi}_x\right)\\
d_1 &= \frac{a_5}{8}\left(u_{xxx} + 4u_x\bar{\psi}\bar{\psi}_x + 2u\bar{\psi}\bar{\psi}_{xx} - 6u^2u_x\right),\\
a_1 &= -\frac{a_5}{8}\left(8u^2\bar{\psi}\bar{\psi}_x - 3u^4 - u_x^2 + 2uu_{xx} - \bar{\psi}\bar{\psi}_{xxx} + \bar{\psi}_x\bar{\psi}_{xx}\right),\\
b_1 &= \frac{a_5}{8}\left(4u^2\bar{\psi}\bar{\psi}_x - \bar{\psi}\bar{\psi}_{xxx} + \bar{\psi}_x\bar{\psi}_{xx}\right),\\
e_{1/2} &= -\frac{a_5}{16}\left(2u_x^2\bar{\psi} + 6\bar{\psi}uu_{xx} + 4u^2\bar{\psi}_{xx} + 8uu_x\bar{\psi}_x - 6u^4\bar{\psi} - \bar{\psi}_{xxxx}\right),\\
f_{1/2} &= -\frac{a_5}{16}\left(4u^3\bar{\psi}_x - 4u^2u_x\bar{\psi} + u_{xxx}\bar{\psi} - u\bar{\psi}_{xxx} - u_{xx}\bar{\psi}_x + u_x\bar{\psi}_{xx}\right),\\
c_0 &= \frac{a_5}{16}\left(u_{xxxx} + 5u_{xx}\bar{\psi}\bar{\psi}_x + 5u_x\bar{\psi}\bar{\psi}_{xx} + 5u\bar{\psi}\bar{\psi}_{xxx} - 10uu_x^2 - 10u^2u_{xx} - 20u^3\bar{\psi}\bar{\psi}_x + 6u^5\right),\\
a_{-1} &= \text{constante},\\
e_{-1/2} &= c_{-3}\psi\cosh(\phi),\\
f_{-1/2} &= -c_{-3}\psi\sinh(\phi),\\
d_{-1} &= c_{-3}\sinh(2\phi),\\
c_{-2} &= c_{-3}\cosh(2\phi),
\end{aligned}
\tag{4.18}
$$

o que implica a equação de movimento

$$
\begin{aligned}
\frac{a_5}{16}\left(u_{xxxxx} + 5u_{xxx}\bar{\psi}\bar{\psi}_x + 10u_{xx}\bar{\psi}\bar{\psi}_{xx} + 5u_x\bar{\psi}_x\bar{\psi}_{xx} + 10u_x\bar{\psi}\bar{\psi}_{xxx} + 5u\bar{\psi}_x\bar{\psi}_{xxx}\right. & \\
\left. +5u\bar{\psi}\bar{\psi}_{xxxx} - 10u_x^3 - 40uu_xu_{xx} - 10u^2u_{xxx} - 60u^2u_x\bar{\psi}\bar{\psi}_x - 20u^3\bar{\psi}\bar{\psi}_{xx} + 30u^4u_x\right) & \\
-2c_{-3}\left[\sinh(2\phi) + \bar{\psi}\psi\sinh(\phi)\right] - u_{t_5} &= 0, \\
-\frac{a_5}{16}\left(4u_xu_{xx} + 10u_x^2\bar{\psi}_x + 15uu_{xx}\bar{\psi}_x + 6\bar{\psi}u_xu_{xx} + 5\bar{\psi}uu_{xxx} + 15uu_x\bar{\psi}_{xx} + 4u^2\bar{\psi}_{xx}\right. & \\
\left. -20u^3u_x\bar{\psi} - 6u^4\bar{\psi}_x - \bar{\psi}_{xxxxx} - 4u^4\bar{\psi}_x + u^2\bar{\psi}_{xxx}\right) + 2c_{-3}\psi\cosh(\phi) - \bar{\psi}_{t_5} &= 0, \\
\psi_x - 2\bar{\psi}\cosh(\phi) &= 0, \qquad (4.19)
\end{aligned}
$$

onde $u = -\partial_x\phi$. Esta é a versão supersimétrica da equação (3.11).

Verificamos que as equações (4.14) e (4.19) são invariantes pela transformação de supersimetria:

$$
\begin{aligned}
\delta\bar{\psi} &= \epsilon u, \\
\delta u &= \epsilon\partial_x\bar{\psi}, \qquad (4.20)
\end{aligned}
$$

onde ϵ é um parâmetro fermiônico.

4.1 Sólitons

Os operadores de vértice são dados por,

$$
F_-(\gamma) = \sum_{n=-\infty}^{+\infty}\gamma^{-(2n+1)}M_1^{(2n+1)} + \gamma^{-2n}(M_2^{(2n)} - \frac{1}{2}\delta_{n,0}\hat{c}), \qquad (4.21)
$$

$$
F_+(\gamma) = \sum_{n=-\infty}^{+\infty}\gamma^{-2n}G_1^{(2n+1/2)} + \gamma^{-(2n+1)}G_2^{(2n+3/2)}. \qquad (4.22)
$$

A função $F_-(\gamma)$ é uma função de vértice bosônica, pois é escrita em termos de geradores bosônicos, enquanto que $F_+(\gamma)$ é uma função de vértice fermiônica pois é escrita em termos de geradores fermiônicos. Elas satisfazem a seguinte relação:

$$
[\epsilon_+, F_\pm(\gamma)] = \pm 2\gamma^{(1)}F_\pm(\gamma),
$$

$$
[\epsilon_-, F_\pm(\gamma)] = \pm 2\left(a_n\gamma^{(n)} + c_{-3}\gamma^{(-1)}\right)F_\pm(\gamma),
$$

onde

$$
\epsilon_+ = K_1^{(1)} + K_2^{(1)}. \qquad (4.23)
$$

$$\epsilon_- = a_n\left(K_1^{(n)} + K_2^{(n)}\right) + c_{-3}K_2^{(-1)}. \tag{4.24}$$

Os campos u, $\bar{\psi}$ e η são dados por

$$u = -\left(\frac{\partial_x \tau_1}{\tau_1} - \frac{\partial_x \tau_0}{\tau_0}\right), \tag{4.25}$$

$$\bar{\psi} = \frac{\tau_3}{\tau_1} + \frac{\tau_2}{\tau_0}, \tag{4.26}$$

$$\eta = -\frac{\partial_x \tau_0}{\tau_0}, \tag{4.27}$$

e as funções τ por

$$\tau_0 = e^{\nu} = \langle \lambda_0 \mid T_{vac}\, g\, T_{vac}^{-1} \mid \lambda_0 \rangle, \tag{4.28}$$

$$\tau_1 = e^{\phi+\nu} = \langle \lambda_1 \mid T_{vac}\, g\, T_{vac}^{-1} \mid \lambda_1 \rangle. \tag{4.29}$$

$$\tau_2 = \frac{1}{2}(\bar{\psi} - \chi)e^{\nu} = \langle \lambda_0 \mid G_1^{(1/2)} T_{vac}\, g\, T_{vac}^{-1} \mid \lambda_0 \rangle, \tag{4.30}$$

$$\tau_3 = \frac{1}{2}(\bar{\psi} + \chi)e^{\phi+\nu} = \langle \lambda_1 \mid G_1^{(1/2)} T_{vac}\, g\, T_{vac}^{-1} \mid \lambda_1 \rangle. \tag{4.31}$$

Para a solução 1-vértice bosônico + 1-vértice fermiônico usaremos:

$$g = e^{b_1 F_-(\gamma_1)} e^{c_1 F_+(\gamma_2)}. \tag{4.32}$$

Portanto,

$$T_{vac} F_{\pm}(\gamma) T_{vac}^{-1} = e^{\mp\left(2\gamma x + 2\left(a_n \gamma^{(n)} + c_{-3}\gamma^{(-1)}\right)t_n\right)} F_{\pm}(\gamma), \tag{4.33}$$

com

$$T_{vac} = \exp(-\epsilon_+ x - \epsilon_- t_n), \tag{4.34}$$

e,

$$\rho_i^{\pm} = e^{\pm\left(2\gamma_i x + 2\left(a_n \gamma_i^{(n)} + c_{-3}\gamma_i^{(-1)}\right)t_n\right)}. \tag{4.35}$$

Desta maneira

$$\begin{aligned}
\tau_0 &= e^{\nu} = 1 - \frac{1}{2} b_1 \rho_1^{+}, \\
\tau_1 &= e^{\phi+\nu} = 1 + \frac{1}{2} b_1 \rho_1^{+}, \\
\tau_2 &= \frac{1}{2}(\bar{\psi} - \chi)e^{\nu} = c_1 \rho_2^{-} \gamma_2 + b_1 c_1 \rho_1^{+} \rho_2^{-} \sigma_{1,2}, \\
\tau_3 &= \frac{1}{2}(\bar{\psi} + \chi)e^{\phi+\nu} = c_1 \rho_2^{-} \gamma_2 - b_1 c_1 \rho_1^{+} \rho_2^{-} \sigma_{1,2},
\end{aligned} \tag{4.36}$$

onde

$$\sigma_{1,2} = \frac{\gamma_2}{2}\frac{(\gamma_1 + \gamma_2)}{(\gamma_1 - \gamma_2)}. \tag{4.37}$$

A solução 1-sóliton da hierarquia super mKdV/sinh-Gordon é dada por:

$$\begin{aligned}
\phi &= \ln\left(\frac{1+\frac{1}{2}b_1\rho_1^+}{1-\frac{1}{2}b_1\rho_1^+}\right), \\
\bar{\psi} &= \frac{c_1\rho_2^-\gamma_2 - b_1c_1\rho_1^+\rho_2^-\sigma_{1,2}}{1+\frac{1}{2}b_1\rho_1^+} + \frac{c_1\rho_2^-\gamma_2 + b_1c_1\rho_1^+\rho_2^-\sigma_{1,2}}{1-\frac{1}{2}b_1\rho_1^+}, \\
\nu &= \ln\left(1-\frac{1}{2}b_1\rho_1^+\right),
\end{aligned} \tag{4.38}$$

onde

$$\rho_i^{\pm} = e^{\pm\left(2\gamma_i x + 2\left(a_5\gamma_i^{(5)} + c_{-3}\gamma_i^{(-1)}\right)t_5\right)}. \tag{4.39}$$

para $n = 5$ e

$$\rho_i^{\pm} = e^{\pm\left(2\gamma_i x + 2\left(a_3\gamma_i^{(3)} + c_{-3}\gamma_i^{(-1)}\right)t_3\right)}. \tag{4.40}$$

para $n = 3$.

Para a solução 2-vértices bosônicos + 2-vértices fermiônicos usaremos:

$$g = e^{b_1F_-(\gamma_1)}e^{b_2F_-(\gamma_2)}e^{c_1F_+(\gamma_3)}e^{c_2F_+(\gamma_4)}. \tag{4.41}$$

E as funções τ ficam

$$\begin{aligned}
\tau_0 &= e^{\nu} \\
&= 1 - \frac{1}{2}b_1\rho_1^+ - \frac{1}{2}b_2\rho_2^+ + b_1b_2\rho_1^+\rho_2^+\alpha_{1,2} \\
&+ c_1c_2\rho_3^-\rho_4^-\left(\beta_{3,4} - b_1\rho_1^+\delta_{1,3,4} - b_2\rho_2^+\delta_{2,3,4} + b_1b_2\rho_1^+\rho_2^+\theta_{1,2,3,4}\right), \\
\tau_1 &= e^{\phi+\nu} \\
&= 1 + \frac{1}{2}b_1\rho_1^+ + \frac{1}{2}b_2\rho_2^+ + b_1b_2\rho_1^+\rho_2^+\alpha_{1,2} \\
&+ c_1c_2\rho_3^-\rho_4^-\left(\beta_{3,4} + b_1\rho_1^+\delta_{1,3,4} + b_2\rho_2^+\delta_{2,3,4} + b_1b_2\rho_1^+\rho_2^+\theta_{1,2,3,4}\right), \\
\tau_2 &= \frac{1}{2}(\bar{\psi} - \chi)e^{\nu} \\
&= c_1\rho_3^-\left(\gamma_3 + b_1\rho_1^+\sigma_{1,3} + b_2\rho_2^+\sigma_{2,3} + b_1b_2\rho_1^+\rho_2^+\lambda_{1,2,3}\right) \\
&+ c_2\rho_4^-\left(\gamma_4 + b_1\rho_1^+\sigma_{1,4} + b_2\rho_2^+\sigma_{2,4} + b_1b_2\rho_1^+\rho_2^+\lambda_{1,2,4}\right), \\
\tau_3 &= \frac{1}{2}(\bar{\psi} + \chi)e^{\phi+\nu} \\
&= c_1\rho_3^-\left(\gamma_3 - b_1\rho_1^+\sigma_{1,3} - b_2\rho_2^+\sigma_{2,3} + b_1b_2\rho_1^+\rho_2^+\lambda_{1,2,3}\right) \\
&+ c_2\rho_4^-\left(\gamma_4 - b_1\rho_1^+\sigma_{1,4} - b_2\rho_2^+\sigma_{2,4} + b_1b_2\rho_1^+\rho_2^+\lambda_{1,2,4}\right),
\end{aligned} \tag{4.42}$$

onde

$$\begin{aligned}
\alpha_{1,2} &= \frac{1}{4}\frac{(\gamma_1-\gamma_2)^2}{(\gamma_1+\gamma_2)^2}, \\
\beta_{3,4} &= \gamma_3\gamma_4\frac{(\gamma_3-\gamma_4)}{(\gamma_3+\gamma_4)^2}, \\
\delta_{j,3,4} &= \frac{\gamma_3\gamma_4}{2}\frac{(\gamma_3-\gamma_4)}{(\gamma_3+\gamma_4)^2}\frac{(\gamma_j+\gamma_3)}{(\gamma_j-\gamma_3)}\frac{(\gamma_j+\gamma_4)}{(\gamma_j-\gamma_4)} \qquad (j=1,2), \\
\sigma_{j,k} &= \frac{\gamma_k}{2}\frac{(\gamma_j+\gamma_k)}{(\gamma_j-\gamma_k)} \qquad (j=1,2) \qquad (k=3,4), \\
\lambda_{1,2,j} &= \frac{\gamma_j}{4}\frac{(\gamma_1-\gamma_2)^2}{(\gamma_1+\gamma_2)^2}\frac{(\gamma_1+\gamma_j)}{(\gamma_1-\gamma_j)}\frac{(\gamma_2+\gamma_j)}{(\gamma_2-\gamma_j)}, \qquad (j=3,4), \\
\theta_{1,2,3,4} &= \frac{\gamma_3\gamma_4}{4}\frac{(\gamma_1-\gamma_2)^2}{(\gamma_1+\gamma_2)^2}\frac{(\gamma_1+\gamma_3)}{(\gamma_1-\gamma_3)}\frac{(\gamma_2+\gamma_3)}{(\gamma_2-\gamma_3)}\frac{(\gamma_3-\gamma_4)}{(\gamma_3+\gamma_4)^2}\frac{(\gamma_1+\gamma_4)}{(\gamma_1-\gamma_4)}\frac{(\gamma_2+\gamma_4)}{(\gamma_2-\gamma_4)}.
\end{aligned} \tag{4.43}$$

A solução 2-sólitons da hierarquia super mKdV/sinh-Gordon é dada por:

$$\begin{aligned}
\phi &= \ln\left(\frac{1+\frac{1}{2}b_1\rho_1^+ + \frac{1}{2}b_2\rho_2^+ + b_1b_2\rho_1^+\rho_2^+\alpha_{1,2}}{1-\frac{1}{2}b_1\rho_1^+ - \frac{1}{2}b_2\rho_2^+ + b_1b_2\rho_1^+\rho_2^+\alpha_{1,2}}\right) \\
&+ \frac{c_1c_2\rho_3^-\rho_4^-\left(\beta_{3,4}+b_1\rho_1^+\delta_{1,3,4}+b_2\rho_2^+\delta_{2,3,4}+b_1b_2\rho_1^+\rho_2^+\theta_{1,2,3,4}\right)}{1+\frac{1}{2}b_1\rho_1^+ + \frac{1}{2}b_2\rho_2^+ + b_1b_2\rho_1^+\rho_2^+\alpha_{1,2}} \\
&- \frac{c_1c_2\rho_3^-\rho_4^-\left(\beta_{3,4}-b_1\rho_1^+\delta_{1,3,4}-b_2\rho_2^+\delta_{2,3,4}+b_1b_2\rho_1^+\rho_2^+\theta_{1,2,3,4}\right)}{1-\frac{1}{2}b_1\rho_1^+ - \frac{1}{2}b_2\rho_2^+ + b_1b_2\rho_1^+\rho_2^+\alpha_{1,2}},
\end{aligned} \tag{4.44}$$

$$\begin{aligned}
\bar{\psi} &= \frac{c_1\rho_3^-\left(\gamma_3-b_1\rho_1^+\sigma_{1,3}-b_2\rho_2^+\sigma_{2,3}+b_1b_2\rho_1^+\rho_2^+\lambda_{1,2,3}\right)}{1+\frac{1}{2}b_1\rho_1^+ + \frac{1}{2}b_2\rho_2^+ + b_1b_2\rho_1^+\rho_2^+\alpha_{1,2}} \\
&+ \frac{c_2\rho_4^-\left(\gamma_4-b_1\rho_1^+\sigma_{1,4}-b_2\rho_2^+\sigma_{2,4}+b_1b_2\rho_1^+\rho_2^+\lambda_{1,2,4}\right)}{1+\frac{1}{2}b_1\rho_1^+ + \frac{1}{2}b_2\rho_2^+ + b_1b_2\rho_1^+\rho_2^+\alpha_{1,2}} \\
&+ \frac{c_1\rho_3^-\left(\gamma_3+b_1\rho_1^+\sigma_{1,3}+b_2\rho_2^+\sigma_{2,3}+b_1b_2\rho_1^+\rho_2^+\lambda_{1,2,3}\right)}{1-\frac{1}{2}b_1\rho_1^+ - \frac{1}{2}b_2\rho_2^+ + b_1b_2\rho_1^+\rho_2^+\alpha_{1,2}} \\
&+ \frac{c_2\rho_4^-\left(\gamma_4+b_1\rho_1^+\sigma_{1,4}+b_2\rho_2^+\sigma_{2,4}+b_1b_2\rho_1^+\rho_2^+\lambda_{1,2,4}\right)}{1-\frac{1}{2}b_1\rho_1^+ - \frac{1}{2}b_2\rho_2^+ + b_1b_2\rho_1^+\rho_2^+\alpha_{1,2}},
\end{aligned} \tag{4.45}$$

$$\begin{aligned}
\nu &= \ln\left(1-\frac{1}{2}b_1\rho_1^+ - \frac{1}{2}b_2\rho_2^+ + b_1b_2\rho_1^+\rho_2^+\alpha_{1,2}\right) \\
&+ \frac{c_1c_2\rho_3^-\rho_4^-\left(\beta_{3,4}-b_1\rho_1^+\delta_{1,3,4}-b_2\rho_2^+\delta_{2,3,4}+b_1b_2\rho_1^+\rho_2^+\theta_{1,2,3,4}\right)}{1-\frac{1}{2}b_1\rho_1^+ - \frac{1}{2}b_2\rho_2^+ + b_1b_2\rho_1^+\rho_2^+\alpha_{1,2}}.
\end{aligned} \tag{4.46}$$

onde

$$\rho_i^{\pm} = e^{\pm\left(2\gamma_i x + 2\left(a_5\gamma_i^{(5)} + c_{-3}\gamma_i^{(-1)}\right)t_5\right)}, \tag{4.47}$$

para $n = 5$ e

$$\rho_i^{\pm} = e^{\pm\left(2\gamma_i x + 2\left(a_3\gamma_i^{(3)} + c_{-3}\gamma_i^{(-1)}\right)t_3\right)}, \tag{4.48}$$

para $n = 3$.

5 Hierarquia NLS/Lund-Regge

Considere a equação de curvatura nula,

$$[\partial_x + A_x, \partial_{t_n} + A_{t_n}] = 0, \tag{5.1}$$

com os potenciais A_x e A_{t_n} da forma,

$$\begin{aligned} A_x &= E^{(1)} + A_0, \\ A_{t_n} &= D_n^{(n)} + D_n^{(n-1)} + \cdots + D_n^{(0)} + D_n^{(-1)}. \end{aligned} \tag{5.2}$$

Substituindo (5.2) em (5.1) e separando grau a grau obtemos

$$\begin{aligned} \left[E^{(1)}, D_n^{(n)}\right] &= 0, \\ \partial_x D_n^{(n)} + \left[A_0, D_n^{(n)}\right] + \left[E^{(1)}, D_n^{(n-1)}\right] &= 0, \\ \vdots &= \vdots \\ \partial_x D_n^{(1)} + \left[A_0, D_n^{(1)}\right] + \left[E^{(1)}, D_n^{(0)}\right] &= 0, \\ \partial_x D_n^{(0)} + \left[A_0, D_n^{(0)}\right] + \left[E^{(1)}, D_n^{(-1)}\right] - \partial_{t_n} A_0 &= 0, \\ \partial_x D_n^{(-1)} + \left[A_0, D_n^{(-1)}\right] &= 0. \end{aligned} \tag{5.3}$$

Para a hierarquia NLS/Lund-Regge considere a estrutura algébrica:

$$\begin{aligned} \hat{\mathcal{G}} &= \hat{sl}(2), \\ E^{(1)} &= H^{(1)}, \\ \mathcal{Q} &= d. \end{aligned} \tag{5.4}$$

O operador de graduação (5.4) e as relações de comutação da álgebra implicam,

$$\begin{aligned} \left[\mathcal{Q}, E_{\pm\alpha}^{(n)}\right] &= n E_{\pm\alpha}^{(n)}, \\ \left[\mathcal{Q}, H^{(n)}\right] &= n H^{(n)}, \end{aligned} \tag{5.5}$$

e a decomposição da álgebra

$$\hat{\mathcal{G}}^{(m)} = \left\{H^{(m)}, E_{\alpha}^{(m)}, E_{-\alpha}^{(m)}\right\}. \tag{5.6}$$

O elemento semi-simples $E \equiv E^{(1)}$ decompõe a álgebra em

$$\begin{aligned} \mathcal{K}(E) &= \left\{ H^{(m)} \right\}, \\ \mathcal{M}(E) &= \left\{ E_{\alpha}^{(m)}, E_{-\alpha}^{(m)} \right\}. \end{aligned} \tag{5.7}$$

Das relações (5.5) e (5.7) obtemos,

$$A_0 = q(x,t_n)E_{\alpha}^{(0)} + r(x,t_n)E_{-\alpha}^{(0)}. \tag{5.8}$$

Para construirmos a equação NLS/Lund-Regge considere $n = 2$ na relação (5.2). Os operadores graduados $D_2^{(i)}$ podem ser escritos em suas componentes núcleo e imagem como

$$\begin{aligned} D_2^{(2)} &= a_2 E_{\alpha}^{(2)} + b_2 E_{-\alpha}^{(2)} + c_2 H^{(2)}, \\ D_2^{(1)} &= a_1 E_{\alpha}^{(1)} + b_1 E_{-\alpha}^{(1)} + c_1 H^{(1)}, \\ D_2^{(0)} &= a_0 E_{\alpha}^{(0)} + b_0 E_{-\alpha}^{(0)} + c_0 H^{(0)}, \\ D_2^{(-1)} &= a_{-1} E_{\alpha}^{(-1)} + b_{-1} E_{-\alpha}^{(-1)} + c_{-1} H^{(-1)}. \end{aligned} \tag{5.9}$$

Substituindo (5.9) em (5.3) obtemos o conjunto de equações:

$$\begin{aligned} c_2 &= \text{constante}, & \mathrm{a}_2 = \mathrm{b}_2 = 0, \\ a_1 - c_2 q &= 0, & b_1 - c_2 r = 0, \\ \partial_x a_1 - 2c_1 q + 2a_0 &= 0, & \partial_x b_1 + 2c_1 r - 2b_0 = 0, \\ \partial_x c_1 + q b_1 - r a_1 &= 0, & \partial_x c_0 + q b_0 - r a_0 = 0, \end{aligned} \tag{5.10}$$

cuja solução é dada por

$$\begin{aligned} c_1 &= 0, \\ c_2 &= \text{constante}, \\ a_2 &= b_2 = 0, \\ a_1 &= c_2 q, \\ b_1 &= c_2 r, \\ a_0 &= -\frac{1}{2} c_2 q_x, \\ b_0 &= \frac{1}{2} c_2 r_x, \\ c_0 &= -\frac{1}{2} c_2 r q. \end{aligned} \tag{5.11}$$

Para os coeficientes associados aos operadores de grau -1 as equações resultantes são:

$$\partial_x a_{-1} - 2qc_{-1} = 0, \tag{5.12}$$

$$\partial_x b_{-1} + 2rc_{-1} = 0, \tag{5.13}$$

$$\partial_x c_{-1} + qb_{-1} - ra_{-1} = 0. \tag{5.14}$$

Os coeficientes a_{-1}, b_{-1} e c_{-1} que resolvem a equação (5.14) podem ser encontrados através das relações

$$A_0 = -\partial_x B\, B^{-1}, \tag{5.15}$$

$$D_2^{(-1)} = \eta B E^{(-1)} B^{-1}, \tag{5.16}$$

onde $E^{(-1)} = H^{(-1)}$. Para tal, o elemento de grupo B é definido como

$$B = e^{\chi E_{-\alpha}^{(0)}} e^{\phi H^{(0)}} e^{\psi E_{\alpha}^{(0)}}, \tag{5.17}$$

onde χ, ϕ, ψ são os campos físicos relativísticos da hierarquia AKNS. Usando a relação

$$e^L T e^{-L} = T + [L, T] + \frac{1}{2!}[L, [L, T]] + \frac{1}{3!}[L, [L, [L, T]]] + \cdots \tag{5.18}$$

com L e T pertencentes a álgebra, podemos calcular

$$\begin{aligned} A_0 &= -\partial_x B B^{-1} \\ &= -e^{2\phi}\partial_x\psi E_{\alpha}^{(0)} + \left(e^{2\phi}\chi^2\partial_x\psi - 2\chi\partial_x\phi - \partial_x\chi\right) E_{-\alpha}^{(0)} \\ &+ \left(e^{2\phi}\chi\partial_x\psi - \partial_x\phi\right) H^{(0)}, \end{aligned} \tag{5.19}$$

$$\begin{aligned} D_2^{(-1)} &= \eta B E^{(-1)} B^{-1} \\ &= -2\eta\psi e^{2\phi} E_{\alpha}^{(-1)} + 2\eta\left(\chi + \psi\chi^2 e^{2\phi}\right) E_{-\alpha}^{(-1)} + \eta\left(1 + 2\psi\chi e^{2\phi}\right) H^{(-1)}, \end{aligned} \tag{5.20}$$

$$\begin{aligned} B^{-1}\partial_t B &= \left(-2\psi^2\partial_t\chi e^{2\phi} + 2\psi\partial_t\phi + \partial_t\psi\right) E_{\alpha}^{(0)} \\ &+ \partial_t\chi e^{2\phi} E_{-\alpha}^{(0)} + \left(\partial_t\phi - \psi\partial_t\chi e^{2\phi}\right) H^{(0)}. \end{aligned} \tag{5.21}$$

Então, definindo novas variáveis:

$$\psi = \tilde{\psi} e^{-\phi}, \quad \text{e} \quad \chi = \tilde{\chi} \mathrm{e}^{-\phi}, \tag{5.22}$$

então comparando (5.9) com (5.20) obtemos os coeficientes associados aos operadores de grau -1

$$\begin{aligned}
a_{-1} &= -2\eta\tilde{\psi}e^{\phi}, \\
b_{-1} &= 2\eta\tilde{\chi}e^{-\phi}\left(1+\tilde{\psi}\tilde{\chi}\right), \\
c_{-1} &= \eta\left(1+2\tilde{\psi}\tilde{\chi}\right),
\end{aligned} \tag{5.23}$$

onde $\Delta = 1+\tilde{\psi}\tilde{\chi}$. Desta forma, a equação NLS / Lund-Regge é dada por [19]

$$q_t + \frac{c_2}{2}q_{xx} - c_2 q^2 r - 2\eta\tilde{\psi}e^{\phi} = 0, \tag{5.24}$$

$$r_t - \frac{c_2}{2}r_{xx} + c_2 r^2 q + 4\eta\tilde{\chi}e^{-\phi}\left(1+\tilde{\psi}\tilde{\chi}\right) = 0. \tag{5.25}$$

Fazendo $\eta = 0$ obtemos a equação Schrodinger não-linear (NLS) nas variáveis q e r e para $c_2 = 0$ obtemos a equação Lund-Regge [8]:

$$\partial_t\left(\frac{\partial_x\tilde{\psi}}{\Delta}\right) + \tilde{\psi}\frac{\partial_t\tilde{\chi}\partial_x\tilde{\psi}}{\Delta^2} + 4\eta\tilde{\psi} = 0, \tag{5.26}$$

$$\partial_x\left(\frac{\partial_t\tilde{\chi}}{\Delta}\right) + \tilde{\chi}\frac{\partial_t\tilde{\chi}\partial_x\tilde{\psi}}{\Delta^2} + 4\eta\tilde{\chi} = 0. \tag{5.27}$$

onde usamos o resultado $\partial_t\phi = \frac{\tilde{\psi}\partial_t\tilde{\chi}}{\Delta}$ que resulta de $B^{-1}\partial_t B$ em termos dos campos transformados (5.22).

Comparando (5.19) e (5.8) obtemos as relações entre os campos NLS e Lund-Regge:

$$q = -\frac{e^{\phi}\partial_x\tilde{\psi}}{\Delta}, \tag{5.28}$$

$$r = -e^{-\phi}\partial_x\tilde{\chi}, \tag{5.29}$$

$$\partial_x\phi = \frac{\tilde{\chi}\partial_x\tilde{\psi}}{\Delta}. \tag{5.30}$$

Tomando operadores graduados de ordem mais alta, obtemos outros modelos da hierarquia. Por exemplo, para $n = 3$ temos

$$\begin{aligned}
D_3^{(3)} &= a_3E_{\alpha}^{(3)} + b_3E_{-\alpha}^{(3)} + c_3H^{(3)}, \\
D_3^{(2)} &= a_2E_{\alpha}^{(2)} + b_2E_{-\alpha}^{(2)} + c_2H^{(2)}, \\
D_3^{(1)} &= a_1E_{\alpha}^{(1)} + b_1E_{-\alpha}^{(1)} + c_1H^{(1)}, \\
D_3^{(0)} &= a_0E_{\alpha}^{(0)} + b_0E_{-\alpha}^{(0)} + c_0H^{(0)}, \\
D_3^{(-1)} &= a_{-1}E_{\alpha}^{(-1)} + b_{-1}E_{-\alpha}^{(-1)} + c_{-1}H^{(-1)}.
\end{aligned} \tag{5.31}$$

Substituindo (5.31) em (5.3) obtemos o conjunto de equações:

$$\begin{aligned}
c_3 &= \text{constante}, & a_3 = b_3 = 0, \\
a_2 - c_3 q &= 0, & b_2 - c_3 r = 0, \\
\partial_x a_2 - 2c_2 q + 2a_1 &= 0, & \partial_x b_2 + 2c_2 r - 2b_1 = 0, \\
\partial_x c_2 + q b_2 - r a_2 &= 0, & \\
\partial_x a_1 - 2c_1 q + 2a_0 &= 0, & \partial_x b_1 + 2c_1 r - 2b_0 = 0, \\
\partial_x c_1 + q b_1 - r a_1 &= 0, & \partial_x c_0 + q b_0 - r a_0 = 0,
\end{aligned} \tag{5.32}$$

cuja solução é dada por

$$\begin{aligned}
c_3 &= \text{constante}, \\
c_2 &= 0, \\
a_3 &= b_3 = 0, \\
a_1 &= c_2 q - \frac{1}{2} c_3 q_x, \\
b_1 &= c_2 r + \frac{1}{2} c_3 r_x, \\
c_1 &= -\frac{1}{2} c_3 q r, \\
a_0 &= -\frac{1}{2} c_3 q^2 r - \frac{1}{2} c_2 q_x + \frac{1}{4} c_3 q_{xx}, \\
b_0 &= -\frac{1}{2} c_3 q r^2 + \frac{1}{2} c_2 r_x + \frac{1}{4} c_3 r_{xx}, \\
c_0 &= -\frac{1}{2} c_2 r q - \frac{1}{4} c_3 \left(q r_x - r q_x \right), \\
a_{-1} &= -2\eta \tilde{\psi} e^{\phi}, \\
b_{-1} &= 2\eta \tilde{\chi} e^{-\phi} \left(1 + \tilde{\psi}\tilde{\chi} \right), \\
c_{-1} &= \eta \left(1 + 2\tilde{\psi}\tilde{\chi} \right).
\end{aligned} \tag{5.33}$$

Portanto a equação de movimento pode ser escrita como

$$q_t + \frac{3}{2} c_3 q r q_x - \frac{1}{4} c_3 q_{xxx} - 2\eta \tilde{\psi} e^{\phi} = 0, \tag{5.34}$$

$$r_t + \frac{3}{2} c_3 q r r_x - \frac{1}{4} c_3 r_{xxx} + \eta 2 \tilde{\chi} e^{-\phi} \left(1 + \tilde{\psi}\tilde{\chi} \right) = 0, \tag{5.35}$$

com

$$q = -\frac{e^{\phi} \partial_x \tilde{\psi}}{\Delta}, \tag{5.36}$$

$$r = -e^{-\phi} \partial_x \tilde{\chi}, \tag{5.37}$$

$$\partial_x \phi = \frac{\tilde{\chi} \partial_x \tilde{\psi}}{\Delta}. \tag{5.38}$$

onde $\Delta = 1 + \tilde{\psi}\tilde{\chi}$.

5.1 Sólitons

O operador de vértice é dado por,

$$V_{\pm}(\gamma) = \sum_{x \in Z} E_{\pm\alpha}^{(n)} \gamma^{(-n)}, \tag{5.39}$$

e possui os autovalores,

$$\left[E^{(1)}, V_{\pm}(\gamma)\right] = \mp 2\gamma V_{\pm}(\gamma), \tag{5.40}$$

$$\left[E^{(n)}, V_{\pm}(\gamma)\right] = \mp 2\left(c_n \gamma^n + \frac{\eta}{\gamma}\right) V_{\pm}(\gamma), \tag{5.41}$$

onde

$$E^{(1)} = H^{(1)}, \tag{5.42}$$

$$E^{(n)} = c_n H^{(n)} + \eta H^{(-1)}. \tag{5.43}$$

As funções τ são dadas por,

$$\tau_i = < \lambda_i|(\Theta_-)^{-1}\Theta_+|\lambda_i >= \left\langle \lambda_i|T_{vac} g T_{vac}^{-1}|\lambda_i\right\rangle. \tag{5.44}$$

Escrevendo $B = e^{\chi E_{-\alpha}^{(0)}} e^{\phi H^{(0)} + \nu \hat{c}} e^{\psi E_{\alpha}^{(0)}}$ podemos reescrever as funções τ como,

$$\tau_0 = e^{\nu} = \left\langle \lambda_0|T_{vac} g T_{vac}^{-1}|\lambda_0\right\rangle, \tag{5.45}$$

$$\tau_1 = e^{\nu+\phi} = \left\langle \lambda_1|T_{vac} g T_{vac}^{-1}|\lambda_1\right\rangle, \tag{5.46}$$

$$\tau_2 = \psi e^{\nu+\phi} = \left\langle \lambda_1|T_{vac} g T_{vac}^{-1} E_{-\alpha}^{(0)}|\lambda_1\right\rangle, \tag{5.47}$$

$$\tau_3 = \chi e^{\nu+\phi} = \left\langle \lambda_1|E_{\alpha}^{(0)} T_{vac} g T_{vac}^{-1}|\lambda_1\right\rangle. \tag{5.48}$$

Para a solução 1-sóliton temos

$$g = e^{V_-(\gamma_1)} e^{V_+(\gamma_2)}, \tag{5.49}$$

logo,

$$\begin{aligned} T_{vac} g T_{vac}^{-1} &= e^{-E^{(1)}x - (c_n E^{(n)} + \eta E^{(-1)})t_n} \left(1 + V_-(\gamma_1)\right)\left(1 + V_+(\gamma_2)\right) e^{E^{(1)}x + (c_n E^{(n)} + \eta E^{(-1)})t_n}, \\ &= 1 + \rho_1^+ V_-(\gamma_1) + \rho_2^- V_+(\gamma_2) + \rho_1^+ \rho_2^- V_-(\gamma_1) V_+(\gamma_2), \end{aligned} \tag{5.50}$$

com

$$\rho_i^{\pm} = e^{\pm\left(2\gamma_i x + 2\left(c_n \gamma_i^{(n)} + \eta\, \gamma_i^{(-1)}\right) t_n\right)}. \tag{5.51}$$

Desta forma,

$$\tau_0 = e^{\nu} = 1 + \frac{\gamma_1\gamma_2}{(\gamma_1 - \gamma_2)^2}\rho_1^+\rho_2^-, \tag{5.52}$$

$$\tau_1 = e^{\nu+\phi} = 1 + \frac{\gamma_2^2}{(\gamma_1 - \gamma_2)^2}\rho_1^+\rho_2^-, \tag{5.53}$$

$$\tau_2 = \psi e^{\nu+\phi} = \rho_2^-, \tag{5.54}$$

$$\tau_3 = \chi e^{\nu+\phi} = \rho_1^+. \tag{5.55}$$

A solução 1-sóliton da hierarquia NLS/Lund-Regge é dada por

$$\phi = \ln\frac{\tau_1}{\tau_0} = \ln\left[\frac{1 + \frac{\xi_2^2}{(\xi_1-\xi_2)^2}\rho_1^+\rho_2^-}{1 + \frac{\xi_1\xi_2}{(\xi_1-\xi_2)^2}\rho_1^+\rho_2^-}\right], \tag{5.56}$$

$$\psi = \frac{\tau_2}{\tau_1} = \frac{\rho_2^-}{1 + \frac{\xi_2^2}{(\xi_1-\xi_2)^2}\rho_1^+\rho_2^-}, \tag{5.57}$$

$$\chi = \frac{\tau_3}{\tau_1} = \frac{\rho_1^+}{1 + \frac{\xi_2^2}{(\xi_1-\xi_2)^2}\rho_1^+\rho_2^-}, \tag{5.58}$$

onde

$$\rho_i^{\pm} = e^{\pm\left(2\gamma_i x + 2\left(c_3\gamma_i^{(3)} + \eta\,\gamma_i^{(-1)}\right)t_3\right)}, \tag{5.59}$$

para $n = 3$ e

$$\rho_i^{\pm} = e^{\pm\left(2\gamma_i x + 2\left(c_2\gamma_i^{(2)} + \eta\,\gamma_i^{(-1)}\right)t_2\right)}, \tag{5.60}$$

para $n = 2$.

6 Hierarquia super NLS/Lund-Regge

Considere a equação de curvatura nula,

$$[\partial_x + A_x, \partial_{t_n} + A_{t_n}] = 0, \tag{6.1}$$

com potenciais A_x e A_{t_n} da forma,

$$\begin{aligned} A_x &= E^{(1)} + A_0, \\ A_{t_n} &= D_n^{(n)} + D_n^{(n-1)} + \cdots + D_n^{(0)} + D_n^{(-1)}. \end{aligned} \tag{6.2}$$

Substituindo (6.2) em (6.1) e separando grau a grau obtemos

$$\begin{aligned} \left[E^{(1)}, D_n^{(n)}\right] &= 0, \\ \partial_x D_n^{(n)} + \left[A_0, D_n^{(n)}\right] + \left[E^{(1)}, D_n^{(n-1)}\right] &= 0, \\ \vdots &= \vdots \\ \partial_x D_n^{(1)} + \left[A_0, D_n^{(1)}\right] + \left[E^{(1)}, D_n^{(0)}\right] &= 0, \\ \partial_x D_n^{(0)} + \left[A_0, D_n^{(0)}\right] + \left[E^{(1)}, D_{-m}^{(-1)}\right] - \partial_{t_n} A_0 &= 0, \\ \partial_x D_n^{(-1)} + \left[A_0, D_n^{(-1)}\right] &= 0. \end{aligned} \tag{6.3}$$

Para a hierarquia super NLS/Lund-Regge considere a estrutura algébrica:

$$\begin{aligned} \hat{\mathcal{G}} &= \hat{sl}(2,1), \\ E^{(1)} &= (\alpha_1 + \alpha_2) \cdot H^{(1)}, \\ \mathcal{Q} &= d. \end{aligned} \tag{6.4}$$

O operador de graduação (6.4) e as relações de comutação da álgebra implicam,

$$\left[\mathcal{Q}, G^{(m)}\right] = mG^{(m)}, \tag{6.5}$$

onde $G^{(m)}$ é um elemento qualquer da álgebra. Portanto a álgebra é decomposta como

$$\hat{\mathcal{G}}^{(m)} = \left\{(\alpha_1 + \alpha_2) \cdot H^{(m)}, \alpha_2 \cdot H^{(m)}, E_{\pm\alpha_2}^{(m)}, E_{\pm(\alpha_1+\alpha_2)}^{(m)}, E_{\pm\alpha_1}^{(m)}\right\}. \tag{6.6}$$

O elemento semi-simples $E^{(1)}$ impõe outra decomposição na álgebra, isto é:

$$\begin{aligned}\mathcal{K}\,(E) &= \{H_1^{(n)}, H_2^{(n)}, E_{(\alpha_1+\alpha_2)}^{(n)}, E_{-(\alpha_1+\alpha_2)}^{(n)}\},\\ \mathcal{M}\,(E) &= \{E_{\alpha_1}^{(n)}, E_{-\alpha_1}^{(n)}, E_{-\alpha_2}^{(n)}, E_{\alpha_2}^{(n)}\}.\end{aligned} \tag{6.7}$$

Das relações (6.5) e (6.7) obtemos,

$$A_0 = b_1 E_{\alpha_1}^{(0)} + b_2 E_{-\alpha_1}^{(0)} + f_1 E_{\alpha_2}^{(0)} + f_2 E_{-\alpha_2}^{(0)}.$$

Para construirmos a versão supersimétrica da equação NLS/Lund-Regge considere $n = 2$ na relação (6.2). Os operadores graduados $D_2^{(i)}$ podem ser escritos em suas componentes núcleo e imagem como

$$\begin{aligned}D_2^{(2)} &= a_2\,(\alpha_1+\alpha_2)\cdot H^{(2)},\\ D_2^{(1)} &= a_1\,(\alpha_1+\alpha_2)\cdot H^{(1)} - c_1\alpha_2\cdot H^{(1)} + d_1 E_{(\alpha_1+\alpha_2)}^{(1)} + e_1 E_{-(\alpha_1+\alpha_2)}^{(1)}\\ &+ g_1 E_{\alpha_1}^{(1)} + m_1 E_{-\alpha_1}^{(1)} + n_1 E_{-\alpha_2}^{(1)} + o_1 E_{\alpha_2}^{(1)},\\ D_2^{(0)} &= a_0\,(\alpha_1+\alpha_2)\cdot H^{(0)} - c_0\alpha_2\cdot H^{(0)} + d_0 E_{(\alpha_1+\alpha_2)}^{(0)} + e_0 E_{-(\alpha_1+\alpha_2)}^{(0)}\\ &+ g_0 E_{\alpha_1}^{(0)} + m_0 E_{-\alpha_1}^{(0)} + n_0 E_{-\alpha_2}^{(0)} + o_0 E_{\alpha_2}^{(0)},\\ D_2^{(-1)} &= a_{-1}\,(\alpha_1+\alpha_2)\cdot H^{(-1)} - c_{-1}\alpha_2\cdot H^{(-1)} + d_{-1} E_{(\alpha_1+\alpha_2)}^{(-1)} + e_{-1} E_{-(\alpha_1+\alpha_2)}^{(-1)}\\ &+ g_{-1} E_{\alpha_1}^{(-1)} + m_{-1} E_{-\alpha_1}^{(-1)} + n_{-1} E_{-\alpha_2}^{(-1)} + o_{-1} E_{\alpha_2}^{(-1)}.\end{aligned} \tag{6.8}$$

Substituindo (6.8) em (6.3) obtemos o sistema de equações:

$$\begin{aligned}\partial_x a_2 &= 0, & g_1 - a_2 b_1 = 0,\\ a_2 b_2 - m_1 &= 0, & a_2 f_1 - o_1 = 0,\\ n_1 - a_2 f_2 &= 0, & g_0 + a_2\partial_x b_1 = 0,\\ m_0 - a_2\partial_x b_2 &= 0, & n_0 + a_2\partial_x f_2 = 0,\\ o_0 - a_2\partial_x b_1 &= 0, & \\ \partial_x d_0 + b_1 o_0 - f_1 g_0 &= 0, & \partial_x e_0 - b_2 n_0 + f_2 m_0 = 0,\\ \partial_x a_0 + \partial_x c_0 + 2b_1 m_0 - 2b_2 g_0 - f_1 n_0 - f_2 o_0 &= 0, & \partial_x a_0 - \partial_x c_0 + f_1 n_0 + f_2 o_0 = 0.\end{aligned}$$

cuja solução é dada por,

$$\begin{aligned}
a_2 &= \text{constante}, \\
g_1 &= a_2 b_1, \\
m_1 &= a_2 b_2, \\
o_1 &= a_2 f_1, \\
n_1 &= a_2 f_2, \\
a_1 &= 0, \\
c_1 &= 0, \\
d_1 &= 0, \\
e_1 &= 0, \\
g_0 &= -a_2 \partial_x b_1, \\
m_0 &= a_2 \partial_x b_2, \\
n_0 &= -a_2 \partial_x f_2, \\
o_0 &= a_2 \partial_x f_1, \\
d_0 &= -a_2 f_1 b_1, \\
e_0 &= -a_2 f_2 b_2, \\
a_0 &= -a_2 b_1 b_2, \\
c_0 &= -a_2 \left(b_1 b_2 + f_1 f_2\right). \qquad (6.9)
\end{aligned}$$

Análogo ao caso bosônico feito no capítulo anterior, iremos obter os coeficientes associados aos operadores de grau -1, dado em (6.8), através das relações:

$$A_0 = -\partial_x B \, B^{-1}, \qquad (6.10)$$

$$D_2^{(-1)} = \eta B E^{(-1)} B^{-1}, \qquad (6.11)$$

Desta forma, o elemento de grupo B é escrito como

$$B = e^{\tilde{\chi} E^{(0)}_{-\alpha_1}} e^{\tilde{f}_1 E^{(0)}_{-(\alpha_1+\alpha_2)}} e^{\tilde{f}_2 E^{(0)}_{\alpha_2}} e^{\phi_1(\alpha_1+\alpha_2)\cdot H^{(0)} - \phi_2 \alpha_2 \cdot H^{(0)} + \nu \hat{c}} e^{\tilde{g}_2 E^{(0)}_{-\alpha_2}} e^{\tilde{g}_1 E^{(0)}_{(\alpha_1+\alpha_2)}} e^{\tilde{\psi} E^{(0)}_{\alpha_1}}, \qquad (6.12)$$

onde $\tilde{\chi}, \tilde{\phi}, \tilde{\psi}, \cdots$ são os campos físicos relativísticos da hierarquia sAKNS. Usando a relação

$$e^L T e^{-L} = T + [L, T] + \frac{1}{2!}[L, [L, T]] + \frac{1}{3!}[L, [L, [L, T]]] + \cdots \qquad (6.13)$$

podemos calcular:

$$
\begin{aligned}
B^{-1}\partial_t B &= E^{(0)}_{\alpha_1}\Big[\partial_t\tilde{\psi}+\tilde{\psi}(\partial_t\phi_1+\partial_t\phi_2)+\tilde{\psi}\tilde{g}_2\partial_t\tilde{f}_2e^{\phi_1}-\tilde{\psi}^2(\partial_t\tilde{\chi}-\partial_t\tilde{f}_1\tilde{f}_2)e^{\phi_1+\phi_2}\\
&+\Big(\partial_t\tilde{g}_2+\partial_t\phi_1 g_2+\tilde{\psi}\partial_t\tilde{f}_1e^{\phi_2}-\tilde{\psi}(\partial_t\tilde{\chi}+\partial_t\tilde{f}_1\tilde{f}_2)\tilde{g}_2e^{\phi_1+\phi_2}\Big)\tilde{g}_1\Big]\\
&+E^{(0)}_{(\alpha_1+\alpha_2)}\Big[\partial_t\tilde{g}_1-\tilde{\psi}\partial_t\tilde{f}_2e^{\phi_1}+\Big(\partial_t\phi_2+\tilde{g}_2\partial_t\tilde{f}_2e^{\phi_1}-\tilde{\psi}(\partial_t\tilde{f}_1\tilde{f}_2+\partial_t\tilde{\chi})e^{\phi_1+\phi_2}\Big)\tilde{g}_1\Big]\\
&+E^{(0)}_{-\alpha_2}\Big[\partial_t\tilde{g}_2+\partial_t\phi_1\tilde{g}_2-\tilde{\psi}\Big(\partial_t\tilde{f}_1\tilde{f}_2+\partial_t\tilde{\chi}\Big)\tilde{g}_2e^{\phi_1+\phi_2}+\tilde{\psi}\partial_t\tilde{f}_1e^{\phi_2}\Big]\\
&+(\alpha_1+\alpha_2)\cdot H^{(0)}\Big[\partial_t\phi_1+\partial_t\tilde{f}_1\tilde{g}_1e^{\phi_2}+\Big(\partial_t\tilde{\chi}+\partial_t\tilde{f}_1\tilde{f}_2\Big)\tilde{g}_1\tilde{g}_2e^{\phi_1+\phi_2}-\tilde{\psi}\Big(\partial_t\tilde{\chi}+\partial_t\tilde{f}_1\tilde{f}_2\Big)e^{\phi_1+\phi_2}\Big]\\
&-\alpha_2\cdot H^{(0)}\Big[\partial_t\phi_2+\tilde{g}_2\partial_t\tilde{f}_2e^{\phi_1}-\tilde{\psi}\Big(\partial_t\tilde{\chi}+\partial_t\tilde{f}_1\tilde{f}_2\Big)e^{\phi_1+\phi_2}\Big]\\
&+E^{(0)}_{\alpha_2}\Big[\partial_t\tilde{f}_2e^{\phi_1}+\Big(\partial_t\tilde{\chi}+\partial_t\tilde{f}_1\tilde{f}_2\Big)\tilde{g}_1e^{\phi_1+\phi_2}\Big]\\
&+E^{(0)}_{-(\alpha_1+\alpha_2)}\Big[\partial_t\tilde{f}_1e^{\phi_2}-\Big(\partial_t\tilde{\chi}+\partial_t\tilde{f}_1\tilde{f}_2\Big)\tilde{g}_2e^{\phi_1+\phi_2}\Big]\\
&+E^{(0)}_{-\alpha_1}\Big(\partial_t\tilde{\chi}+\partial_t\tilde{f}_1\tilde{f}_2\Big)e^{\phi_1+\phi_2}. \qquad (6.14)
\end{aligned}
$$

$$
\begin{aligned}
D^{(-1)}_{2\,\mathcal{M}} &= -\tilde{\psi}e^{\phi_1+\phi_2}E^{(-1)}_{\alpha_1}+\Big[\tilde{\chi}\Big(1+\tilde{\psi}e^{\phi_1+\phi_2}\tilde{f}_2\tilde{f}_1+\tilde{g}_2e^{\phi_1}\tilde{f}_2\Big)+\tilde{f}_2\tilde{f}_1+\tilde{\psi}e^{\phi_1+\phi_2}\tilde{\chi}^2\Big]E^{(-1)}_{-\alpha_1}\\
&- \Big(\tilde{g}_2e^{\phi_1}+\tilde{\psi}e^{\phi_1+\phi_2}\tilde{f}_1\Big)E^{(-1)}_{-\alpha_2}+\Big(\tilde{f}_2+\tilde{\psi}e^{\phi_1+\phi_2}\tilde{f}_2\tilde{\chi}\Big)E^{(-1)}_{\alpha_2}. \qquad (6.15)
\end{aligned}
$$

$$
\begin{aligned}
A_0 &= -\partial_x BB^{-1}\\
&= -E^{(0)}_{-\alpha_1}\Big[\partial_x\tilde{\chi}+\tilde{\chi}(\partial_x\phi_1+\partial_x\phi_2)+\tilde{\chi}\tilde{f}_2\partial_x\tilde{g}_2e^{\phi_1}-\tilde{\chi}^2(\partial_x\tilde{\psi}-\partial_x\tilde{g}_1\tilde{g}_2)e^{\phi_1+\phi_2}\\
&+\tilde{f}_1\Big(\partial_x\tilde{f}_2+\tilde{f}_2\partial_x\phi_1+\tilde{\chi}\partial_x\tilde{g}_1e^{\phi_2}-\tilde{\chi}\tilde{f}_2(\partial_x\tilde{\psi}+\tilde{g}_2\partial_x\tilde{g}_1)e^{\phi_1+\phi_2}\Big)\Big]\\
&-E^{(0)}_{-(\alpha_1+\alpha_2)}\Big[\partial_x\tilde{f}_1-\tilde{\chi}\partial_x\tilde{g}_2e^{\phi_1}+\tilde{f}_1\Big(\partial_x\phi_2-\tilde{f}_2\partial_x\tilde{g}_2e^{\phi_1}-\tilde{\chi}(\tilde{g}_2\partial_x\tilde{g}_1+\partial_x\tilde{\psi})e^{\phi_1+\phi_2}\Big)\Big]\\
&-E^{(0)}_{\alpha_2}\Big[\partial_x\tilde{f}_2+\tilde{f}_2\partial_x\phi_1-\tilde{\chi}\tilde{f}_2\Big(\tilde{g}_2\partial_x\tilde{g}_1+\partial_x\tilde{\psi}\Big)e^{\phi_1+\phi_2}+\tilde{\chi}\partial_x\tilde{g}_1e^{\phi_2}\Big]\\
&-(\alpha_1+\alpha_2)\cdot H^{(0)}\Big[\partial_x\phi_1-\partial_x\tilde{g}_1\tilde{f}_1e^{\phi_2}+\tilde{f}_2\tilde{f}_1\Big(\partial_x\tilde{\psi}+\tilde{g}_2\partial_x\tilde{g}_1\Big)e^{\phi_1+\phi_2}-\tilde{\chi}\Big(\partial_x\tilde{\psi}+\tilde{g}_2\partial_x\tilde{g}_1\Big)e^{\phi_1+\phi_2}\Big]\\
&+\alpha_2\cdot H^{(0)}\Big[\partial_x\phi_2+\partial_x\tilde{g}_2\tilde{f}_2e^{\phi_1}-\tilde{\chi}\Big(\partial_x\tilde{\psi}+\tilde{g}_2\partial_x\tilde{g}_1\Big)e^{\phi_1+\phi_2}\Big]\\
&-E^{(0)}_{-\alpha_2}\Big[\partial_x\tilde{g}_2e^{\phi_1}+\tilde{f}_1\Big(\partial_x\tilde{\psi}+\tilde{g}_2\partial_x\tilde{g}_1\Big)e^{\phi_1+\phi_2}\Big]\\
&-E^{(0)}_{(\alpha_1+\alpha_2)}\Big[\partial_x\tilde{g}_1e^{\phi_2}-\tilde{f}_2\Big(\partial_x\tilde{\psi}+\tilde{g}_2\partial_x\tilde{g}_1\Big)e^{\phi_1+\phi_2}\Big]\\
&-E^{(0)}_{\alpha_1}\Big(\partial_x\tilde{\psi}+\tilde{g}_2\partial_x\tilde{g}_1\Big)e^{\phi_1+\phi_2}. \qquad (6.16)
\end{aligned}
$$

Introduzindo novas variáveis:

$$
\begin{aligned}
\tilde{\psi} &= \psi e^{-\frac{\phi_1+\phi_2}{2}}, \quad \tilde{g}_1=g_1e^{-\frac{\phi_2}{2}},\\
\tilde{f}_1 &= \bar{f}_1e^{-\frac{\phi_2}{2}}, \qquad \tilde{\chi}=\chi e^{-\frac{\phi_1+\phi_2}{2}}\\
\tilde{g}_2 &= g_2e^{-\frac{\phi_1}{2}}, \qquad \tilde{f}_2=\bar{f}_2e^{-\frac{\phi_1}{2}}, \qquad (6.17)
\end{aligned}
$$

onde $\psi, \chi, \phi_i, i = 1, 2$ e $f_i, g_i, i = 1, 2$ são campos bosônicos e fermiônicos respectivamente obtemos,

$$\begin{aligned} J &= B^{-1}\partial_t B \\ &= J_{-\alpha_1} E^{(0)}_{\alpha_1} + J_{\alpha_2} E^{(0)}_{-\alpha_2} + J_{-(\alpha_1+\alpha_2)} E^{(0)}_{(\alpha_1+\alpha_2)} + J_{(\alpha_1+\alpha_2)\cdot H^{(0)}} (\alpha_1+\alpha_2)\cdot H^{(0)} \\ &\quad -J_{-\alpha_2\cdot H^{(0)}} - \alpha_2 \cdot H^{(0)} + J_{\alpha_1} E^{(0)}_{-\alpha_1} + J_{-\alpha_2} E^{(0)}_{\alpha_2} + J_{(\alpha_1+\alpha_2)} E^{(0)}_{-(\alpha_1+\alpha_2)}. \end{aligned} \tag{6.18}$$

$$\begin{aligned} \bar{J} &= A_0 = -\partial_x B B^{-1} \\ &= \bar{J}_{-\alpha_1} E^{(0)}_{\alpha_1} + \bar{J}_{\alpha_2} E^{(0)}_{-\alpha_2} + \bar{J}_{-(\alpha_1+\alpha_2)} E^{(0)}_{(\alpha_1+\alpha_2)} + \bar{J}_{(\alpha_1+\alpha_2)\cdot H^{(0)}} (\alpha_1+\alpha_2)\cdot H^{(0)} \\ &\quad -\bar{J}_{-\alpha_2\cdot H^{(0)}} - \alpha_2 \cdot H^{(0)} + \bar{J}_{\alpha_1} E^{(0)}_{-\alpha_1} + \bar{J}_{-\alpha_2} E^{(0)}_{\alpha_2} + \bar{J}_{(\alpha_1+\alpha_2)} E^{(0)}_{-(\alpha_1+\alpha_2)}, \end{aligned} \tag{6.19}$$

onde

$$\begin{aligned} J_{\alpha_1} &= e^{\frac{1}{2}(\phi_1+\phi_2)} \left(\partial_t \chi - \frac{1}{2}\chi \left(\partial_t\phi_1 + \partial_t\phi_2\right) + \partial_t \bar{f}_1 \bar{f}_2 - \frac{1}{2}\bar{f}_1 \bar{f}_2 \partial_t \phi_2 \right), \\ J_{-\alpha_2} &= e^{-\frac{1}{2}\phi_1} \left(\partial_t g_2 + \frac{1}{2} g_2 \partial_t \phi_1 + \partial_t \bar{f}_1 \psi - \frac{1}{2}\bar{f}_1 \partial_t \phi_2 \psi - \psi e^{-\frac{1}{2}(\phi_1+\phi_2)} \bar{J}_{\alpha_1} g_2 \right), \\ J_{(\alpha_1+\alpha_2)} &= e^{\frac{1}{2}\phi_2} \left(\partial_t \bar{f}_1 - \frac{1}{2}\bar{f}_1 \partial_t \phi_2 - e^{-\frac{1}{2}(\phi_1+\phi_2)} \bar{J}_{-\alpha_2} g_2 \right), \\ J_{-\alpha_1} &= e^{-\frac{1}{2}(\phi_1+\phi_2)} \left(\partial_t \psi + \frac{1}{2}\psi \left(\partial_t\phi_1 + \partial_t\phi_2\right) + \psi g_2 \partial_t \bar{f}_2 - \frac{1}{2}\psi \partial_t \phi_1 g_2 \bar{f}_2 + e^{\frac{1}{2}\phi_1} \bar{J}_{-\alpha_2} g_1 \right. \\ &\quad \left. -\psi^2 e^{-\frac{1}{2}(\phi_1+\phi_2)} \bar{J}_{\alpha_1} \right), \\ J_{\alpha_2} &= e^{\frac{\phi_1}{2}} \left(\partial_t \bar{f}_2 - \frac{1}{2}\bar{f}_2 \partial_t \phi_1 + e^{\frac{1}{2}(\phi_1+\phi_2)} \bar{J}_{\alpha_1} g_1 \right), \\ J_{-(\alpha_1+\alpha_2)} &= e^{\frac{1}{2}\phi_2} \left(\partial_t g_1 - \frac{1}{2} g_1 \partial_t \phi_2 - \psi \partial_t \bar{f}_2 + \frac{1}{2}\psi \partial_t \phi_1 \bar{f}_2 + g_1 \bar{J}_{-\alpha_2\cdot H^{(0)}} \right), \\ J_{-\alpha_2\cdot H^{(0)}} &= \partial_t \phi_2 - \partial_t \bar{f}_2 g_2 + \frac{1}{2}\bar{f}_2 g_2 \partial_t \phi_1 - \psi e^{-\frac{1}{2}(\phi_1+\phi_2)} \bar{J}_{\alpha_1}, \\ J_{(\alpha_1+\alpha_2)\cdot H^{(0)}} &= \partial_t \phi_1 + \partial_t \bar{f}_1 g_1 - \frac{1}{2}\bar{f}_1 g_1 \partial_t \phi_2 - \left(\psi + g_2 g_1\right) e^{-\frac{1}{2}(\phi_1+\phi_2)} \bar{J}_{\alpha_1}. \end{aligned} \tag{6.20}$$

e

$$\begin{aligned}
\bar{J}_{-\alpha_1} &= -e^{\frac{1}{2}(\phi_1+\phi_2)}\left(\partial_x\psi - \frac{1}{2}\psi\left(\partial_x\phi_1+\partial_x\phi_2\right) + g_2\partial_x g_1 - \frac{1}{2}g_2 g_1\partial_x\phi_2\right),\\
\bar{J}_{\alpha_2} &= -e^{-\frac{1}{2}\phi_1}\left(\partial_x\bar{f}_2 + \frac{1}{2}\bar{f}_2\partial_x\phi_1 + \partial_x g_1\chi - \frac{1}{2}g_1\partial_x\phi_2\chi + \chi e^{-\frac{1}{2}(\phi_1+\phi_2)}\bar{J}_{-\alpha_1}\bar{f}_2\right),\\
\bar{J}_{-(\alpha_1+\alpha_2)} &= -e^{\frac{1}{2}\phi_2}\left(\partial_x g_1 - \frac{1}{2}g_1\partial_x\phi_2 + e^{-\frac{1}{2}(\phi_1+\phi_2)}\bar{J}_{-\alpha_1}\bar{f}_2\right),\\
\bar{J}_{\alpha_1} &= -e^{-\frac{1}{2}(\phi_1+\phi_2)}\partial_x\chi - \frac{1}{2}\chi\left(\partial_x\phi_1+\partial_x\phi_2\right) + \chi\bar{f}_2\partial_x g_2 + \frac{1}{2}\chi\partial_x\phi_1 g_2\bar{f}_2 + e^{\frac{1}{2}\phi_1}\bar{J}_{\alpha_2}\bar{f}_1\\
&\quad -\chi^2 e^{-\frac{1}{2}(\phi_1+\phi_2)}\bar{J}_{-\alpha_1},\\
\bar{J}_{-\alpha_2} &= -e^{\frac{\phi_1}{2}}\left(\partial_x g_2 - \frac{1}{2}g_2\partial_x\phi_1 - e^{-\frac{1}{2}(\phi_1+\phi_2)}\bar{J}_{\alpha_1} - \frac{1}{2}\bar{f}_1\right),\\
\bar{J}_{(\alpha_1+\alpha_2)} &= -e^{\frac{1}{2}\phi_2}\left(\partial_x\bar{f}_1 - \frac{1}{2}\bar{f}_1\partial_x\phi_2 - \chi\partial_x g_2 + \frac{1}{2}\chi\partial_x\phi_1 g_2 - \bar{f}_1\bar{J}_{-\alpha_2\cdot H^{(0)}}\right),\\
\bar{J}_{-\alpha_2\cdot H^{(0)}} &= -\partial_x\phi_2 - \partial_x g_2\bar{f}_2 - \frac{1}{2}\bar{f}_2 g_2\partial_x\phi_1 - \chi e^{-\frac{1}{2}(\phi_1+\phi_2)}\bar{J}_{-\alpha_1},\\
\bar{J}_{(\alpha_1+\alpha_2)\cdot H^{(0)}} &= -\partial_x\phi_1 + \partial_x g_1\bar{f}_1 + \frac{1}{2}\bar{f}_1 g_1\partial_x\phi_2 - \left(\chi + \bar{f}_1\bar{f}_2\right)e^{-\frac{1}{2}(\phi_1+\phi_2)}\bar{J}_{-\alpha_1}. \qquad (6.21)
\end{aligned}$$

Os geradores $E^{(0)}_{(\alpha_1+\alpha_2)}, E^{(0)}_{-(\alpha_1+\alpha_2)}, (\alpha_1+\alpha_2)\cdot H^{(0)}, -\alpha_2\cdot H^{(0)}$ comutam com $(\alpha_1+\alpha_2)\cdot H^{(n)}$, desta forma,

$$\begin{aligned}
\partial_t\bar{f}_1 &= \frac{1}{2}\bar{f}_1\partial_t\phi_2 + g_2\left[\partial_t\chi - \frac{1}{2}\chi\left(\partial_t\phi_1+\partial_t\phi_2\right)\right],\\
\partial_t g_1 &= \psi\partial_t\bar{f}_2 + \frac{1}{2}g_1\partial_t\phi_2 - \frac{1}{2}\psi\bar{f}_2\partial_t\phi_1,\\
\partial_x\bar{f}_1 &= \chi\partial_x g_2 + \frac{1}{2}\bar{f}_1\partial_x\phi_2 - \frac{1}{2}\chi\bar{g}_2\partial_x\phi_1,\\
\partial_x g_1 &= \frac{1}{2}g_1\partial_x\phi_2 + \bar{f}_2\left[\partial_x\psi - \frac{1}{2}\psi\left(\partial_x\phi_1+\partial_x\phi_2\right)\right],\\
\partial_t\phi_1 &= \frac{\psi\left[\partial_t\chi\left(1+g_2\bar{f}_2\right) + \frac{1}{2}\chi g_2\partial_t\bar{f}_2\right]}{1+\psi\chi\left(1+\frac{5}{4}g_2\bar{f}_2\right)},\\
\partial_t\phi_2 &= \frac{\psi\partial_t\chi\left(1+\frac{3}{2}g_2\bar{f}_2\right) - g_2\partial_t\bar{f}_2 - \frac{1}{2}\psi\chi g_2\partial_t\bar{f}_2}{1+\psi\chi\left(1+\frac{5}{4}g_2\bar{f}_2\right)},\\
\partial_x\phi_1 &= \frac{\chi\left[\partial_x\psi\left(1+g_2\bar{f}_2\right) + \frac{1}{2}\psi\partial_x g_2\bar{f}_2\right]}{1+\psi\chi\left(1+\frac{5}{4}g_2\bar{f}_2\right)},\\
\partial_x\phi_2 &= \frac{\chi\partial_x\psi\left(1+\frac{3}{2}g_2\bar{f}_2\right) + \left(1+\frac{1}{2}\psi\chi\right)\bar{f}_2\partial_x g_2}{1+\psi\chi\left(1+\frac{5}{4}g_2\bar{f}_2\right)}. \qquad (6.22)
\end{aligned}$$

Do que resulta as equações de movimento [19]:

$$-\partial_{t_2}b_2 + a_2\partial_x^2 b_2 - 2a_2\left(b_2 b_1 + f_1 f_2\right)b_2 - m_{-1} = 0, \qquad (6.23)$$

$$-\partial_{t_2}b_1 - a_2\partial_x^2 b_1 + 2a_2\left(b_2 b_1 + f_1 f_2\right)b_1 + g_{-1} = 0, \qquad (6.24)$$

$$-\partial_{t_2}f_2 - a_2\partial_x^2 f_2 + 2a_2 b_1 b_2 f_2 + n_{-1} = 0, \qquad (6.25)$$

$$-\partial_{t_2}f_1 + a_2\partial_x^2 f_1 - 2a_2 b_1 b_2 f_1 - o_{-3} = 0. \qquad (6.26)$$

onde,

$$
\begin{aligned}
g_{-1} &= -\eta\psi \exp\left(\frac{\phi_1+\phi_2}{2}\right), \\
m_{-1} &= \eta\left(\chi + \chi\psi\bar{f}_2\bar{f}_1 + \chi g_2\bar{f}_2 + \bar{f}_2\bar{f}_1 + \psi\chi^2\right)\exp\left(-\frac{\phi_1+\phi_2}{2}\right), \\
n_{-1} &= -\eta\left(g_2 + \psi\bar{f}_1\right)\exp\left(\frac{\phi_1}{2}\right), \\
o_{-1} &= \eta\bar{f}_2\left(1+\psi\chi\right)\exp\left(-\frac{\phi_1}{2}\right).
\end{aligned}
$$

Fazendo $a_2 = 0$ obtemos a equação super Lund-Regge e $\eta = 0$ a equação super NLS [2].

As relações entre as variáveis sAKNS e super Lund-Regge são dadas por,

$$
\begin{aligned}
f_1 &= -\exp\left(-\frac{\phi_1}{2}\right)\left(\partial_x\bar{f}_2 + \frac{1}{2}\bar{f}_2\partial_x\phi_1\right), \\
f_2 &= \exp\left(\frac{\phi_1}{2}\right)\left[\frac{1}{2}g_2\partial_x\phi_1 - \partial_x g_2 + \bar{f}_1\left(1+g_2\bar{f}_2\right)\left(\frac{1}{2}\psi\left(\partial_x\phi_1 + \partial_x\phi_2\right) - \partial_x\psi\right)\right], \\
b_1 &= \exp\left(\frac{\phi_1+\phi_2}{2}\right)\left(1+g_2\bar{f}_2\right)\left(\frac{1}{2}\psi\left(\partial_x\phi_1+\partial_x\phi_2\right) - \partial_x\psi\right), \\
b_2 &= \exp\left(-\frac{\phi_1+\phi_2}{2}\right)\left[-\partial_x\chi - \frac{1}{2}\chi\left(\partial_x\phi_1+\partial_x\phi_2\right) - \chi\bar{f}_2\left(\frac{1}{2}g_2\partial_x\phi_1 - \partial_x g_2\right)\right. \\
&\quad \left. -\chi^2\left(1+g_2\bar{f}_2\right)\left[\frac{1}{2}\psi\left(\partial_x\phi_1+\partial_x\phi_2\right) - \partial_x\psi\right] - \bar{f}_1\left(\partial_x\bar{f}_2 + \frac{1}{2}\bar{f}_2\partial_x\phi_1\right)\right].
\end{aligned}
$$

Tomando operadores graduados de ordem mais alta, obtemos outros modelos da hierarquia. Considere por exemplo $n = 3$ em (6.2):

$$
\begin{aligned}
D_3^{(3)} &= a_3\left(\alpha_1+\alpha_2\right)\cdot H^{(3)}, \\
D_3^{(2)} &= a_2\left(\alpha_1+\alpha_2\right)\cdot H^{(2)} - c_2\alpha_2\cdot H^{(2)} + d_2E^{(2)}_{(\alpha_1+\alpha_2)} + e_2E^{(2)}_{-(\alpha_1+\alpha_2)} \\
&+ g_2E^{(2)}_{\alpha_1} + m_2E^{(2)}_{-\alpha_1} + n_2E^{(2)}_{-\alpha_2} + o_2E^{(2)}_{\alpha_2}, \\
D_3^{(1)} &= a_1\left(\alpha_1+\alpha_2\right)\cdot H^{(1)} - c_1\alpha_2\cdot H^{(1)} + d_1E^{(1)}_{(\alpha_1+\alpha_2)} + e_1E^{(1)}_{-(\alpha_1+\alpha_2)} \\
&+ g_1E^{(1)}_{\alpha_1} + m_1E^{(1)}_{-\alpha_1} + n_1E^{(1)}_{-\alpha_2} + o_1E^{(1)}_{\alpha_2}, \\
D_3^{(0)} &= a_0\left(\alpha_1+\alpha_2\right)\cdot H^{(0)} - c_0\alpha_2\cdot H^{(0)} + d_0E^{(0)}_{(\alpha_1+\alpha_2)} + e_0E^{(0)}_{-(\alpha_1+\alpha_2)} \\
&+ g_0E^{(0)}_{\alpha_1} + m_0E^{(0)}_{-\alpha_1} + n_0E^{(0)}_{-\alpha_2} + o_0E^{(0)}_{\alpha_2}, \\
D_3^{(-1)} &= a_{-1}\left(\alpha_1+\alpha_2\right)\cdot H^{(-1)} - c_{-1}\alpha_2\cdot H^{(-1)} + d_{-1}E^{(-1)}_{(\alpha_1+\alpha_2)} + e_{-1}E^{(-1)}_{-(\alpha_1+\alpha_2)} \\
&+ g_{-1}E^{(-1)}_{\alpha_1} + m_{-1}E^{(-1)}_{-\alpha_1} + n_{-1}E^{(-1)}_{-\alpha_2} + o_{-1}E^{(-1)}_{\alpha_2}, \qquad (6.27)
\end{aligned}
$$

Substituindo (6.27) em (6.3) obtemos o sistema de equações:

$$
\begin{aligned}
\partial_x a_3 &= 0, \qquad g_2 - a_3 b_1 = 0,\\
a_3 b_2 - m_2 &= 0, \qquad a_3 f_1 - o_2 = 0,\\
n_2 - a_3 f_2 &= 0,\\
\partial_x d_2 + b_1 o_2 - f_1 g_2 &= 0, \qquad \partial_x e_2 - b_2 n_2 + f_2 m_2 = 0,\\
\partial_x g_2 - b_1 a_2 - b_1 c_2 + f_2 d_2 + g_1 &= 0, \qquad \partial_x m_2 + b_2 a_2 + b_2 c_2 + f_1 e_2 - m_1 = 0,\\
\partial_x n_2 - b_1 e_2 - f_2 a_2 + n_1 &= 0, \qquad \partial_x o_2 + b_2 d_2 + f_1 a_2 - o_1 = 0,\\
\partial_x a_1 + \partial_x c_1 + 2b_1 m_1 - 2b_2 g_1 - f_1 n_1 - f_2 o_1 &= 0, \qquad \partial_x a_1 - \partial_x c_1 + f_1 n_1 + f_2 o_1 = 0,\\
\partial_x d_1 + b_1 o_1 - f_1 g_1 &= 0, \qquad \partial_x e_1 - b_2 n_1 + f_2 m_1 = 0,\\
\partial_x g_1 - b_1 a_1 - b_1 c_1 + f_2 d_1 + g_0 &= 0, \qquad \partial_x m_1 + b_2 a_1 + b_2 c_1 + f_1 e_1 - m_0 = 0,\\
\partial_x n_1 - b_1 e_1 - f_2 a_1 + n_0 &= 0, \qquad \partial_x o_1 + b_2 d_1 + f_1 a_1 - o_0 = 0,\\
\partial_x a_0 + \partial_x c_0 + 2b_1 m_0 - 2b_2 g_0 - f_1 n_0 - f_2 o_0 &= 0, \qquad \partial_x a_0 - \partial_x c_0 + f_1 n_0 + f_2 o_0 = 0,\\
\partial_x d_0 + b_1 o_0 - f_1 g_0 &= 0, \qquad \partial_x e_0 - b_2 n_0 + f_2 m_0 = 0,
\end{aligned}
$$

cuja solução é dada por,

$$
\begin{aligned}
a_3 &= \text{constante}, \\
g_2 &= a_3 b_1, \\
m_2 &= a_3 b_2, \\
o_2 &= a_3 f_1, \\
n_2 &= a_3 f_2, \\
a_2 &= 0, \\
c_2 &= 0, \\
d_2 &= 0, \\
e_2 &= 0, \\
g_1 &= -a_3 \partial_x b_1, \\
m_1 &= a_3 \partial_x b_2, \\
n_1 &= -a_3 \partial_x f_2, \\
o_1 &= a_3 \partial_x f_1, \\
d_1 &= -a_3 f_1 b_1, \\
e_1 &= -a_3 f_2 b_2, \\
a_1 &= -a_3 b_1 b_2, \\
c_1 &= -a_3 \left(b_1 b_2 + f_1 f_2\right), \\
a_0 &= a_3 \left(b_2 \partial_x b_1 - b_1 \partial_x b_2\right), \\
c_0 &= a_3 \left(f_2 \partial_x f_1 + f_1 \partial_x f_2 + b_2 \partial_x b_1 - b_1 \partial_x b_2\right), \\
d_0 &= a_3 \left(\partial_x b_1 f_1 - \partial_x f_1 b_1\right), \\
e_0 &= a_3 \left(\partial_x f_2 b_2 - \partial_x b_2 f_2\right), \\
g_0 &= a_3 \left(\partial_x^2 b_1 - 2 b_2 b_2^1 - 2 b_1 f_1 f_2\right), \\
m_0 &= a_3 \left(\partial_x^2 b_2 - 2 b_1 b_2^2 - 2 b_2 f_1 f_2\right), \\
n_0 &= -a_3 \left(\partial_x^2 f_2 - 2 f_2 b_1 b_2\right), \\
o_0 &= -a_3 \left(\partial_x^2 f_1 - 2 f_1 b_1 b_2\right).
\end{aligned} \tag{6.28}
$$

Logo, as equações de movimento são dadas por:

$$-\partial_{t_3} b_2 + a_3 \partial_x^3 b_2 - 3a_3 \left(2 b_1 b_2 \partial_x b_2 + b_2 \partial_x f_1 f_2 + f_1 f_2 \partial_x b_2\right) - m_{-1} = 0, \tag{6.29}$$

$$-\partial_{t_3} b_1 + a_3 \partial_x^3 b_1 - 3a_3 \left(2 b_1 b_2 \partial_x b_1 + b_1 f_1 \partial_x f_2 + f_1 f_2 \partial_x b_1\right) + g_{-1} = 0, \tag{6.30}$$

$$-\partial_{t_3} f_2 + a_3 \partial_x^3 f_2 - 3a_3 \left(b_1 b_2 \partial_x f_2 + b_2 \partial_x b_1 f_2\right) + n_{-1} = 0, \tag{6.31}$$

$$-\partial_{t_3} f_1 + a_3 \partial_x^3 f_1 - 3a_3 \left(b_1 b_2 \partial_x f_1 + b_1 \partial_x b_2 f_1\right) - o_{-1} = 0, \tag{6.32}$$

com

$$\begin{aligned}
g_{-1} &= -\eta\psi\exp\left(\frac{\phi_1+\phi_2}{2}\right),\\
m_{-1} &= \eta\left(\chi+\chi\psi\bar{f}_2\bar{f}_1+\chi g_2\bar{f}_2+\bar{f}_2\bar{f}_1+\psi\chi^2\right)\exp\left(-\frac{\phi_1+\phi_2}{2}\right),\\
n_{-1} &= -\eta\left(g_2+\psi\bar{f}_1\right)\exp\left(\frac{\phi_1}{2}\right),\\
o_{-1} &= \eta\bar{f}_2\left(1+\psi\chi\right)\exp\left(-\frac{\phi_1}{2}\right).
\end{aligned}$$

Verificamos que as equações de movimento da hierarquia super NLS/Lund-Regge são invariantes pela transformação de supersimetria:

$$\delta b_1 = \epsilon_1 f_2, \tag{6.33}$$

$$\delta b_2 = \epsilon_2 f_1, \tag{6.34}$$

$$\delta f_1 = -\epsilon_1 b_2, \tag{6.35}$$

$$\delta f_2 = \epsilon_2 b_1, \tag{6.36}$$

onde ϵ_1 e ϵ_2 são parâmetros fermiônicos.

6.1 Sólitons

O operador de vértice é dado por,

$$V_+(\gamma) = \sum_{x\in\mathbb{Z}}\left(E^{(n)}_{\alpha_1}+a^+E^{(n)}_{-\alpha_2}\right)\gamma^{(-n)}, \tag{6.37}$$

$$V_-(\gamma) = \sum_{x\in\mathbb{Z}}\left(E^{(n)}_{-\alpha_1}+a^-E^{(n)}_{\alpha_2}\right)\gamma^{(-n)}, \tag{6.38}$$

e possui os autovalores,

$$[\epsilon_1, V_\pm(\gamma)] = \mp 2\gamma V_\pm(\gamma), \tag{6.39}$$

$$[\epsilon_n, V_\pm(\gamma)] = \mp 2\left(a_n\gamma^n+\frac{\eta}{\gamma}\right)V_\pm(\gamma), \tag{6.40}$$

onde

$$\epsilon_1 = (\alpha_1+\alpha_2)\cdot H^{(1)}, \tag{6.41}$$

$$\epsilon_n = a_n(\alpha_1+\alpha_2)\cdot H^{(n)}+\eta\alpha_1\cdot H^{(-1)}. \tag{6.42}$$

As funções τ são dadas por,

$$\tau_i =< \lambda_i|(\Theta_-)^{-1}\Theta_+|\lambda_i >= \left\langle\lambda_i|T_{vac}gT^{-1}_{vac}|\lambda_i\right\rangle. \tag{6.43}$$

Usando o elemento de grau zero B dado em (6.12) podemos reescrever as funções τ como,

$$\tau_0 = e^{\nu} = \left\langle \lambda_0 | T_{vac} g T_{vac}^{-1} | \lambda_0 \right\rangle , \tag{6.44}$$

$$\tau_1 = e^{\nu+\phi} = \left\langle \lambda_1 | T_{vac} g T_{vac}^{-1} | \lambda_1 \right\rangle , \tag{6.45}$$

$$\tau_2 = \psi e^{\nu+\phi} = \left\langle \lambda_1 | T_{vac} g T_{vac}^{-1} E_{-\alpha}^{(0)} | \lambda_1 \right\rangle , \tag{6.46}$$

$$\tau_3 = \chi e^{\nu+\phi} = \left\langle \lambda_1 | E_{\alpha}^{(0)} T_{vac} g T_{vac}^{-1} | \lambda_1 \right\rangle , \tag{6.47}$$

Para a solução 1-sóliton temos

$$g = e^{V_-(\gamma_1)} e^{V_+(\gamma_2)}, \tag{6.48}$$

logo,

$$\begin{aligned} T_{vac} g T_{vac}^{-1} &= e^{-E^{(1)}x-(a_n E^{(n)}+\eta E^{(-1)})t_n} \left(1 + V_-(\gamma_1)\right) \left(1 + V_+(\gamma_2)\right) e^{E^{(1)}x+(a_n E^{(n)}+\eta E^{(-1)})t_n}, \\ &= 1 + \rho_1^+ V_-(\gamma_1) + \rho_2^- V_+(\gamma_2) + \rho_1^+ \rho_2^- V_-(\gamma_1) V_+(\gamma_2), \end{aligned} \tag{6.49}$$

com

$$\rho_i^{\pm} = e^{\pm\left(2\gamma_i x + 2\left(a_n \gamma_i^{(n)} + \eta\, \gamma_i^{(-1)}\right) t_n\right)}. \tag{6.50}$$

A solução 1-sóliton da hierarquia super NLS/Lund-Regge é dada por

$$b_1 = \frac{\gamma_1\rho_1^+}{\tau_0}, \tag{6.51}$$

$$b_2 = -\frac{\gamma_2\rho_2^-}{\tau_0}, \tag{6.52}$$

$$f_1 = -a^-\frac{\gamma_2\rho_2^-}{\tau_0}, \tag{6.53}$$

$$f_2 = a^+\frac{\gamma_1\rho_1^+}{\tau_0}, \tag{6.54}$$

$$\psi = \frac{\rho_1^+}{\tau_0}\left(1 - \frac{b\gamma_1\rho_1^+\rho_2^-}{2(\gamma_1-\gamma_2)(1+\frac{\gamma_1}{\gamma_2}\rho_1^+\rho_2^-)}\right), \tag{6.55}$$

$$\chi = \frac{\rho_2^-}{\tau_0}\left(1 - \frac{b\gamma_2\rho_1^+\rho_2^-}{2(\gamma_1-\gamma_2)(1+\frac{\gamma_1}{\gamma_2}\rho_1^+\rho_2^-)}\right), \tag{6.56}$$

$$g_1 = a^-\frac{\gamma_1\rho_1^+\rho_2^-}{(\gamma_1-\gamma_2)\tau_0}e^{-\frac{1}{2}\phi_1}, \tag{6.57}$$

$$\bar{f}_1 = a^+\frac{\gamma_1\rho_1^+\rho_2^-}{(\gamma_1-\gamma_2)\tau_0}e^{-\frac{1}{2}\phi_1}, \tag{6.58}$$

$$g_2 = a^+\frac{\rho_1^+}{\tau_0}e^{-\frac{1}{2}\phi_2}, \tag{6.59}$$

$$\bar{f}_2 = a^-\frac{\rho_2^-}{\tau_0}e^{-\frac{1}{2}\phi_2}, \tag{6.60}$$

$$e^{\frac{1}{2}(\phi_1+\phi_2)} = \frac{1+a_3\rho_1^+\rho_2^-}{\tau_0}, \tag{6.61}$$

$$e^{\frac{1}{2}(\phi_1-\phi_2)} = \frac{1+\bar{a_3}\rho_1^+\rho_2^-}{\tau_0}, \tag{6.62}$$

onde

$$\begin{aligned} \tau_0 &= 1+\Gamma\rho_1^+\rho_2^-, \qquad (6.63)\\ \Gamma &= 1+a^+a^-\frac{\gamma_1\gamma_2}{(\gamma_1-\gamma_2)^2},\\ b &= a^+a^-,\\ a_3 &= \frac{\gamma_1}{\gamma_2}\Gamma_0 - \frac{1}{2}\frac{\gamma_1(\gamma_1+\gamma_2)\,b}{(\gamma_1-\gamma_2)^2},\\ \bar{a_3} &= \Gamma_0 + \frac{b}{2}\frac{\gamma_1(\gamma_1-3\gamma_2)\,b}{(\gamma_1-\gamma_2)^2},\\ \Gamma_0 &= \frac{\gamma_1\gamma_2}{(\gamma_1-\gamma_2)^2}, \end{aligned}$$

onde a^+, a^- e b são constantes fermiônicas e bosônica respectivamente. Ademais,

$$\rho_1^+ = e^{-t_3\left(a_3\gamma_1{}^3+\frac{\eta}{\gamma_1}\right)-\gamma_1 x}, \tag{6.64}$$

$$\rho_2^- = e^{t_3\left(a_3\gamma_2{}^3+\frac{\eta}{\gamma_2}\right)+\gamma_2 x}, \tag{6.65}$$

para $n = 3$ e

$$\rho_1^+ = e^{-t_2\left(a_2{\gamma_1}^2+\frac{\eta}{\gamma_1}\right)-\gamma_1 x}, \tag{6.66}$$

$$\rho_2^- = e^{t_2\left(a_2{\gamma_2}^2+\frac{\eta}{\gamma_2}\right)+\gamma_2 x}, \tag{6.67}$$

para $n = 2$.

7 Conclusões e perspectivas

Nesta tese, estudamos o método de curvatura nula para a construção de hierarquias integráveis e o subseqüente uso do método de dressing para a obtenção de soluções sólitons usando como estrutura subjacente uma de álgebra de Lie de dimensão infinita graduada. Uma importante característica desta construção é a obtenção de equações de movimento e suas soluções sólitons de uma maneira algébrica sistemática.

No capítulo dois nós obtemos a hierarquia mKdV par negativa e suas soluções multi-sólitons usando a álgebra $\hat{sl}(2)$ e a graduação $\mathcal{Q} = \frac{1}{2}H^{(0)} + 2d$. Embora esta equação tenha sido introduzida na literatura em [10], apenas em [11] suas soluções sólitons foram encontradas de forma clara, para este fim, o método dressing foi modificado para permitir soluções de vácuo constantes diferente de zero. Nós obtemos ainda, as duas primeiras equações da hierarquia, correspondentes a t_{-2} e t_{-4}, e encontramos explicitamente as soluções um e dois sólitons. Mesmo modificando o operador de vértice, nós conservamos a propriedade de nilpotência para este novo operador, pois esta propriedade é necessária para a obtenção das soluções sóliton.

No capítulo três a importante equação mista mKdV/sinh-Gordon é obtida através da formulação de curvatura nula com o uso da álgebra $\hat{sl}(2)$ e a graduação $\mathcal{Q} = \frac{1}{2}H^{(0)} + 2d$. Para isso, introduzimos na curvatura nula um termo de grau menos um. Isto permitiu-nos acoplar os modelo não-relativísticos, caracterizados por tempos positivos (t_n para $n > 0$), e o modelo relativístico da hierarquia (t_{-1}). Com a estrutura algébrica determinada, usamos o método dressing para encontrar as soluções sólitons da hierarquia.

No capítulo quatro a versão supersimétrica da hierarquia mKdV/sinh-Gordon é determinada através do uso da superálgebra $\hat{sl}(2,1)$ e suas soluções sólitons são encontradas. Pode-se verificar que a versão supersimétrica das equações encontradas neste capítulo são reduzidas aos respectivos casos bosônicos encontrado no capítulo anterior ao fazermos a parte fermiônica igual a zero.

Motivados pelos capítulos três e quatro, nós estudamos o acoplamento entre a equação Schrodinger não-linear e Lund-Regge no capítulo cinco e sua versão supersimétrica no capítulo seis. Neste caso os cálculos são mais elaborados devido ao aumento do número dos campos físicos dos modelos.

Como perspectivas de trabalhos futuros, nós pretendemos estudar a extensão dos problemas propostos nesta tese para as hierarquias KP e Boussinesq, através do formalismo de curvatura nula e o método dressing para a álgebra $\hat{sl}(3)$. Além disso, pretendemos estudar soluções tipo breather [26] and wobble [27, 28] para estas novas hierarquias introduzidas, devido a possíveis aplicações tecnológicas destes modelos.

Além disso, futuramente pretendemos estudar os seguintes aspectos:

1. Construção de novas hierarquias acopladas associadas a álgebra $\hat{sl}(3)$.
2. Construção de novas hierarquias associadas a gradações negativas pares para os modelos integráveis construídos através da álgebra $\hat{sl}(3)$.
3. Construção de soluções tipo breather e wobble associadas a estes novos modelos.
4. Estudo das cargas conservadas associadas a estes novos modelos.

APÊNDICE A - A Álgebra de Lie $sl(2)$

A.1 Estrutura de uma Álgebra de Lie e a base de Weyl-Cartan

Uma *álgebra de Lie* $\mathcal{G}$ possui um conjunto de operadores $\{T_a\}$, $a = 1, 2, ..., dim\,\mathcal{G}$, satisfazendo

$i)$ $[T_a, T_b] = i f_{ab}^c T_c,$

$ii)$ $[T_a, [T_b, T_c]] + [T_c, [T_a, T_b]] + [T_b, [T_c, T_a]] = 0$ (id. de Jacobi).

As constantes $f_{ab}^c = -f_{ba}^c$ são chamadas de *constantes de estrutura* e caracterizam a álgebra de Lie. Os operadores T_a são chamados de *geradores* da álgebra e o conjunto de todas as combinações lineares de T_a forma um espaço vetorial no qual $\{T_a\}$ forma uma base nesse espaço.
Um elemento g do grupo de Lie G está associado a $\mathcal{G}$ através de um mapeamento exponencial dos geradores T_a:

$$g = exp\left(i \sum_a \zeta^a T_a\right). \tag{A.1}$$

onde ζ^a, $a = 1, 2, ..., dim\,\mathcal{G}$, são parâmetros contínuos. Quando variamos o parâmetro ζ_a para $\zeta_a + \delta\zeta_a$, variamos de um elemento $g(\zeta_a)$ para um outro elemento $g(\zeta_a + \delta\zeta_a)$ em G.
Para uma álgebra de Lie $\mathcal{G}$ semisimples, podemos escolher uma combinação linear adequada dos geradores T_a e formar uma nova base $\mathcal{G} = \{H_i, E_\alpha\}$. Esta nova base é conhecida como *base de Weyl-Cartan* [29]. O conjunto $\{H_i\}$, $i = 1, 2, ..., r = rank\,\mathcal{G}$, é chamado *sub-álgebra de Cartan* e seus geradores constituem a maior sub-álgebra abeliana de $\mathcal{G}$ os quais podem ser diagonalizados simultâneamente, i.e. os geradores H_i satisfazem

$$[H_i, H_j] = 0, \tag{A.2}$$

para qualquer i e qualquer j. Portanto, podemos encontrar um conjunto de autoestados $\mid \mu\rangle$ satisfazendo

$$H_i \mid \mu\rangle = \mu_i \mid \mu\rangle, \tag{A.3}$$

onde os autovalores μ_i são componentes de um vetor $\mu = (\mu_1, \mu_2, ..., \mu_r)$ que vive num espaço vetorial de mesma dimensão que a sub-álgebra de Cartan.
Os geradores E_α são autovetores de H_i, no sentido de que

$$[H_i, E_\alpha] = \alpha_i E_\alpha, \tag{A.4}$$

onde α_i também são componentes de um vetor α de dimensão $rank\,\mathcal{G}$. O vetor α é chamado de *raíz* da álgebra de Lie. Para cada raíz α de uma álgebra de Lie semisimples existe um único gerador correspondente E_α e uma única raíz negativa $-\alpha$. Portanto o número de raízes e de geradores E_α de uma álgebra de Lie $\mathcal{G}$ é igual a

$$dim\,\mathcal{G} - rank\,\mathcal{G} = \textit{número par}. \tag{A.5}$$

Dentre o conjunto de todas as raízes de uma álgebra de Lie, existe um subconjunto de raízes $\alpha_1, \alpha_2, ..., \alpha_r$ ($r = rank\,\mathcal{G}$), chamadas de *raízes simples*. Todas as outras raízes da álgebra podem ser escritas como uma combinação linear das raízes simples, i.e.

$$\alpha = \sum_{a=1}^{r} n_a \alpha_a, \tag{A.6}$$

onde os n_a são números inteiros, todos sendo ou positivos ou negativos.
Um vetor μ satisfazendo (A.3) tal que

$$\frac{2\alpha \cdot \mu}{\alpha^2} = n, \qquad n \in \mathbb{Z}, \tag{A.7}$$

para qualquer raíz α, é chamado de *peso*. É possível encontrar um conjunto de pesos λ_a, $a = 1, 2, ..., r$, satisfazendo

$$\frac{2\lambda_a \cdot \alpha_b}{\alpha_b^2} = \delta_{ab}, \tag{A.8}$$

para qualquer raíz simples α_b. Estes pesos são chamados de *pesos fundamentais*. Qualquer peso μ pode ser escritos como uma combinação linear dos pesos fundamentais:

$$\mu = \sum_{a=1}^{r} m_a \lambda_a, \qquad m_a \in \mathbb{Z}. \tag{A.9}$$

Considere agora um estado $\mid \mu\rangle$ satisfazendo (A.3). O estado definido por $E_\alpha \mid \mu\rangle$, satisfaz

$$\begin{aligned} H_i E_\alpha \mid \mu\rangle &= (E_\alpha H_i + [H_i, E_\alpha]) \mid \mu\rangle \\ &= (\mu_i + \alpha_i) E_\alpha \mid \mu\rangle, \end{aligned} \tag{A.10}$$

e portanto $E_\alpha \mid \mu\rangle$ tem peso $\mu + \alpha$. Logo, o estado

$$E_{\alpha_1} E_{\alpha_2} ... E_{\alpha_n} \mid \mu\rangle, \tag{A.11}$$

tem peso $\mu + \alpha_1 + ... + \alpha_n$. Por esta razão os geradores E_α são chamados de *operadores de levantamento* (ou *operadores passo*).
Um estado $\mid \lambda\rangle$ que satisfaz

$$E_\alpha \mid \lambda\rangle = 0, \quad \text{para qualquer } \alpha > 0 \tag{A.12}$$

é chamado de *estado de peso mais alto* e define uma representação de $\mathcal{G}$. O peso λ é chamado de *peso mais alto* da representação. Todos os outros estados da representação são obtidos a partir do estado de peso mais alto pela aplicação de uma sequência de operadores $E_{-\alpha_n}$. Por exemplo, o estado definido por

$$\mid \mu\rangle = E_{-\alpha_1} E_{-\alpha_2} ... E_{-\alpha_n} \mid \lambda\rangle, \tag{A.13}$$

tem peso $\lambda - \alpha_1 - \alpha_2 - ... - \alpha_n$.
Agora, desejamos avaliar $[E_\alpha, E_\beta]$. Usando a identidade de Jacobi e a eq. (A.4), obtemos:

$$\begin{aligned} [H_i, [E_\alpha, E_\beta]] &= -[E_\alpha, [E_\beta, H_i]] - [E_\beta, [H_i, E_\alpha]] \\ &= (\alpha_i + \beta_i)[E_\alpha, E_\beta], \end{aligned}$$

Uma vez que a álgebra é fechada sob o comutador segue que $[E_\alpha, E_\beta]$ deve ser um elemento da álgebra, portanto temos três possibilidades:

1- se $\alpha + \beta$ é raíz, então $[E_\alpha, E_\beta] \propto E_{\alpha+\beta}$,

2- se $\alpha + \beta$ não é raíz, então $[E_\alpha, E_\beta] = 0$,

3- se $\alpha + \beta = 0$, então $[E_\alpha, E_\beta]$ deve ser um elemento da sub-álgebra de Cartan, uma vez que ele comuta com todo H_i.

Para finalizar esta seção, deixamos um resumo das relações de comutação de uma álgebra de Lie $\mathcal{G}$ semisimples na base de Weyl-Cartan:

$$[H_i, H_j] = 0, \qquad i, j = 1, 2, ..., r = rank\, \mathcal{G} \tag{A.14}$$

$$[H_i, E_\alpha] = \alpha_i E_\alpha, \tag{A.15}$$

$$[E_\alpha, E_\beta] = \begin{cases} N_{\alpha\beta} E_{\alpha+\beta} & \text{se } \alpha+\beta \text{ é raíz,} \\ \frac{2}{\alpha^2}\alpha \cdot H & \text{se } \alpha+\beta = 0 \text{ ,} \\ 0 & \text{em outro caso.} \end{cases} \tag{A.16}$$

onde $N_{\alpha\beta} = -N_{\beta\alpha}$ são constantes de estruturas.

A.2 Álgebra de Kac-Moody

Podemos estender uma álgebra de Lie $\mathcal{G}$ de dimensão finita para uma álgebra de dimensão infinita adicionando um índice extra, $T_a \rightarrow T_a^{(n)}$, e acrescentando dois novos geradores:

$$\mathcal{G} = \{T_a\} \rightarrow \tilde{\mathcal{G}} = \{T_a^{(n)}, \hat{c}, \hat{d}\}. \tag{A.17}$$

A álgebra $\tilde{\mathcal{G}}$ é chamada *álgebra de Kac-Moody* [30] e seus geradores satistifazem

$$[T_a^{(n)}, T_b^{(m)}] = f_{ab}^c T_c^{(n+m)} + \hat{c} n \delta_{ab} \delta^{n+m,0}, \tag{A.18}$$

$$[\hat{d}, T_a^{(n)}] = n T_a^{(n)}, \tag{A.19}$$

$$[\hat{c}, T_a^{(n)}] = [\hat{c}, \hat{d}] = 0. \tag{A.20}$$

A identidade de Jacobi também é satisfeita. O novo gerador $\hat{c}$ é chamado de *termo central* e comuta com todos os outros geradores. $\hat{d}$ é o *operador derivação* e através dele a álgebra de Kac-Moody $\tilde{\mathcal{G}}$ pode ser decomposta em sub-espaços graduados, isto é

$$\tilde{\mathcal{G}} = \oplus_n \tilde{\mathcal{G}}_n, \tag{A.21}$$

onde

$$\tilde{\mathcal{G}}_n = \{h \in \tilde{\mathcal{G}} \mid [\hat{d}, h] = nh\}, \tag{A.22}$$

e satisfaz

$$[\tilde{\mathcal{G}}_n, \tilde{\mathcal{G}}_m] \subset \tilde{\mathcal{G}}_{n+m}. \tag{A.23}$$

A base de Weyl-Cartan na álgebra de Kac-Moody, agora é dada por

$$\tilde{\mathcal{G}} = \{H_i^{(n)}, E_\alpha^{(n)}, \hat{c}, \hat{D}\}, \tag{A.24}$$

onde $\hat{D}$ é o operador derivação na base de Weyl-Cartan e satisfaz

$$[\hat{D}, E_\alpha^{(n)}] = nE_\alpha^{(n)}, \tag{A.25}$$

$$[\hat{D}, H_i^{(n)}] = nH_i^{(n)}, \qquad i = 1, 2, ..., r \tag{A.26}$$

$$[\hat{D}, \hat{c}] = 0. \tag{A.27}$$

Os demais geradores da base de Weyl-Cartan satisfazem

$$[H_i^{(n)}, H_j^{(m)}] = n\,\delta_{ij}\delta_{n+m,0}\hat{c}, \tag{A.28}$$

$$[H_i^{(n)}, E_{\pm\alpha}^{(m)}] = \pm\alpha_i E_{\pm\alpha}^{(n+m)}, \tag{A.29}$$

$$[E_\alpha^{(n)}, E_{-\alpha}^{(m)}] = \frac{2}{\alpha^2}\alpha \cdot H + \frac{2}{\alpha^2} n\,\delta_{n+m,0}\hat{c}, \tag{A.30}$$

$$[E_\alpha^{(n)}, E_\beta^{(m)}] = n_{\alpha\beta}E_{\alpha+\beta}^{(n+m)} \qquad \text{se } \alpha + \beta \text{ é raíz}, \tag{A.31}$$

$$[\hat{c}, H_i^{(n)}] = [\hat{c}, E_\alpha^{(n)}] = 0, \tag{A.32}$$

onde $n_{\alpha\beta}$ é uma constante de estrutura.
Note que, apesar da álgebra de Kac-Moody $\tilde{\mathcal{G}}$ possuir dimensão infinita a sub-álgebra de Cartan ainda possui dimensão finita e é dada por

$$\tilde{\mathcal{G}}_0 = \{H_i^{(0)}, \hat{c}, \hat{D}\}, \qquad i = 1, 2, ..., r. \tag{A.33}$$

Os geradores $H_i^{(n)}$ com $n \neq 0$ funcionam como operadores de levantamento na álgebra de Kac-Moody e as raízes correspondendentes a $H_i^{(n)}$ são

$$\hat{\alpha} = (0, 0, n), \qquad n \in \mathbb{Z} \setminus \{0\} \tag{A.34}$$

uma vez que,

$$[H_i^{(0)}, H_j^{(n)}] = 0, \qquad [\hat{c}, H_j^{(n)}] = 0, \qquad [\hat{D}, H_j^{(n)}] = n\,H_j^{(n)}. \tag{A.35}$$

As raízes correspondentes aos geradores $E_\alpha^{(n)}$ agora são dadas por

$$\hat{\alpha} = (\alpha, 0, n), \qquad \alpha \in \{\alpha\}_{\mathcal{G}}, \quad n \in \mathbb{Z} \tag{A.36}$$

pois,

$$[H_i^{(0)}, E_\alpha^{(n)}] = \alpha_i E_\alpha^{(n)}, \qquad [\hat{c}, E_\alpha^{(n)}] = 0, \qquad [\hat{D}, E_\alpha^{(n)}] = n\,E_\alpha^{(n)}. \tag{A.37}$$

O conjunto de geradores correspondendo as raízes positivas na álgebra de Kac-Moody agora é $\mathcal{B} = \{E_{\alpha}^{(n)}, E_{-\alpha}^{(n)}, H_i^{(n)}, E_{\alpha}^{(0)}\}$, com $n > 0$. Assim, o estado $\mid \lambda \rangle$ de peso mais alto em $\tilde{\mathcal{G}}$ agora é aniquilado por

$$b \mid \lambda \rangle = 0, \qquad b \in \mathcal{B}. \tag{A.38}$$

Na álgebra de Kac-Moody é introduzido um novo estado $\mid \lambda_0 \rangle$ que satisfaz

$$\hat{c} \mid \lambda_0 \rangle = c \mid \lambda_0 \rangle, \tag{A.39}$$

$$h \mid \lambda_0 \rangle = 0, \qquad h \in \tilde{\mathcal{G}} \setminus \{\hat{c}\} \tag{A.40}$$

onde c é uma constante.

APÊNDICE B - Superálgebra $sl(2,1)$

A superálgebra $sl(2,1)$ é formada por geradores bosônicos B e geradores fermiônicos F que satisfazem as propriedades:

$$[B_1, B_2] = B_3, \tag{B.1}$$

$$\{F_1, F_2\} = B_3, \tag{B.2}$$

$$[B_1, F_2] = F_3, \tag{B.3}$$

onde B_i são geradores bosônicos e F_i são geradores fermiônicos. A superálgebra $\hat{sl}(2,1)$ possui quatro geradores bosônicos e quatro geradores fermiônicos. Podemos representar estes geradores através duma representação matricial 3×3 como:

$$\begin{aligned}
H_1 &= \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & H_2 &= \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
E_{\alpha_1} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & E_{-\alpha_1} &= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
E_{\alpha_2} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, & E_{-\alpha_2} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \\
E_{\alpha_1+\alpha_2} &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & E_{-\alpha_1-\alpha_2} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},
\end{aligned} \tag{B.4}$$

onde $\{H_1, H_2, E_{\pm\alpha_1}\}$ são geradores bosônicos e $\{E_{\pm\alpha_2}, E_{\pm(\alpha_1+\alpha_2)}\}$ são geradores fermiônicos.

As raízes simples da superálgebra $sl(2,1)$ podem ser representadas em termos de vetores unitários e_i, f, $i = 1,2$ ($e_i \cdot e_j = \delta_{ij}$, $e_i \cdot f = 0$, $f \cdot f = -1$) como segue

$$\alpha_1 = e_1 - e_2, \qquad \alpha_2 = e_2 - f. \tag{B.5}$$

Para obter uma superálgebra de Kac-Moody $\tilde{sl}(2,1)$ a partir da superálgebra $\hat{sl}(2,1)$, introduzimos um parâmetro espectral, $h \to h^{(n)} = \lambda^n h$, onde $h \in sl(2,1)$, e usamos a seguinte regra

$$[g^{(n)}, h^{(m)}]_\pm = [g,h]_\pm^{(n+m)} + n\,\delta_{n+m,0} Str(gh)\hat{c}, \qquad g,h \in sl(2,1) \tag{B.6}$$

onde $[g,h]_+ = \{g,h\}$, $[g,h]_- = [g,h]$ e $Str(m) = m_{11} + m_{22} - m_{33}$, é chamado de *supertraço*. Seguindo este procedimento, obtemos as seguintes relações de comutação e anticomutação para a superálgebra de Kac-Moody $\tilde{sl}(2,1)$:

$$\begin{aligned}
&[h_+^{(n)}, E_{\pm\alpha_1}^{(m)}] = \pm E_{\pm\alpha_1}^{(n+m)}, \qquad && [h_+^{(n)}, E_{\mp\alpha_2}^{(m)}] = \pm E_{\mp\alpha_2}^{(n+m)},\\
&[h_2^{(n)}, E_{\pm\alpha_1}^{(m)}] = \mp E_{\pm\alpha_1}^{(n+m)}, && [h_2^{(n)}, E_{\pm(\alpha_1+\alpha_2)}^{(m)}] = \mp E_{\pm(\alpha_1+\alpha_2)}^{(n+m)},\\
&[h_+^{(n)}, E_{\pm(\alpha_1+\alpha_2)}^{(m)}] = 0, && [h_2^{(n)}, E_{\mp\alpha_2}^{(m)}] = 0,\\
&[E_{\pm\alpha_1}^{(n)}, E_{\mp\alpha_2}^{(m)}] = 0, && [E_{\pm\alpha_1}^{(n)}, E_{\pm(\alpha_1+\alpha_2)}^{(m)}] = 0,\\
&\{E_{\mp\alpha_2}^{(n)}, E_{\mp\alpha_2}^{(m)}\} = 0, && \{E_{\pm(\alpha_1+\alpha_2)}^{(n)}, E_{\pm(\alpha_1+\alpha_2)}^{(m)}\} = 0,\\
&[E_{\alpha_1}^{(n)}, E_{-\alpha_1}^{(m)}] = h_1^{(n+m)} + n\,\delta_{n+m,0}\hat{c}, &&\\
&\{E_{\alpha_1+\alpha_2}^{(n)}, E_{-\alpha_1-\alpha_2}^{(m)}\} = h_+^{(n+m)} + n\,\delta_{n+m,0}\hat{c}, &&\\
&[E_{\pm\alpha_1}^{(n)}, E_{\mp(\alpha_1+\alpha_2)}^{(m)}] = \mp E_{\mp\alpha_2}^{(n+m)}, && [E_{\pm\alpha_1}^{(n)}, E_{\pm\alpha_2}^{(m)}] = \pm E_{\pm(\alpha_1+\alpha_2)}^{(n+m)},\\
&\{E_{-\alpha_2}^{(n)}, E_{\alpha_2}^{(m)}\} = h_2^{(n+m)} - n\,\delta_{n+m,0}\hat{c}, &&\\
&\{E_{\mp\alpha_2}^{(n)}, E_{\mp(\alpha_1+\alpha_2)}^{(m)}\} = 0, && \{E_{\pm(\alpha_1+\alpha_2)}^{(n)}, E_{\mp\alpha_2}^{(m)}\} = E_{\pm\alpha_1}^{(n+m)},\\
&[h_+^{(n)}, h_2^{(m)}] = -n\,\delta_{n+m,0}\hat{c},
\end{aligned} \tag{B.7}$$

onde temos definido

$$\begin{aligned}
h_1 &= 2H_1 = \begin{pmatrix} 1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0 \end{pmatrix}, \quad h_2 = -H_1 + H_2 = \begin{pmatrix} 0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1 \end{pmatrix},\\
h_+ &= h_1 + h_2.
\end{aligned} \tag{B.8}$$

APÊNDICE C - Superálgebra $\tilde{sl}(2,1)$

$$
\begin{aligned}
[K_1^{(2n+1)}, K_2^{(2m+1)}] &= 0, \\
\{F_1^{(2n+3/2)}, F_2^{(2m+1/2)}\} &= [(2n+1) - 2m]\delta_{n+m+1,0}\hat{c}, \\
[F_1^{(2n+3/2)}, K_1^{(2m+1)}] &= F_2^{2(n+m+1)+1/2}, \\
[F_1^{(2n+3/2)}, K_2^{(2m+1)}] &= -F_2^{2(n+m+1)+1/2}, \\
[F_2^{(2n+1/2)}, K_1^{(2m+1)}] &= F_1^{2(n+m)+3/2}, \\
[F_2^{(2n+1/2)}, K_2^{(2m+1)}] &= -F_1^{2(n+m)+3/2}, \\
\{F_1^{(2n+3/2)}, F_1^{(2m+3/2)}\} &= 2(K_2^{2(n+m+1)+1} + K_1^{2(n+m+1)+1}), \\
\{F_2^{(2n+1/2)}, F_2^{(2m+1/2)}\} &= -2(K_2^{2(n+m)+1} + K_1^{2(n+m)+1}), \\
\{F_2^{(2n+1/2)}, G_1^{(2m+1/2)}\} &= 2M_1^{2(n+m)+1}, \\
\{F_1^{(2n+3/2)}, G_2^{(2m+3/2)}\} &= -2M_1^{2(n+m+1)+1}, \\
\{F_1^{(2n+3/2)}, G_1^{(2m+1/2)}\} &= 2M_2^{2(n+m+1)} + [(2n+1) + 2m]\delta_{n+m+1,0}\hat{c}, \\
\{F_2^{(2n+1/2)}, G_2^{(2m+3/2)}\} &= -2M_2^{2(n+m+1)} - [2n + (2m+1)]\delta_{n+m+1,0}\hat{c}, \\
[M_1^{(2n+1)}, F_1^{(2m+3/2)}] &= G_1^{2(n+m+1)+1/2}, \\
[M_1^{(2n+1)}, F_2^{(2m+1/2)}] &= G_2^{2(n+m)+3/2}, \\
[M_2^{(2n)}, F_1^{(2m+3/2)}] &= -G_2^{2(n+m)+3/2}, \\
[M_2^{(2n)}, F_2^{(2m+1/2)}] &= -G_1^{2(n+m)+1/2}, \\
[M_1^{(2n+1)}, K_1^{(2m+1)}] &= 2M_2^{2(n+m+1)} + (n+m)\delta_{n+m+1,0}\hat{c},
\end{aligned}
$$

$$
\begin{aligned}
[M_1^{(2n+1)}, K_2^{(2m+1)}] &= 0, \\
[K_2^{(2n+1)}, K_2^{(2m+1)}] &= -(n-m)\delta_{n+m+1,0}\hat{c}, \\
[M_2^{(2n)}, K_1^{(2m+1)}] &= 2M_1^{2(n+m)+1}, \\
[M_2^{(2n)}, K_2^{(2m+1)}] &= 0, \\
[G_1^{(2n+1/2)}, K_1^{(2m+1)}] &= -G_2^{2(n+m)+3/2}, \\
[G_1^{(2n+1/2)}, K_2^{(2m+1)}] &= -G_2^{2(n+m)+3/2}, \\
[G_2^{(2n+3/2)}, K_1^{(2m+1)}] &= -G_1^{2(n+m+1)+1/2}, \\
[G_2^{(2n+3/2)}, K_2^{(2m+1)}] &= -G_1^{2(n+m+1)+1/2}, \\
\{G_1^{(2n+1/2)}, G_2^{(2m+3/2)}\} &= [2n-(2m+1)]\delta_{n+m+1,0}\hat{c}, \\
\{G_1^{(2n+1/2)}, G_1^{(2m+1/2)}\} &= 2(K_2^{2(n+m)+1} - K_1^{2(n+m)+1}), \\
\{G_2^{(2n+3/2)}, G_2^{(2m+3/2)}\} &= -2(K_2^{2(n+m+1)+1} - K_1^{2(n+m+1)+1}), \\
[M_1^{(2n+1)}, G_1^{(2m+1/2)}] &= -F_1^{2(n+m)+3/2}, \\
[M_1^{(2n+1)}, G_2^{(2m+3/2)}] &= -F_2^{2(n+m+1)+1/2}, \\
[M_2^{(2n)}, G_1^{(2m+1/2)}] &= -F_2^{2(n+m)+1/2}, \\
[M_2^{(2n)}, G_2^{(2m+3/2)}] &= -F_1^{2(n+m)+3/2}, \\
[M_1^{(2n+1)}, M_2^{(2m)}] &= -2K_1^{2(n+m)+1}, \\
[M_1^{(2n+1)}, M_1^{(2m+1)}] &= -(n-m)\delta_{n+m+1,0}\hat{c}, \\
[M_2^{(2n)}, M_2^{(2m)}] &= (n-m)\delta_{n+m,0}\hat{c}, \\
[K_1^{(2n+1)}, K_1^{(2m+1)}] &= (n-m)\delta_{n+m+1,0}\hat{c}. \qquad \text{(C.1)}
\end{aligned}
$$

onde

$$
\begin{aligned}
F_1^{(2n+3/2)} &= (E_{\alpha_1+\alpha_2}^{(n+1/2)} - E_{\alpha_2}^{(n+1)}) + (E_{-\alpha_1-\alpha_2}^{(n+1)} - E_{-\alpha_2}^{(n+1/2)}), \\
F_2^{(2n+1/2)} &= -(E_{\alpha_1+\alpha_2}^{(n)} - E_{\alpha_2}^{(n+1/2)}) + (E_{-\alpha_1-\alpha_2}^{(n+1/2)} - E_{-\alpha_2}^{(n)}), \\
G_1^{(2n+1/2)} &= (E_{\alpha_1+\alpha_2}^{(n)} + E_{\alpha_2}^{(n+1/2)}) + (E_{-\alpha_1-\alpha_2}^{(n+1/2)} + E_{-\alpha_2}^{(n)}), \\
G_2^{(2n+3/2)} &= -(E_{\alpha_1+\alpha_2}^{(n+1/2)} + E_{\alpha_2}^{(n+1)}) + (E_{-\alpha_1-\alpha_2}^{(n+1)} + E_{-\alpha_2}^{(n+1/2)}), \\
K_1^{(2n+1)} &= -E_{-\alpha_1}^{(n+1)} - E_{\alpha_1}^{(n)}, \\
K_2^{(2n+1)} &= h_+^{(n+1/2)} + h_2^{(n+1/2)}, \\
M_1^{(2n+1)} &= E_{-\alpha_1}^{(n+1)} - E_{\alpha_1}^{(n)}, \\
M_2^{(2n)} &= h_1^{(n)}. \qquad \text{(C.2)}
\end{aligned}
$$

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A class of mixed integrable models

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Abstract
The algebraic structure of the integrable mixed mKdV/sinh-Gordon model is discussed and extended to the AKNS/Lund–Regge model and to its corresponding supersymmetric versions. The integrability of the models is guaranteed from the zero curvature representation and some soliton solutions are discussed.

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1. Introduction

The mKdV and sine-Gordon equations are nonlinear differential equations belonging to the same integrable hierarchy representing different time evolutions [1]. The structure of its soliton solutions present the same functional form in terms of

$$\rho = e^{kx+k^n t_n}, \tag{1.1}$$

which carries the spacetime dependence. Solutions of different equations within the same hierarchy differ only by the factor $k^n t_n$ in ρ. For instance $n = 3$ corresponds to the mKdV equation and $n = -1$ to the sinh-Gordon. For $n > 0$ a systematic construction of integrable hierarchies can be solved and classified according to a decomposition of an affine Lie algebra, $\hat{\mathcal{G}}$ and a choice of a semi-simple constant element E (see [2] for review). Such a framework was shown to be derived from the Riemann–Hilbert decomposition which was later shown to incorporate negative grade isospectral flows $n < 0$ [3] as well.

The mixed system

$$\phi_{xt} = \frac{\alpha_3}{4}\left(\phi_{xxxx} - 6\phi_x^2\phi_{xx}\right) + 2\eta\sinh(2\phi) \tag{1.2}$$

is a nonlinear differential equation which represents the well-known mKdV equation for $\eta = 0$ $(v = -\partial_x\phi)$ and the sinh-Gordon equation for $\alpha_3 = 0$. It was introduced in [4] where, employing the inverse scattering method, multi-soliton solutions were constructed by modification of time dependence in ρ. Solutions (multi-soliton) were also considered in [5] by Hirota's method. Moreover, a two-breather solution was discussed in [6] in connection with few-optical-cycle pulses in transparent media. The soliton solutions obtained in [4–6] indicates integrability of the mixed model (1.2).

In this paper, we consider the mixed system mKdV/sinh-Gordon (1.2) within the zero curvature representation. We show that a systematic solution for the mixed model is obtained by the dressing method and a specific choice of vacuum solution. Such formalism is extended to the mixed AKNS/Lund–Regge and to its supersymmetric versions as well.

In the last section, we discuss the coupling of higher positive and negative flows generalizing the examples given previously.

2. The mixed mKdV/sinh-Gordon model

Let us consider a nonlinear system composed of a mixed sinh-Gordon and mKdV equation given by equation (1.2) and the following zero curvature representation,

$$\left[\partial_x + E^{(1)} + A_0, \partial_t + D_3^{(3)} + D_3^{(2)} + D_3^{(1)} + D_3^{(0)} + D_3^{(-1)}\right] = 0 \tag{2.1}$$

where $E^{(2n+1)} = \lambda^n(E_\alpha + \lambda E_{-\alpha})$, $A_0 = vh$ and $E_{\pm\alpha}$ and h are $sl(2)$ generators satisfying $[h, E_{\pm\alpha}] = \pm 2E_{\pm\alpha}$, $\quad [E_\alpha, E_{-\alpha}] = h$. According to the grading operator $Q = 2\lambda\frac{\mathrm{d}}{\mathrm{d}\lambda} + \frac{1}{2}h$, $D_3^{(j)}$ is a graded j Lie algebra valued and equation (2.1) decomposes into six independent equations (decomposing grade by grade):

$$\begin{aligned}
&\left[E, D_3^{(3)}\right] = 0,\\
&\left[E, D_3^{(2)}\right] + \left[A_0, D_3^{(3)}\right] + \partial_x D_3^{(3)} = 0,\\
&\left[E, D_3^{(1)}\right] + \left[A_0, D_3^{(2)}\right] + \partial_x D_3^{(2)} = 0,\\
&\left[E, D_3^{(0)}\right] + \left[A_0, D_3^{(1)}\right] + \partial_x D_3^{(1)} = 0,\\
&\left[E, D_3^{(-1)}\right] + \left[A_0, D_3^{(0)}\right] + \partial_x D_3^{(0)} - \partial_t A_0 = 0,\\
&\left[A_0, D_3^{(-1)}\right] + \partial_x D_3^{(-1)} = 0.
\end{aligned} \tag{2.2}$$

where $E \equiv E^{(1)}$. In order to solve (2.2) let us propose

$$\begin{aligned}
D_3^{(3)} &= \alpha_3(\lambda E_\alpha + \lambda^2 E_{-\alpha}) + \beta_3(\lambda E_\alpha - \lambda^2 E_{-\alpha}),\\
D_3^{(2)} &= \sigma_2\lambda h,\\
D_3^{(1)} &= \alpha_1(E_\alpha + \lambda E_{-\alpha}) + \beta_1(E_\alpha - \lambda E_{-\alpha}),\\
D_3^{(0)} &= \sigma_0 h.
\end{aligned} \tag{2.3}$$

Substituting (2.3) into (2.2) we obtain $\beta_3 = 0$, $\quad \alpha_3 = \text{const}$ and

$$\beta_1 = \frac{\alpha_3}{2}v_x, \qquad \alpha_1 = -\frac{\alpha_3}{2}v^2, \qquad \sigma_0 = \frac{\alpha_3}{4}(v_{xx} - 2v^3), \qquad \sigma_2 = \alpha_3 v. \tag{2.4}$$

In order to solve the last equation in (2.2) we parametrize

$$A_0 = -\partial_x B B^{-1} = -\partial_x\phi h, \qquad B = \mathrm{e}^{\phi h} \tag{2.5}$$

and

$$D_3^{(-1)} = \eta B E^{(-1)} B^{-1} = \eta\lambda^{-1}(\mathrm{e}^{2\phi}E_\alpha + \lambda\,\mathrm{e}^{-2\phi}E_{-\alpha}). \tag{2.6}$$

The zero grade projection in (2.2) yields the time evolution equation (1.2). Note that in order to solve the last equation (2.3) we have introduced the sinh-Gordon variable ϕ in (2.5) and (2.6) such that $v = -\partial_x\phi$.

Let us now recall some basic aspects of the dressing method which provides systematic construction of soliton solutions. The zero curvature representation implies in a pure gauge configuration. In particular, the vacuum is obtained by setting $\phi_{\text{vac}} = 0$ or $v_{\text{vac}} = 0$ which, when in (2.1) implies

$$\partial_x T_0 T_0^{-1} = E^{(1)}, \qquad \partial_t T_0 T_0^{-1} = \alpha_3 E^{(3)} + \eta E^{(-1)} \tag{2.7}$$

and after integration

$$T_0 = \exp(t(\alpha_3 E^{(3)} + \eta E^{(-1)})) \exp(x E^{(1)}), \qquad E^{(2n+1)} = \lambda^n (E_\alpha + \lambda E_{-\alpha}). \tag{2.8}$$

If we identify $v = -\partial_x \phi$ equation (1.2) represents a coupling of mKdV and sinh-Gordon equations and becomes a pure mKdV when $\eta = 0$ and pure sinh-Gordon when $\alpha_3 = 0$. Tracing back those two limits from (2.4) and (2.6) it becomes clear that the sinh-Gordon limit ($\eta = 0$) in (1.2) is responsible for the vanishing of $D_3^{(-1)}$. On the other hand, $\alpha_3 = 0$ implies $D_3^{(j)} = 0$, $j = 0, \ldots, 3$. Inspired by the dressing method for constructing soliton solutions of integrable hierarchies (see for instance [7]) and the fact that the nth member of the hierarchy is associated with the time evolution parameter $k_i^n t_n$ ($n = 3$ for mKdV and $n = -1$ for sinh-Gordon) it is natural to propose soliton solutions based on the modified spacetime dependence

$$\rho_i = \exp\left(2k_i x + 2\left(\alpha_3 k_i^3 + \eta/k_i\right) t\right). \tag{2.9}$$

It therefore follows that the general structure of the 1-, 2- and 3-soliton solutions is respectively given by (after $\phi \to \mathrm{i}\phi$)

$$\begin{aligned}
\phi_{1\text{-sol}} &= \mathrm{i}ln\left(\frac{1 - a_1\rho_1}{1 + a_1\rho_1}\right), \\
\phi_{2\text{-sol}} &= \mathrm{i}ln\left(\frac{1 - a_1\rho_1 - a_2\rho_2 + a_1a_2a_{12}\rho_1\rho_2}{1 + a_1\rho_1 + a_2\rho_2 + a_1a_2a_{12}\rho_1\rho_2}\right), \\
\phi_{3\text{-sol}} &= \mathrm{i}ln\left(\frac{1 - \sum_{i=1}^3 a_i\rho_i + \sum_{i<j=1}^3 a_ia_ja_{ij}\rho_i\rho_j - a_1a_2a_3a_{12}a_{13}a_{23}\rho_1\rho_2\rho_3}{1 + \sum_{i=1}^3 a_i\rho_i + \sum_{i<j=1}^3 a_ia_ja_{ij}\rho_i\rho_j + a_1a_2a_3a_{12}a_{13}a_{23}\rho_1\rho_2\rho_3}\right)
\end{aligned} \tag{2.10}$$

where a_1, a_2 are constants and $a_{ij} = \left(\frac{k_i - k_j}{k_i + k_j}\right)^2$.

More general solutions (N-solitons and breathers) were found in [4–6] with same time dependence as in (2.9).

3. The mixed AKNS/Lund–Regge model

Let us consider another example involving $\mathcal{G} = \hat{sl}(2)$ and homogeneous gradation $Q = \lambda\frac{\mathrm{d}}{\mathrm{d}\lambda}$, $E^{(n)} = \lambda^n h$, $E = E^{(1)}$ and $A_0 = qE_\alpha + rE_{-\alpha}$ and the zero curvature representation of the form

$$\left[\partial_x + E + A_0, \partial_t + D_2^{(2)} + D_2^{(1)} + D_2^{(0)} + D_2^{(-1)}\right] = 0. \tag{3.1}$$

According to gradation Q, propose

$$D_2^{(j)} = \lambda^j(\alpha_j E_\alpha + \beta_j E_{-\alpha} + \sigma_j h), \qquad j = -1, 0, 1, 2 \tag{3.2}$$

In order to find solution for (3.1) we introduce variables $\tilde{\psi}$ and $\tilde{\chi}$ [8],

$$A_0 = qE_\alpha + rE_{-\alpha} = -\partial_x BB^{-1}, \qquad D_2^{(-1)} = \eta BE^{(-1)}B^{-1}, \qquad B = \mathrm{e}^{\tilde{\chi}E_{-\alpha}}\,\mathrm{e}^{\phi h}\,\mathrm{e}^{\tilde{\psi}E_\alpha} \tag{3.3}$$

which defines

$$q = -\partial_x\tilde{\psi}\,\mathrm{e}^{2\phi}, \qquad r = \tilde{\chi}^2\partial_x\tilde{\psi}\,\mathrm{e}^{2\phi} - \partial_x\tilde{\chi} \tag{3.4}$$

together with the subsidiary conditions for the non-local auxiliary field ϕ,

$$Tr(\partial_x B B^{-1} h) = \partial_x \phi - \tilde{\chi} \partial_x \tilde{\psi}\, e^{2\phi} = 0, \qquad Tr(B^{-1} \partial_t B h) = \partial_t \phi - \tilde{\psi} \partial_t \tilde{\chi}\, e^{2\phi} = 0. \quad (3.5)$$

Solution of constraints (3.5) leads to natural variables [9]

$$\psi = \tilde{\psi}\, e^{\phi}, \qquad \chi = \tilde{\chi}\, e^{\phi}. \quad (3.6)$$

Inserting (3.2) into (3.1) and collecting powers of λ, we find solution in terms of non-local fields ψ and χ

$$\begin{array}{llll} \sigma_2 = \text{const}, & \beta_2 = \alpha_2 = 0, & \sigma_1 = 0, & \sigma_0 = -1/2\sigma_2 r q \\ \beta_1 = \sigma_2 r, & \alpha_1 = \sigma_2 q, & \alpha_0 = -1/2\sigma_2 q_x, & \beta_0 = 1/2\sigma_2 r_x, \\ \alpha_{-1} = -2\eta\psi\, e^{\phi}, & \beta_{-1} = 2\eta(\chi + \psi\chi^2)\, e^{-\phi}, & \sigma_{-1} = \eta(1 + 2\psi\chi) & \end{array} \quad (3.7)$$

leading to the equations of motion

$$q_t + \tfrac{1}{2}\sigma_2(q_{xx} - 2q^2 r) - 2\alpha_{-1} = 0, \qquad r_t - \tfrac{1}{2}\sigma_2(r_{xx} - 2r^2 q) + 2\beta_{-1} = 0, \quad (3.8)$$

where q and r in variables ψ and χ reads

$$q = -\frac{\partial_x \psi}{1 + \psi\chi}\, e^{\phi} \qquad r = -\partial_x \chi\, e^{-\phi}. \quad (3.9)$$

Equations (3.8) represent a mixed system of AKNS (for $\eta = 0, \alpha_{-1} = \beta_{-1} = 0$) in variables q, r and the relativistic Lund–Regge (for $\sigma_2 = 0$) in variables ψ, χ.

$$\partial_t \left(\frac{\partial_x \psi}{\Delta}\right) + \psi \frac{\partial_t \chi \partial_x \psi}{\Delta^2} + 4\eta\psi = 0, \qquad \partial_x \left(\frac{\partial_t \chi}{\Delta}\right) + \chi \frac{\partial_t \chi \partial_x \psi}{\Delta^2} + 4\eta\chi = 0. \quad (3.10)$$

Again the terms proportional to α_{-1} and β_{-1} originate from the contribution of $D_2^{(-1)} = \eta B E^{(-1)} B^{-1}$ in (3.1) and the vacuum configuration is obtained for $\psi_{\text{vac}} = \chi_{\text{vac}} = q_{\text{vac}} = r_{\text{vac}} = 0$. The model is now characterized by $E^{(n)} = \lambda^n h$ and the vacuum solution of (3.1) yield

$$T_0 = \exp(t(\sigma_2 E^{(2)} + \eta E^{(-1)})) \exp(x E^{(1)}). \quad (3.11)$$

and therefore the spacetime dependence in ρ_i comes in the form

$$\rho_i = \exp\left(2k_i x + 2\left(\sigma_2 k_i^2 + \eta/k_i\right) t\right). \quad (3.12)$$

We have checked the solution for the composite model (3.8) to agree with the functional form of the one proposed in [9] with modified spacetime dependence given by (3.12), i.e.,

$$\psi = \frac{b\rho_2}{1 + \frac{k_1}{k_2}\Gamma\rho_1^{-1}\rho_2}, \qquad \chi = \frac{a\rho_1^{-1}}{1 + \frac{k_1}{k_2}\Gamma\rho_1^{-1}\rho_2}, \qquad e^{-\phi} = \frac{1 + \frac{k_1}{k_2}\Gamma\rho_1^{-1}\rho_2}{1 + \Gamma\rho_1^{-1}\rho_2} \quad (3.13)$$

where a and b are constants, $\Gamma = \frac{abk_2^2}{(k_1 - k_2)}$. In terms of AKNS field variables, from (3.9) we find

$$r = -\frac{2ak_1\rho_1^{-1}}{1 + \frac{abk_1k_2}{(k_1-k_2)^2}\rho_1^{-1}\rho_2}, \qquad q = \frac{2bk_2\rho_2}{1 + \frac{abk_1k_2}{(k_1-k_2)^2}\rho_1^{-1}\rho_2}. \quad (3.14)$$

4. The supersymmetric mKdV/sinh-Gordon model

Following the same line of reasoning, we now consider algebraic structures with half integer gradation [10]. Let $\hat{\mathcal{G}} = \hat{sl}(2,1)$, $Q = 2\lambda\frac{d}{d\lambda} + \frac{1}{2}h$ and $E^{(1)} = \lambda^{1/2}(h_1 + 2h_2) - (E_{\alpha_1} + \lambda E_{-\alpha_1})$. The graded structure can be decomposed as follows (see the appendix of [11]) for instance),

$$\begin{aligned}
\mathcal{K}_{\rm Bose} &= \left\{K_1^{(2n+1)} = -\left(E_{\alpha_1}^{(n)} + E_{-\alpha_1}^{(n+1)}\right), K_2^{(2n+1)} = \mu_2 \cdot H^{(n+1/2)}\right\},\\
\mathcal{M}_{\rm Bose} &= \left\{M_1^{(2n+1)} = -E_{\alpha_1}^{(n)} + E_{-\alpha_1}^{(n+1)}, M_2^{(2n)} = h_1^{(n)} = \alpha_1 \cdot H^{(n)}\right\},\\
\mathcal{K}_{\rm Fermi} &= \left\{F_1^{(2n+3/2)} = \left(E_{\alpha_1+\alpha_2}^{(n+1/2)} - E_{\alpha_2}^{(n+1)}\right) + \left(E_{-\alpha_1-\alpha_2}^{(n+1)} - E_{-\alpha_2}^{(n+1/2)}\right),\right.\\
&\qquad \left. F_2^{(2n+1/2)} = -\left(E_{\alpha_1+\alpha_2}^{(n)} - E_{\alpha_2}^{(n+1/2)}\right) + \left(E_{-\alpha_1-\alpha_2}^{(n+1/2)} - E_{-\alpha_2}^{(n)}\right)\right\},\\
\mathcal{M}_{\rm Fermi} &= \left\{G_1^{(2n+1/2)} = \left(E_{\alpha_1+\alpha_2}^{(n)} + E_{\alpha_2}^{(n+1/2)}\right) + \left(E_{-\alpha_1-\alpha_2}^{(n+1/2)} + E_{-\alpha_2}^{(n)}\right),\right.\\
&\qquad \left. G_2^{(2n+3/2)} = -\left(E_{\alpha_1+\alpha_2}^{(n+1/2)} + E_{\alpha_2}^{(n+1)}\right) + \left(E_{-\alpha_1-\alpha_2}^{(n+1)} + E_{-\alpha_2}^{(n+1/2)}\right)\right\},
\end{aligned} \tag{4.1}$$

where we have denoted $E_{\pm\alpha}^{(n)} = \lambda^n E_{\pm\alpha}$ and $H^{(n)} = \lambda^n H$ and $\alpha_i, \mu_i, i = 1, 2$ are respectively the simple roots and fundamental weights of $sl(2,1)$. In (4.1) we have denoted $\mathcal{K} = \mathcal{K}_{\rm Bose} \cup \mathcal{K}_{\rm Fermi}$ to be the Kernel of $E^{(1)}$, i.e., $[E^{(1)}, \mathcal{K}] = 0$ and $\mathcal{M}$ is its complement. The Lax operator is constructed as

$$L = \partial_x + E^{(1)} + A_{1/2} + A_0, \qquad A_0 = vM_2^{(0)}, \qquad A_{1/2} = \bar{\psi} G_1^{(1/2)}, \tag{4.2}$$

and the zero curvature representation reads

$$\left[\partial_x + E^{(1)} + A_{1/2} + A_0, \partial_t + D_3^{(3)} + D_3^{(5/2)} + \cdots + D_3^{(-1/2)} + D_3^{(-1)}\right] = 0. \tag{4.3}$$

In order to solve for the lowest grades $-1, -1/2$ of equation (4.3) we introduce the parametrization

$$D_3^{(-1)} = \eta B E^{(-1)} B^{-1}, \qquad A_0 = -\partial_x B B^{-1}, \qquad B = e^{\phi M_2^{(0)}} \tag{4.4}$$

together with the change of variables

$$D_3^{(-1/2)} = B j_{-1/2} B^{-1}, \qquad j_{-1/2} = \psi G_2^{(-1/2)}. \tag{4.5}$$

We propose the solution of the form

$$\begin{aligned}
D_3^{(3)} &= \alpha_3\left(h_1^{(3/2)} + 2h_2^{(3/2)} - E_{\alpha_1}^{(1)} - E_{-\alpha_1}^{(2)}\right),\\
D_3^{(0)} &= \alpha_1 M_2^{(0)},\\
D_3^{(1/2)} &= \beta_1 G_1^{(1/2)} + \beta_2 F_2^{(1/2)},\\
D_3^{(1)} &= \sigma_1 M_1^{(1)} + \sigma_2 K_1^{(1)} + \sigma_3 K_2^{(1)},\\
D_3^{(3/2)} &= \delta_1 G_2^{(3/2)} + \delta_2 F_1^{(3/2)},\\
D_3^{(2)} &= \mu_1 M_2^{(2)},\\
D_3^{(5/2)} &= \nu_1 G_1^{(5/2)} + \nu_2 F_2^{(5/2)},\\
D_3^{(-1/2)} &= \beta_{-1} G_1^{(-1/2)} + \beta_{-2} F_1^{(-1/2)},\\
D_3^{(-1)} &= \sigma_{-1} M_1^{(-1)} + \sigma_{-2} K_1^{(-1)} + \sigma_{-3} K_2^{(-1)}.
\end{aligned} \tag{4.6}$$

where the coefficients are given by

$$\begin{aligned}
&\alpha_1 = \tfrac{1}{4}\partial_x^2 v + \tfrac{3}{4} v\bar\psi\partial_x\bar\psi - \tfrac{1}{2}v^3, \qquad \beta_1 = \tfrac{1}{4}\partial_x^2\bar\psi - \tfrac{1}{2}v^2\bar\psi, \qquad \beta_2 = \tfrac{1}{4}(v\partial_x\bar\psi - \bar\psi\partial_x v),\\
&\sigma_1 = \tfrac{1}{2}\partial_x v, \qquad \sigma_2 = \tfrac{1}{2}(\bar\psi\partial_x\bar\psi - v^2), \qquad \sigma_3 = -\tfrac{1}{2}\bar\psi\partial_x\psi \qquad \delta_1 = -\tfrac{1}{2}\partial_x\bar\psi,\\
&\delta_2 = -\tfrac{1}{2}v\bar\psi, \qquad \mu_1 = v, \qquad \nu_1 = \bar\psi, \qquad \nu_2 = 0, \qquad \beta_{-1} = \psi\cosh\phi,\\
&\beta_{-2} = -\psi\sinh\phi, \qquad \sigma_{-1} = \eta\sinh 2\phi, \qquad \sigma_{-2} = \eta\cosh 2\phi, \qquad \sigma_{-3} = \eta,
\end{aligned} \tag{4.7}$$

where α_3 and η are arbitrary constants. The equations of motion are given by grades 0, $\pm 1/2$ projections of (4.3), i.e.,

$$\begin{aligned}
\partial_t\partial_x\phi &= \frac{\alpha_3}{4}\left[\partial_x^4\phi - 6(\partial_x\phi)^2\partial_x^2\phi + 3\bar\psi\partial_x(\partial_x\phi\partial_x\bar\psi)\right] + 2\eta[\sinh(2\phi) + \bar\psi\psi\sinh(\phi)],\\
\partial_{t_3}\bar\psi &= \frac{\alpha_3}{4}\left[\partial_x^3\bar\psi - 3\partial_x\phi\partial_x(\partial_x\phi\bar\psi)\right] + 2\eta\psi\cosh(\phi),\\
\partial_x\psi &= 2\bar\psi\cosh(\phi).
\end{aligned} \tag{4.8}$$

Observe that for $\eta = 0$ equations (4.8) corresponds to the $N = 1$ super mKdV equation if we identify $v = -\partial_x\phi$ and for $\alpha_3 = 0$ they correspond to the $N = 1$ super sinh-Gordon.

The soliton solutions are parametrized in terms of tau functions as

$$\phi = ln\left(\frac{\tau_1}{\tau_0}\right), \qquad \bar\psi = \frac{\tau_3}{\tau_1} + \frac{\tau_2}{\tau_0}. \tag{4.9}$$

The one-soliton solution for the $N = 1$ super sinh-Gordon and mKdV equations is given by

$$\begin{aligned}
&\tau_0 = 1 - \tfrac{1}{2}b_1\rho_1, \qquad \tau_1 = 1 + \tfrac{1}{2}b_1\rho_1,\\
&\tau_2 = c_1k_2\rho_2^{-1} + b_1c_1\sigma_{1,2}\rho_1\rho_2^{-1}, \qquad \tau_3 = c_1k_2\rho_2^{-1} - b_1c_1\sigma_{1,2}\rho_1\rho_2^{-1},
\end{aligned} \tag{4.10}$$

where $\sigma_{1,2} = \frac{1}{2}k_2\frac{(k_1+k_2)}{(k_1-k_2)}$, b_1, c_1 are bosonic and Grassmaniann constants respectively and ρ_i carries the spacetime dependence for the sinh-Gordon and mKdV, respectively,

$$\rho_i^{\text{mKdV}} = \exp\left(2k_ix + 2\left(\alpha_3k_i^3\right)t\right), \qquad \rho_i^{s-G} = \exp\left(2k_ix + 2\left(\frac{\eta}{k_i}\right)t\right). \tag{4.11}$$

Note however that the introduction of the $D_{-1}^{(-1/2)}$ and $D_{-1}^{(-1)}$ terms changes the vacuum configuration such that

$$T_0 = \exp(xE^{(1)})\exp(\alpha_3E^{(3)} + \eta E^{(-1)})t) \tag{4.12}$$

which induces modification in the spacetime dependence of equations (4.8) as

$$\rho_i = \exp(2k_ix)\exp\left(2\left(\alpha_3k_i^3 + \frac{\eta}{k_i}\right)t\right). \tag{4.13}$$

In fact we have verified explicitly that (4.10) with (4.13) satisfies the equations of motion (4.8). The same was verified for the two soliton solution

$$\begin{aligned}
\tau_0 &= 1 - \tfrac{1}{2}b_1\rho_1 - \tfrac{1}{2}b_2\rho_2 + b_1b_2\rho_1\rho_2\alpha_{1,2}\\
&\quad + c_1c_2\rho_3^{-1}\rho_4^{-1}(\beta_{3,4} - b_1\rho_1\delta_{1,3,4} - b_2\rho_2\delta_{2,3,4} + b_1b_2\rho_1\rho_2\theta_{1,2,3,4}),\\
\tau_1 &= 1 + \tfrac{1}{2}b_1\rho_1 + \tfrac{1}{2}b_2\rho_2 + b_1b_2\rho_1\rho_2\alpha_{1,2}\\
&\quad + c_1c_2\rho_3^{-1}\rho_4^{-1}(\beta_{3,4} + b_1\rho_1\delta_{1,3,4} + b_2\rho_2\delta_{2,3,4} + b_1b_2\rho_1\rho_2\theta_{1,2,3,4}),\\
\tau_2 &= c_1\rho_3^{-1}(k_3 + b_1\rho_1\sigma_{1,3} + b_2\rho_2\sigma_{2,3} + b_1b_2\rho_1\rho_2\lambda_{1,2,3})\\
&\quad + c_2\rho_4^{-1}(k_4 + b_1\rho_1\sigma_{1,4} + b_2\rho_2\sigma_{2,4} + b_1b_2\rho_1\rho_2\lambda_{1,2,4}),\\
\tau_3 &= c_1\rho_3^{-1}(k_3 - b_1\rho_1\sigma_{1,3} - b_2\rho_2\sigma_{2,3} + b_1b_2\rho_1\rho_2\lambda_{1,2,3})\\
&\quad + c_2\rho_4^{-1}(k_4 - b_1\rho_1\sigma_{1,4} - b_2\rho_2\sigma_{2,4} + b_1b_2\rho_1\rho_2\lambda_{1,2,4}),
\end{aligned} \tag{4.14}$$

where

$$\alpha_{1,2} = \frac{1}{4}\frac{(k_1-k_2)^2}{(k_1+k_2)^2}, \qquad \beta_{3,4} = k_3k_4\frac{(k_3-k_4)}{(k_3+k_4)^2},$$
$$\delta_{j,3,4} = \frac{k_3k_4}{2}\frac{(k_3-k_4)}{(k_3+k_4)^2}\frac{(k_j+k_3)}{(k_j-k_3)}\frac{(k_j+k_4)}{(k_j-k_4)} \qquad (j=1,2),$$
$$\sigma_{j,k} = \frac{k_k}{2}\frac{(k_j+k_k)}{(k_j-k_k)} \qquad (j=1,2) \qquad (k=3,4), \tag{4.15}$$
$$\lambda_{1,2,j} = \frac{k_j}{4}\frac{(k_1-k_2)^2}{(k_1+k_2)^2}\frac{(k_1+k_j)}{(k_1-k_j)}\frac{(k_2+k_j)}{(k_2-k_j)}, \qquad (j=3,4),$$
$$\theta_{1,2,3,4} = \frac{k_3k_4}{4}\frac{(k_1-k_2)^2}{(k_1+k_2)^2}\frac{(k_1+k_3)}{(k_1-k_3)}\frac{(k_2+k_3)}{(k_2-k_3)}\frac{(k_3-k_4)}{(k_3+k_4)^2}\frac{(k_1+k_4)}{(k_1-k_4)}\frac{(k_2+k_4)}{(k_2-k_4)},$$

b_1, b_2 are bosonic constants and c_1, c_2 are Grassmaniann constants with ρ_i given by (4.13).

5. The supersymmetric Lund–Regge/AKNS model

In this section we consider the Lie superalgebra $\hat{\mathcal{G}} = \hat{sl}(2,1)$ with homogeneous gradation, $Q = \lambda\frac{\mathrm{d}}{\mathrm{d}\lambda}$ and (see for instance [12])

$$E^{(n)} = (\alpha_1+\alpha_2)\cdot H^{(n)}, \qquad \alpha_1, \alpha_2 \quad \text{are simple roots of sl(2,1).} \tag{5.1}$$

The Lax operator is then

$$L = \partial_x + E^{(1)} + A_0, \qquad A_0 = b_1E_{\alpha_1} + \bar{b}_1E_{-\alpha_1} + F_1E_{\alpha_2} + \bar{F}_1E_{-\alpha_2}. \tag{5.2}$$

We search for the solution of

$$\left[\partial_x + E^{(1)} + A_0, \partial_t + D_2^{(2)} + D_2^{(1)} + D_2^{(0)} + D_2^{(-1)}\right] = 0. \tag{5.3}$$

Decomposing (5.3) grade by grade, we find

$$\begin{aligned} D_2^{(2)} &= a_2\lambda^2\alpha_1\cdot H, \\ D_2^{(1)} &= g_1\lambda E_{\alpha_1} + m_1\lambda E_{-\alpha_1} + n_1\lambda E_{-\alpha_2} + o_1\lambda E_{\alpha_2} \\ D_{2\mathcal{M}}^{(0)} &= g_0E_{\alpha_1} + m_0E_{-\alpha_1} + n_0E_{-\alpha_2} + o_0E_{\alpha_2}, \\ D_{2\mathcal{K}}^{(0)} &= a_0\alpha_1\cdot H + c_0\alpha_2\cdot H + d_0E_{\alpha_1+\alpha_2} + e_0E_{-\alpha_1-\alpha_2}. \end{aligned} \tag{5.4}$$

where $D_2^{(0)} = D_{2\mathcal{M}}^{(0)} + D_{2\mathcal{K}}^{(0)}$ and

$$\begin{aligned} &g_1 = a_2b_1 && m_1 = a_2\bar{b}_1, && o_1 = a_2F_1, && n_1 = a_2\bar{F}_1, \\ &g_0 = a_2\partial_xb_1, && m_0 = -a_2\partial_x\bar{b}_1, && n_0 = a_2\partial_x\bar{F}_1, && o_0 = -a_2\partial_xF_1, \\ &d_0 = -a_2F_1b_1, && e_0 = -a_2\bar{F}_1\bar{b}_1, && a_0 = -a_2b_1\bar{b}_1, && c_0 = -a_2(b_1\bar{b}_1 + F_1\bar{F}_1). \end{aligned}$$

In order to solve the grade -1 projection of equation (5.3) we introduce the $sl(2,1)$ variables [12] as

$$A_0 = -\partial_xBB^{-1} = b_1E_{\alpha_1} + \bar{b}_1E_{-\alpha_1} + F_1E_{\alpha_2} + \bar{F}_1E_{-\alpha_2}, \tag{5.5}$$

where

$$B = \mathrm{e}^{\tilde{\chi}E_{-\alpha_1}}\,\mathrm{e}^{\tilde{f}_1E_{-\alpha_1-\alpha_2}}\,\mathrm{e}^{\tilde{f}_2E_{\alpha_2}}\,\mathrm{e}^{\varphi_1(\alpha_1+\alpha_2)\cdot H-\varphi_2\alpha_2\cdot H}\,\mathrm{e}^{\tilde{g}_2E_{-\alpha_2}}\,\mathrm{e}^{\tilde{g}_1E_{\alpha_1+\alpha_2}}\,\mathrm{e}^{\tilde{\psi}E_{\alpha_1}} \tag{5.6}$$

and

$$\begin{aligned} D_{2\mathcal{M}}^{(-1)} = \eta BE^{(-1)}B^{-1} = &-\eta\psi\,\mathrm{e}^{\frac{1}{2}(\phi_1+\phi_2)}\lambda^{-1}E_{\alpha_1} + \eta f_2(1+\psi\chi)\,\mathrm{e}^{-\frac{1}{2}\phi_1}\lambda^{-1}E_{\alpha_2}, \\ &+\eta(\chi + f_1f_2 + \psi\chi^2 + \psi\chi f_1f_2)\,\mathrm{e}^{-\frac{1}{2}(\phi_1+\phi_2)}\lambda^{-1}E_{-\alpha_1} \\ &-\eta(g_2+\psi f_1)\,\mathrm{e}^{\frac{1}{2}\phi_1}\lambda^{-1}E_{-\alpha_2} \end{aligned} \tag{5.7}$$

written in the natural variables

$$\tilde{\psi} = \psi\, e^{-\frac{\varphi_1+\varphi_2}{2}}, \qquad \tilde{g_1} = g_1\, e^{-\frac{\varphi_2}{2}}, \qquad \tilde{f_1} = f_1\, e^{-\frac{\varphi_2}{2}}$$
$$\tilde{\chi} = \chi\, e^{-\frac{\varphi_1+\varphi_2}{2}}, \qquad \tilde{g_2} = g_2\, e^{-\frac{\varphi_1}{2}}, \qquad \tilde{f_2} = f_2\, e^{-\frac{\varphi_1}{2}}. \tag{5.8}$$

Here, $\psi, \chi, \varphi_i, i = 1, 2$ and $f_i, g_i, i = 1, 2$ are bosonic and fermionic fields, respectively. The absence of Cartan subalgebra h_1, h_2 and $E_{\pm(\alpha_1+\alpha_2)}$ (i.e. in $\mathcal{K}$) in the rhs of (5.5) leads to the following subsidiary constraints:

$$\begin{aligned}
\partial_t f_1 &= \frac{1}{2} f_1 \partial_t \varphi_2 + g_2 \left[\partial_t \chi - \frac{1}{2}\chi(\partial_t\varphi_1 + \partial_t\varphi_2)\right],\\
\partial_t g_1 &= \psi \partial_t f_2 + \frac{1}{2} g_1 \partial_t\varphi_2 - \frac{1}{2}\psi f_2 \partial_t \varphi_1,\\
\partial_x f_1 &= \chi \partial_x g_2 + \frac{1}{2} f_1 \partial_x\varphi_2 - \frac{1}{2}\chi g_2 \partial_x\varphi_1,\\
\partial_x g_1 &= \frac{1}{2} g_1 \partial_x\varphi_2 + f_2 \left[\partial_x\psi - \frac{1}{2}\psi(\partial_x\varphi_1 + \partial_x\varphi_2)\right],\\
\partial_t\varphi_1 &= \frac{\psi\left[\partial_t\chi(1+g_2f_2) + \frac{1}{2}\chi g_2 \partial_t f_2\right]}{1+\psi\chi\left(1+\frac{5}{4}g_2f_2\right)},\\
\partial_t\varphi_2 &= \frac{\psi\partial_t\chi\left(1+\frac{3}{2}g_2f_2\right) - g_2\partial_t f_2 - \frac{1}{2}\psi\chi g_2\partial_t f_2}{1+\psi\chi\left(1+\frac{5}{4}g_2f_2\right)},\\
\partial_x\varphi_1 &= \frac{\chi\left[\partial_x\psi(1+g_2f_2) + \frac{1}{2}\psi\partial_x g_2 f_2\right]}{1+\psi\chi\left(1+\frac{5}{4}g_2f_2\right)},\\
\partial_x\varphi_2 &= \frac{\chi\partial_x\psi\left(1+\frac{3}{2}g_2f_2\right) + \left(\frac{1}{2}\psi\chi+1\right)f_2\partial_x g_2}{1+\psi\chi\left(1+\frac{5}{4}g_2f_2\right)}.
\end{aligned} \tag{5.9}$$

Moreover equation (5.5) yields

$$\begin{aligned}
\bar{b}_1 &= \bar{J}_{-\alpha_1} = -\frac{e^{\frac{1}{2}(\varphi_1+\varphi_2)}}{1+f_2g_2}\left(\partial_x\psi - \frac{1}{2}\psi(\partial_x\varphi_1+\partial_x\varphi_2)\right),\\
F_1 &= \bar{J}_{-\alpha_2} = -e^{-\frac{1}{2}\varphi_1}\left(\partial_x f_2 + \frac{1}{2}f_2\partial_x\varphi_1\right),\\
b_1 &= -e^{-\frac{1}{2}(\varphi_1+\varphi_2)}\Big(\partial_x\chi + \frac{1}{2}\chi(\partial_x\varphi_1+\partial_x\varphi_2) - \chi f_2\partial_x g_2 - \frac{1}{2}\chi\partial_x\varphi_1 g_2 f_2\\
&\qquad - e^{\frac{1}{2}\varphi_1} f_1 \bar{J}_{-\alpha_2} + \chi^2 e^{-\frac{1}{2}(\varphi_1-\varphi_2)}\bar{J}_{-\alpha_1}\Big),\\
\bar{F}_1 &= -e^{\frac{1}{2}\varphi_1}\left(\partial_x g_2 - \frac{1}{2}g_2\partial_x\varphi_1 + e^{-\frac{1}{2}(\varphi_1-\varphi_2)} f_1 \bar{J}_{-\alpha_1}\right).
\end{aligned} \tag{5.10}$$

Solving the zero grade component of (5.3), we find the equations of motion,

$$\begin{aligned}
&\partial_{t_2} b_1 + a_2(\partial_x^2 b_1 - 2\left(b_1\bar{b}_1 + F_1\bar{F}_1\right)b_1) + m_{-1} = 0,\\
&\partial_{t_2}\bar{b}_1 - a_2\left(\partial_x^2\bar{b}_1 - 2(b_1\bar{b}_1 + F_1\bar{F}_1)\bar{b}_1\right) - g_{-1} = 0,\\
&\partial_{t_2} F_1 - a_2\left(\partial_x^2 F_1 - 2b_1\bar{b}_1F_1\right) - n_{-1} = 0,\\
&\partial_{t_2}\bar{F}_1 + a_2\left(\partial_x^2\bar{F}_1 - 2b_1\bar{b}_1\bar{F}_1\right) + o_{-3} = 0,
\end{aligned} \tag{5.11}$$

where

$$g_{-1} = -\eta\psi\, e^{\frac{1}{2}(\phi_1+\phi_2)}, \tag{5.12}$$

$$
\begin{aligned}
m_{-1} &= \eta(\chi + f_1 f_2 + \psi\chi f_1 f_2 + \chi f_2 g_2 + \psi\chi^2)\,\mathrm{e}^{-\frac{1}{2}(\phi_1+\phi_2)},\\
n_{-1} &= -\eta(g_2 + \psi f_1)\,\mathrm{e}^{\frac{1}{2}\phi_1},\\
o_{-1} &= \eta f_2(1+\psi\chi)\,\mathrm{e}^{-\frac{1}{2}\phi_1}.
\end{aligned} \tag{5.13}
$$

Following the same argument as in the pure bosonic case, the vacuum configuration is obtained from

$$
T_0 = \exp(xE^{(1)})\exp((\alpha_2 E^{(2)} + \eta E^{(-1)})t) \tag{5.14}
$$

which leads to spacetime dependence

$$
\rho_i = \exp(k_i x)\exp\left(-\left(\alpha_2 k_i^2 + \frac{\eta}{k_i}\right)t\right). \tag{5.15}
$$

Following the soliton solutions for the Lund–Regge model obtained in [12] we have verified solutions for equations (5.11) to be

$$
\begin{aligned}
&b_1 = \frac{k_1\rho_1^{-1}}{\tau_0}, \qquad \bar{b}_1 = -\frac{k_2\rho_2}{\tau_0}, \qquad F_1 = -a_2\frac{k_2\rho_2}{\tau_0}, \qquad \bar{F}_1 = a_1\frac{k_1\rho_1^{-1}}{\tau_0},\\
&\psi = \frac{\rho_1}{\tau_0}\left(1 - \frac{bk_1\rho_1^{-1}\rho_2}{2(k_1-k_2)\left(1+\frac{k_1}{k_2}\rho_1^{-1}\rho_2\right)}\right), \qquad \chi = \frac{\rho_2}{\tau_0}\left(1 - \frac{bk_2\rho_1^{-1}\rho_2}{2(k_1-k_2)\left(1+\frac{k_1}{k_2}\rho_1^{-1}\rho_2\right)}\right),\\
&g_1 = a_2\frac{k_1\rho_1^{-1}\rho_2}{(k_1-k_2)\tau_0}\,\mathrm{e}^{-\frac{1}{2}\phi_1}, \qquad f_1 = a_1\frac{k_1\rho_1^{-1}\rho_2}{(k_1-k_2)\tau_0}\,\mathrm{e}^{-\frac{1}{2}\phi_1}, \qquad g_2 = a_1\frac{\rho_1^{-1}}{\tau_0}\,\mathrm{e}^{-\frac{1}{2}\phi_2},\\
&f_2 = a_2\frac{\rho_2}{\tau_0}\,\mathrm{e}^{-\frac{1}{2}\phi_2}, \qquad \mathrm{e}^{\frac{1}{2}(\phi_1+\phi_2)} = \frac{1+a_3\rho_1\rho_2}{\tau_0}, \qquad \mathrm{e}^{\frac{1}{2}(\phi_1-\phi_2)} = \frac{1+\bar{a}_3\rho_1^{-1}\rho_2}{\tau_0},
\end{aligned} \tag{5.16}
$$

where a_1, a_2 and b are Grassmaniann and bosonic constants respectively, ρ_i, $i = 1, 2$ are given by (5.15) and

$$
\begin{aligned}
&a_3 = \frac{k_1}{k_2}\Gamma_0\left(1 - b\frac{(k_1+k_2)}{2k_1}\right), \qquad \bar{a}_3 = \Gamma_0\left(1 + b\frac{(k_1-3k_2)}{2k_2}\right),\\
&\Gamma = (1-a_1a_2)\Gamma_0, \qquad \Gamma_0 = \frac{k_1k_2}{(k_1-k_2)^2}, \qquad \tau_0 = 1 + \Gamma\rho_1^{-1}\rho_2.
\end{aligned} \tag{5.17}
$$

6. General case

We now consider a mixed hierarchy associated with a general affine Lie algebra $\hat{\mathcal{G}} = \oplus\mathcal{G}_i, [Q, \mathcal{G}_i] = \mathrm{i}\mathcal{G}_i$ and constant grade one semi-simple element E such that $\hat{\mathcal{G}} = \mathcal{M} \oplus \mathcal{K}, [E, \mathcal{K}] = 0$ with the symmetric space structure,

$$
[\mathcal{K},\mathcal{K}] \subset \mathcal{K}, \qquad [\mathcal{K},\mathcal{M}] \subset \mathcal{M}, \qquad [\mathcal{M},\mathcal{M}] \subset \mathcal{K} \tag{6.1}
$$

with equations of motion involving time evolution with two indices, $t_{n,m}$ defined from the zero curvature representation

$$
\left[\partial_x + E + A_0, \partial_{t_{n,m}} + D^{(n)} + D^{(n-1)} + \cdots + D^{(0)} + D^{(-1)} + \cdots + D^{(-m+1)} + D^{(-m)}\right] = 0. \tag{6.2}
$$

Equation (6.2) leads to

$$
[E, D^{(n)}] = 0, \tag{6.3}
$$

$$
[E, D^{(n-1)}] + [A_0, D^{(n)}] + \partial_x D^{(n)} = 0, \tag{6.4}
$$

$$\vdots$$

$$[E, D^{(n-i)}] + [A_0, D^{(n-i+1)}] + \partial_x D^{(n-i+1)} = 0, \tag{6.5}$$

$$\vdots$$

$$[E, D^{(-1)}] + [A_0, D^{(0)}] + \partial_x D^{(0)} - \partial_{t_{n,m}} A_0 = 0, \tag{6.6}$$

$$[E, D^{(-2)}] + [A_0, D^{(-1)}] + \partial_x D^{(-1)} = 0, \tag{6.7}$$

$$\vdots$$

$$[E, D^{(-j-1)}] + [A_0, D^{(-j)}] + \partial_x D^{(-j)} = 0, \tag{6.8}$$

$$\vdots$$

$$[A_0, D^{(-m)}] + \partial_x D^{(-m)} = 0. \tag{6.9}$$

In order to solve equations (6.3)–(6.9) we have to start from both ends, i.e. from (6.3) towards (6.6), using the symmetric space structure (6.1), we project each equation into $\mathcal{K}$ and $\mathcal{M}$ subspaces to obtain $D_{\mathcal{K}}^{(i)}, i = 1, \ldots, n$ and $D_{\mathcal{M}}^{(i)}, i = 0, \ldots, n$. On the other hand, starting from (6.9) upwards, we find a solution for $D_{\mathcal{K}}^{(-j)}$ and $D_{\mathcal{M}}^{(-j)}$, $j = 1, \ldots, m$ which is non-local in the fields in A_0. For the particular case when $m = 1$, we have seen that there is a set of variables within a group element B that solves (6.9) locally for $m = 1$.

Inserting $D^{(-1)}$ in (6.6) and projecting in $\mathcal{K}$ we find $D_{\mathcal{K}}^{(0)}$ which in turn determines the time evolution as the projection of (6.6) in $\mathcal{M}$. Following the same arguments given before, the spacetime dependence of such generalized mixed model is expected to be of the form

$$\rho_i = \exp(k_i x) \exp\left(\left(\alpha_i k_i^n + \eta k_i^{-m}\right) t\right). \tag{6.10}$$

As a conclusion, we have proposed a zero curvature representation for mixed integrable models associated with $\hat{sl}(2)$ and $\hat{sl}(2,1)$ affine Lie algebras. We have also shown that their soliton solutions follow from the dressing method and with spacetime dependence specified from its vacuum structure. Other more complicated examples deserve to be investigated following the same line of thought.

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Negative Even Grade mKdV Hierarchy and its Soliton Solutions

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ABSTRACT

In this paper we provide an algebraic construction for the negative even mKdV hierarchy which gives rise to time evolutions associated to even graded Lie algebraic structure. We propose a modification of the dressing method, in order to incorporate a non-trivial vacuum configuration and construct a deformed vertex operator for $\hat{sl}(2)$, that enable us to obtain explicit and systematic solutions for the whole negative even grade equations.

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1 Introduction

The odd mKdV hierarchy consists of a series of non-linear equations of motion associated to certain odd graded Lie algebraic structure such that, each equation correspond to a time evolution according to time $t = t_{2n+1}$ [1]. Since these are directly associated to odd graded operators, they are dubbed *odd order mKdV hierarchy.*

In this paper we employ the algebraic technique which, for positive order, the graded structure of the zero curvature representation imposes severe restrictions so that only odd times are allowed. For negative order however, the structure is less restrictive and gives also rise to a subclass of equations of motion, described by time evolutions associated to *negative even grades*. These are constructed by the Lax operator in terms of an affine graded Lie algebra, $\hat{sl}(2)$ which, from the zero curvature representation generates systematically a series of nonlinear integrable equations.

By considering a special case of Zakharov-Shabat AKNS spectral problem and using recursion techniques, a class of integrable equations were considered [2] in order to develop negative order mKdV hierarchy as well as to obtain some parametric type of solutions.

Here, in our approach, the simplest case of the even mKdV where $t = t_{-2}$, is studied in detail and a crucial observation that a trivial zero constant solution is not admissible, lead us to extend the dressing method to incorporate non-zero constant vacuum solutions. This implies in deforming the usual vertex operators preserving the nilpotency property peculiar in solving for soliton solutions. Employing the modified dressing formalism, we construct multi-soliton solutions for the whole negative even grade mKdV hierarchy.

In Sect. 2 we discuss the algebraic formalism for positive and negative hierarchies [3, 4]. In particular, the construction of the equation of motion for $t = t_{-2}$. Such equation agrees with the one proposed in [2] using recursion operator techniques. In Sect. 3 we discuss the dressing formalism [5, 6, 7, 8, 9] to construct soliton solutions for the odd hierarchy. In this case the formalism is based upon a constant zero vacuum solution which, by gauge transformation, generate multi-soliton solutions. In Sect. 4 we extend the dressing formalism to the negative even hierarchy, by introducing a non-zero constant vacuum configuration. We then construct the deformed vertex operators which generate explicitly the multi-soliton solutions. Some details involving the explicit calculation of matrix elements of products of vertex operators and the proof of their nilpotency are described in the appendix.

2 Positive and Negative Hierarchies

Consider the *positive mKdV hierarchy* given by the zero curvature representation

$$[\partial_x + E^{(1)} + A_0, \partial_{t_n} + D^{(n)} + D^{(n-1)} + \cdots + D^{(0)}] = 0, \tag{2.1}$$

where $E^{(2n+1)} = \lambda^n \left(E_\alpha + \lambda E_{-\alpha}\right), \; A_0 = vh$ contains the field variable $v = v\left(x, t_n\right)$, and $\{E_{\pm\alpha}, h\}$ are $sl(2)$ generators satisfying $[h, E_{\pm\alpha}] = \pm 2E_{\pm\alpha}, \; [E_\alpha, E_{-\alpha}] = h$. The grading operator $Q = 2\lambda \frac{d}{d\lambda} + \frac{1}{2}h$ decomposes the affine Lie algebra $\hat{sl}(2)$ into graded subspaces, $\hat{\mathcal{G}} = \oplus_i \mathcal{G}_i$,

$$\mathcal{G}_{2m} = \{h^{(m)} = \lambda^m h\}, \qquad \mathcal{G}_{2m+1} = \{\lambda^m \left(E_\alpha + \lambda E_{-\alpha}\right), \; \lambda^m \left(E_\alpha - \lambda E_{-\alpha}\right)\} \tag{2.2}$$

$m = 0, \pm 1, \pm 2, \ldots$ and $D^{(j)} \in \mathcal{G}_j$. A more subtle structure arises if one consider the decomposition $\hat{\mathcal{G}} = \mathcal{K} \oplus \mathcal{M}$ where $\mathcal{K} = \{\lambda^n (E_\alpha + \lambda E_{-\alpha})\}$ denotes the Kernel of $E \equiv E^{(1)}$, i.e., $\mathcal{K} = \{k \in \hat{\mathcal{G}} \mid [E, k] = 0\}$ and $\mathcal{M}$ is its complement. We assume that E is semi-simple in the sense that this second decomposition is such that

$$[\mathcal{K}, \mathcal{K}] \subset \mathcal{K}, \qquad [\mathcal{K}, \mathcal{M}] \subset \mathcal{M}, \qquad [\mathcal{M}, \mathcal{M}] \subset \mathcal{K}. \tag{2.3}$$

Eqn. (2.1) can be decomposed grade by grade and solved for $D^{(j)}$. For instance, the highest grade in (2.1) yields

$$[E, D^{(n)}] = 0 \Rightarrow D^{(n)} = D^{(n)}_{\mathcal{K}} \in \mathcal{K}. \tag{2.4}$$

Since by (2.2) $\mathcal{K}$ has grade $2m+1$, this last equation implies that $n = 2m+1$ and hence $t_n = t_{2m+1}$, showing that only *odd* grades are admissible for positive mKdV hierarchy (2.1). Moving down grade by grade and using the symmetric space structure (2.3), eqn. (2.1) allows one to solve for all $D^{(j)} = D^{(j)}_{\mathcal{K}} + D^{(j)}_{\mathcal{M}}$, $j = 0 \ldots n$. In particular, the zero grade projection in $\mathcal{M}$ yields the equation of motion

$$\partial_{t_n} A_0 - \partial_x D^{(0)}_{\mathcal{M}} - [A_0, D^{(0)}_{\mathcal{K}}] = 0 \tag{2.5}$$

where we have taken into account that $A_0 \in \mathcal{M}$. Eqn. (2.5) represents a series of nonlinear evolution equations associated with time t_{2m+1}. Choosing $m = 1$ for example, we will obtain the well known mKdV equation [1, 3].

The same, however, does no happen for the *negative mKdV hierarchy* [3, 4], i.e., for $n < 0$. Let us consider the zero curvature representation

$$[\partial_x + E^{(1)} + A_0, \partial_{t_{-n}} + D^{(-n)} + D^{(-n+1)} + \cdots + D^{(-1)}] = 0. \tag{2.6}$$

Here, the lowest grade projection,

$$\partial_x D^{(-n)} + [A_0, D^{(-n)}] = 0 \tag{2.7}$$

yields a nonlocal equation for $D^{(-n)}$. The second lowest projection of grade $-n+1$ leads to

$$\partial_x D^{(-n+1)} + [A_0, D^{(-n+1)}] + [E^{(1)}, D^{(-n)}] = 0 \tag{2.8}$$

which determines $D^{(-n+1)}$. The same mechanism works recursively until we reach the zero grade equation

$$\partial_{t_{-n}} A_0 + [E^{(1)}, D^{(-1)}] = 0 \tag{2.9}$$

which gives the time evolution for the field in A_0 according to time t_{-n}. The simplest model of this sub-hierarchy is obtained for $n = 1$, for which the following equations arrive from the zero curvature (2.6),

$$\begin{aligned} \partial_x D^{(-1)} + [A_0, D^{(-1)}] &= 0, \\ \partial_{t_{-1}} A_0 - [E^{(1)}, D^{(-1)}] &= 0. \end{aligned} \tag{2.10}$$

These equations can be solved in general if we parametrize the fields as

$$D^{(-1)} = B^{-1} E^{(-1)} B, \qquad A_0 = B^{-1} \partial_x B, \qquad B = \exp(\mathcal{G}_0) \tag{2.11}$$

in terms of the zero grade subalgebra $\mathcal{G}_0$. Space-time is associated to the light-cone coordinates $\bar{z}, z$ as $x = \bar{z},\ t_{-1} = z$. The time evolution is then given by the Leznov-Saveliev equation,

$$\partial_{t_{-1}} \left(B^{-1} \partial_x B \right) = [E^{(1)}, B^{-1} E^{(-1)} B] \tag{2.12}$$

which for $\hat{sl}(2)$ with principal gradation $Q = 2\lambda \frac{d}{d\lambda} + \frac{1}{2}h$, yields the sinh-Gordon equation [7]

$$\partial_{t_{-1}} \partial_x \phi = e^{2\phi} - e^{-2\phi}, \qquad B = e^{\phi h}. \tag{2.13}$$

Note that from the definition of A_0 and the parametrization (2.11), we find the following relation between ϕ and v: $A_0 = vh = B^{-1}\partial_x B \ \Rightarrow\ v = \partial_x \phi$.

We now propose the first nontrivial example for the *negative even* sub-hierarchy:

$$\partial_x D^{(-2)} + [A_0, D^{(-2)}] = 0, \tag{2.14}$$
$$\partial_x D^{(-1)} + [A_0, D^{(-1)}] + [E^{(1)}, D^{(-2)}] = 0, \tag{2.15}$$
$$\partial_{t_{-2}} A_0 - [E^{(1)}, D^{(-1)}] = 0. \tag{2.16}$$

From grading (2.2) and decomposition (2.3) we find

$$\begin{aligned} D^{(-2)} &= c_{-2} \lambda^{-1} h, \\ D^{(-1)} &= a_{-1} \left(\lambda^{-1} E_\alpha + E_{-\alpha} \right) + b_{-1} \left(\lambda^{-1} E_\alpha - E_{-\alpha} \right). \end{aligned} \tag{2.17}$$

From eqn. (2.14) and (2.15) we find, $c_{-2} = \text{const.}$ and

$$\begin{aligned} \partial_x (a_{-1} + b_{-1}) + 2v (a_{-1} + b_{-1}) - 2c_{-2} &= 0, \\ \partial_x (a_{-1} - b_{-1}) - 2v (a_{-1} - b_{-1}) + 2c_{-2} &= 0, \end{aligned} \tag{2.18}$$

which are ordinary differential equations with solution

$$\begin{aligned} a_{-1} + b_{-1} &= 2c_{-2} \exp(-2d^{-1}v) d^{-1} \left(\exp(2d^{-1}v) \right), \\ a_{-1} - b_{-1} &= -2c_{-2} \exp(2d^{-1}v) d^{-1} \left(\exp(-2d^{-1}v) \right). \end{aligned} \tag{2.19}$$

In (2.19) we have denoted $d^{-1} f = \int^x f(x') dx'$. Having determined $D^{(-1)}$, the evolution equation associated to time t_{-2} is then given by eqn. (2.16):

$$\partial_{t_{-2}} v + 2c_{-2} e^{-2d^{-1}v} d^{-1} \left(e^{2d^{-1}v} \right) + 2c_{-2} e^{2d^{-1}v} d^{-1} \left(e^{-2d^{-1}v} \right) = 0. \tag{2.20}$$

Differentiating twice with respect to x, and setting $c_{-2} = 1$ for convenience, we find the local equation

$$v_{xxt_{-2}} - 4v^2 v_{t_{-2}} - \frac{v_x v_{xt_{-2}}}{v} - 4\frac{v_x}{v} = 0. \tag{2.21}$$

Eqn. (2.21) was already obtained in [2] using recursion operator techniques.

3 Odd Hierarchy Solutions

In order to employ the dressing method to construct soliton solutions, we now introduce the full $\hat{sl}(2)$ affine Kac-Moody algebra with central extensions:

$$\begin{aligned} [h^{(m)}, h^{(n)}] &= 2m\delta_{m+n,0}\hat{c}, \\ [h^{(m)}, E_{\pm\alpha}^{(n)}] &= \pm 2E_{\pm\alpha}^{(m+n)}, \\ [E_{\alpha}^{(m)}, E_{-\alpha}^{(n)}] &= h^{(m+n)} + m\delta_{m+n,0}\hat{c} \end{aligned} \tag{3.22}$$

together with the derivation operator $\hat{d}$ such that,

$$[\hat{d}, T_a^{(n)}] = nT_a^{(n)}, \qquad T_a^{(n)} = \{h^{(n)}, \;\; E_{\pm\alpha}^{(n)}\}. \tag{3.23}$$

The grading operator now reads $Q = 2\hat{d}+1/2h^{(0)}$. A well established method for determining soliton solutions is to choose a vacuum solution and then to map it into a non trivial solution by gauge transformation (dressing) [6],[7]. The zero curvature condition (2.1) or (2.6) implies pure gauge connections, $A_x = T^{-1}\partial_x T = E + A_0$ and $A_{t_n} = T^{-1}\partial_{t_n} T = D^{(n)} + \cdots + D^{(0)}$ or $A_{t_{-n}} = T^{-1}\partial_{t_{-n}} T = D^{(-n)} + \cdots + D^{(-1)}$, respectively. Suppose there exists a vacuum solution satisfying

$$A_{x,vac} = E^{(1)} - t_k\delta_{k+1,0}\hat{c}, \qquad A_{t_k,vac} = E^{(k)}, \tag{3.24}$$

where now $[E^{(k)}, E^{(l)}] = \frac{1}{2}(k-l)\delta_{k+l,0}\hat{c}$, for (k, l) odd integers.

The solution for $A_{x,vac} = T_0^{-1}\partial_x T_0$ and $A_{t_k,vac} = T_0^{-1}\partial_{t_k} T_0$ is therefore given by

$$T_0 = \exp(xE^{(1)})\exp(t_k E^{(k)}). \tag{3.25}$$

The dressing method is based on the assumption of the existence of two gauge transformations, generated by $\Theta_\pm$, mapping the vacuum into non trivial configuration, i.e.

$$A_x = (\Theta_\pm)^{-1}A_{x,vac}\Theta_\pm + (\Theta_\pm)^{-1}\partial_x\Theta_\pm, \tag{3.26}$$

$$A_{t_k} = (\Theta_\pm)^{-1}A_{t_k,vac}\Theta_\pm + (\Theta_\pm)^{-1}\partial_{t_k}\Theta_\pm. \tag{3.27}$$

As a consequence we relate

$$\Theta_-\Theta_+^{-1} = T_0^{-1}gT_0 \tag{3.28}$$

where g is an arbitrary constant group element. We suppose that $\Theta_\pm$ are group elements of the form

$$\Theta_-^{-1} = e^{p(-1)}e^{p(-2)}\ldots, \qquad \Theta_+^{-1} = e^{q(0)}e^{q(1)}e^{q(2)}\ldots \tag{3.29}$$

where $p^{(-i)}$ and $q^{(i)}$ are linear combinations of grade $(-i)$ and (i) generators, respectively $(i = 0, 1, \ldots)$. In considering Θ_+, the zero grade component of (3.26) admits solution

$$e^{q(0)} = B^{-1}e^{-\nu\hat{c}} \tag{3.30}$$

where we have used $A_x = E^{(1)} + B^{-1}\partial_x B + \partial_x\nu\hat{c} - t_k\delta_{k+1,0}\hat{c}$. From eqn. (3.28) we find

$$\ldots e^{-p(-2)}e^{-p(-1)}B^{-1}e^{-\nu\hat{c}}e^{q(1)}e^{q(2)}\ldots = T_0^{-1}gT_0 \tag{3.31}$$

hence,

$$< \lambda'|B^{-1}|\lambda > e^{-\nu} = < \lambda'|T_0^{-1} g T_0|\lambda > \tag{3.32}$$

where $|\lambda >$ and $< \lambda'|$ are annihilated by $\mathcal{G}_>$ and $\mathcal{G}_<$, respectively. Explicit space time dependence for the field in $\mathcal{G}_0$, defined in (2.13), is given by choosing specific matrix elements (6.64):

$$\begin{aligned} e^{-\nu} &= < \lambda_0|T_0^{-1} g T_0|\lambda_0 >, \\ e^{-\phi-\nu} &= < \lambda_1|T_0^{-1} g T_0|\lambda_1 > . \end{aligned} \tag{3.33}$$

where $|\lambda_i >, i = 0, 1$ correspond to highest weight states, i.e. annihilated by positive grade operators. Suppose we now write the constant group element g as

$$g = \exp\{F(\gamma)\}, \tag{3.34}$$

where γ is a complex parameter and we choose $F(\gamma)$ to be an eigenstate of $E^{(k)}$, i.e.

$$[E^{(k)}, F(\gamma)] = f^{(k)}(\gamma) F(\gamma) \tag{3.35}$$

where $f^{(k)}$ are specific functions of γ. It therefore follows that

$$T_0^{-1} g T_0 = \exp\{\rho(\gamma) F(\gamma)\} \tag{3.36}$$

where

$$\rho(\gamma) = \exp\{-t_k f^{(k)}(\gamma) - x f^{(1)}(\gamma)\}. \tag{3.37}$$

For more general cases in which

$$g = \exp\{F_1(\gamma_1)\} \exp\{F_2(\gamma_2)\} \dots \exp\{F_N(\gamma_N)\} \tag{3.38}$$

with

$$[E^{(k)}, F_i(\gamma_i)] = f_i^{(k)}(\gamma_i) F_i(\gamma_i) \tag{3.39}$$

we find

$$T_0^{-1} g T_0 = \exp\{\rho_1(\gamma_1) F_1(\gamma_1)\} \exp\{\rho_2(\gamma_2) F_2(\gamma_2)\} \dots \exp\{\rho_N(\gamma_N) F_N(\gamma_N)\} \tag{3.40}$$

where

$$\rho_i(\gamma_i) = \exp\{-t_k f_i^{(k)}(\gamma_i) - x f_i^{(1)}(\gamma_i)\}. \tag{3.41}$$

The specific eigenstate, in this case of $\hat{sl}(2)$, is given by

$$F(\gamma) = \sum_{n=-\infty}^{\infty} \left(h^{(n)} - \frac{1}{2}\delta_{n,0}\hat{c}\right) \gamma^{-2n} + \left(E_\alpha^{(n)} - E_{-\alpha}^{(n+1)}\right) \gamma^{-2n-1} \tag{3.42}$$

whose eigenvalues are obtained from

$$[E^{(k)}, F(\gamma)] = -2\gamma^k F(\gamma). \tag{3.43}$$

From eqns. (3.33) we obtain solutions for ϕ (or equivalently $v = \partial_x \phi$) for the whole odd hierarchy, i.e, for all variables t_k in eq. (3.41). Observe that in eqns. (3.35) and (3.39), $F(\gamma)$ is a simultaneous eigenstate of both $E^{(1)}$ and $E^{(k)}$ and belong to the kernel $\mathcal{K}$. The above argument is therefore valid only for $k = 2m + 1,\ m = 0, \pm 1, \pm 2, \dots$ since $\mathcal{K}$ contains only odd grade elements. Then, this method gives explicit solutions for both, eqns. (2.1) and (2.6) for $n = k = 2m + 1$. See for instance [7, 8, 9].

4 Negative Even Hierarchy Solutions

In order to modify the dressing method to construct systematic solutions of equations like (2.20) or (2.21), we notice that $v=0$ cannot be solution of (2.21). Therefore, let us propose the simplest vacuum configuration

$$\begin{aligned} A_{x,vac} &= \left(E_{\alpha}^{(0)}+E_{-\alpha}^{(1)}\right)+v_0 h^{(0)}-\frac{1}{v_0}t_{-2m}\delta_{m-1,0}\hat{c}, \\ A_{t_{-2m},vac} &= \frac{1}{v_0}\left(E_{\alpha}^{(-m)}+E_{-\alpha}^{(1-m)}\right)+h^{(-m)} \end{aligned} \tag{4.44}$$

with $v_0=$ const. $\neq 0$. It is straightforward to verify the zero curvature equation

$$[\partial_x + A_{x,vac}, \partial_{t_{-2m}} + A_{t_{-2m},vac}] = 0. \tag{4.45}$$

This *nontrivial vacuum* leads to the following modification of eqn. (3.25), but now for negative even grades,

$$T_0 = \exp\left\{x\left(E_{\alpha}^{(0)}+E_{-\alpha}^{(1)}+v_0 h^{(0)}\right)\right\}\exp\left\{\frac{t_{-2m}}{v_0}\left(E_{\alpha}^{(-m)}+E_{-\alpha}^{(1-m)}+v_0 h^{(-1)}\right)\right\}. \tag{4.46}$$

The analogous of eqn. (3.30) leads to

$$e^{q(0)} = B^{-1}e^{xv_0 h^{(0)}}e^{-\nu\hat{c}}. \tag{4.47}$$

Observe that consistency of the zero curvature representation with nontrivial vacuum configuration requires terms with mixed gradation in constructing T_0 as in (4.46). The solution is then given by

$$\begin{aligned} e^{-\nu} &= <\lambda_0|T_0^{-1}gT_0|\lambda_0> \;\equiv\; \tau^+, \\ e^{-\phi+xv_0-\nu} &= <\lambda_1|T_0^{-1}gT_0|\lambda_1> \;\equiv\; \tau^- \end{aligned} \tag{4.48}$$

and hence,

$$v = v_0 - \partial_x \ln\left(\frac{\tau^+}{\tau^-}\right), \qquad v=\partial_x\phi. \tag{4.49}$$

In order to construct explicit soliton solutions we need a simultaneous eigenstate of $b_1 \equiv E_{\alpha}^{(0)}+E_{-\alpha}^{(1)}+v_0 h^{(0)}$ and $b_{-2m}\equiv\left(E_{\alpha}^{(-m)}+E_{-\alpha}^{(-m+1)}+v_0 h^{(-m)}\right)$. Let

$$F(\gamma,v_0) = \sum_{n=-\infty}^{\infty}\left(\gamma^2-v_0^2\right)^{-n}\left[h^{(n)}+\frac{v_0-\gamma}{2\gamma}\delta_{n,0}\hat{c}+E_{\alpha}^{(n)}(\gamma+v_0)^{-1}-E_{-\alpha}^{(n+1)}(\gamma-v_0)^{-1}\right]. \tag{4.50}$$

be our deformed vertex operator. A direct calculation shows that

$$\begin{aligned} [b_1, F(\gamma,v_0)] &= -2\gamma F(\gamma,v_0), \\ [b_{-2m}, F(\gamma,v_0)] &= -2\gamma\left(\gamma^2-v_0^2\right)^{-m}F(\gamma,v_0). \end{aligned} \tag{4.51}$$

Therefore from (3.37) we find,

$$\rho\left(\gamma, v_0\right) = \exp\left\{2\gamma x + \frac{2\gamma t_{-2m}}{v_0\left(\gamma^2 - v_0^2\right)^m}\right\}. \tag{4.52}$$

It only remains to calculate the matrix elements in eqns. (4.48). They are shown in Appendix. Note that, because of the nilpotency property of the vertex operator between matrix elements, as discussed in the Appendix, the exponential series in eqn. (4.48) truncates, e.g, if we take,

$$g = \exp\{F(\gamma, v_0)\} \tag{4.53}$$

we have

$$\begin{aligned} < \lambda_a|T_0^{-1} g T_0|\lambda_a > &= < \lambda_a|\exp\left\{\rho\left(\gamma, v_0\right) F\left(\gamma, v_0\right)\right\}|\lambda_a > \\ &= 1 + \rho\left(\gamma, v_0\right) < \lambda_a|F\left(\gamma, v_0\right)|\lambda_a > . \end{aligned} \tag{4.54}$$

Thus, from eqn. (4.49), we obtain *explicit* solutions for the *whole negative even hierachy.* The introduction of a nontrivial vacuum configuration, v_0, in the dressing method, seems to have the same effect as a change in the boundary conditions when looking for solutions of differential equations, as can be noted in eqn. (4.49). In this vein, we can say that our modified dressing approach implements a different boundary condition than the usual dressing of a trivial vacuum, largely used until now.

Let us introduce the shorthand notation:

$$\begin{aligned} c_i^{\pm} &= \frac{v_0 \pm \gamma_i}{2\gamma_i}, \\ a_{ij} &= \left(\frac{\gamma_i - \gamma_j}{\gamma_i + \gamma_j}\right)^2, \\ \rho_i &= \exp\left\{2\gamma_i x + \frac{2\gamma_i t_{-2m}}{v_0\left(\gamma_i^2 - v_0^2\right)^m}\right\}. \end{aligned} \tag{4.55}$$

The 1-soliton solution in eqn. (4.49), is obtained with one vertex as in (4.53). The explicit tau functions are:

$$\tau^{\pm} = 1 + c_1^{\pm}\rho_1. \tag{4.56}$$

The 2-soliton solution is obtained with

$$g = \exp\left\{F\left(\gamma_1, v_0\right)\right\}\exp\left\{F\left(\gamma_2, v_0\right)\right\} \tag{4.57}$$

in (4.48), and then

$$\tau^{\pm} = 1 + c_1^{\pm}\rho_1 + c_2^{\pm}\rho_2 + c_1^{\pm}c_2^{\pm}a_{12}\rho_1\rho_2. \tag{4.58}$$

The 3-soliton solution, obtained as a product of 3 exponential vertices, is given by

$$\begin{aligned} \tau^{\pm} &= 1 + c_1^{\pm}\rho_1 + c_2^{\pm}\rho_2 + c_3^{\pm}\rho_3 + \\ &+ c_1^{\pm}c_2^{\pm}a_{12}\rho_1\rho_2 + c_1^{\pm}c_3^{\pm}a_{13}\rho_1\rho_3 + c_2^{\pm}c_3^{\pm}a_{23}\rho_2\rho_3 + \\ &+ c_1^{\pm}c_2^{\pm}c_3^{\pm}a_{12}a_{13}a_{23}\rho_1\rho_2\rho_3. \end{aligned} \tag{4.59}$$

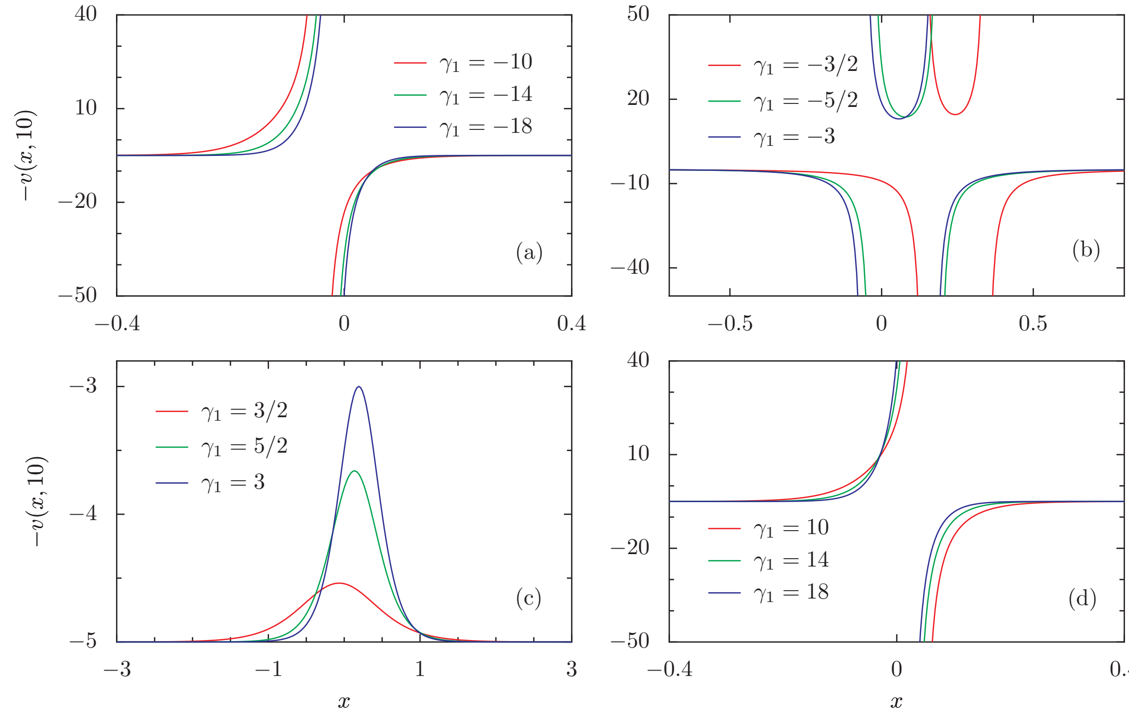

Figure 1: 1-soliton solutions for eqn. (2.21), $t_n = t_{-2}$ in mKdV hierarchy. In all these graphs we set $v_0 = 5$ and choose a fixed time $t_{-2} = 10$.

If we then substitute

$$g = \prod_{i=1}^{n} \exp\{F(\gamma_i, v_0)\} \tag{4.60}$$

in eqns. (4.48) we obtain the general n-soliton solution:

$$\tau^{\pm} = \sum_{J \subset I} \left(\prod_{i \in J} c_i^{\pm} \right) \left(\prod_{i,j \in J,\, i<j} a_{ij} \right) \prod_{i \in J} \rho_i \tag{4.61}$$

where $I = \{1, \ldots, n\}$ and the sum is over all subsets J of I. These solutions present the same structure as those constructed from the trivial vacuum solution. They differ only by the deformation in (4.55) which now incorporates the parameter v_0.

The solutions of eqn. (2.21), $t_{-2m} = t_{-2}$, are obtained by setting $m = 1$ in (4.55). Considering 1-soliton solution (4.56), we see that a critical behavior occurs when $\gamma_1 \to \pm v_0$ or $\gamma_1 \to 0$. So, we have 4 different regions to consider: $\gamma_1 < -v_0$; $-v_0 < \gamma_1 < 0$; $0 < \gamma_1 < v_0$; $\gamma_1 > v_0$. All these regions are considered separately in Fig. (1). These solutions keep their form for any t_{-2}. Note that 3 different types of behavior occur, Fig. (1-a) and Fig. (1-d) are of the same type, and have the same form as the usual *trivial vacuum* solutions of *odd grade* mKdV hierarchy, except from the fact that the solution is displaced by v_0 in the $y-$axis. Fig. (1-b) and Fig. (1-c) are different ones, and their form are not obtained from the trivial vacuum configuration. Also, note that $v \to v_0$ when $x \to \pm\infty$.

In Fig. (2) we show the 2-soliton solution for eqn. (2.21), where we illustrate the mixing of different type of solutions. Again, a different behavior emerges compared with the trivial

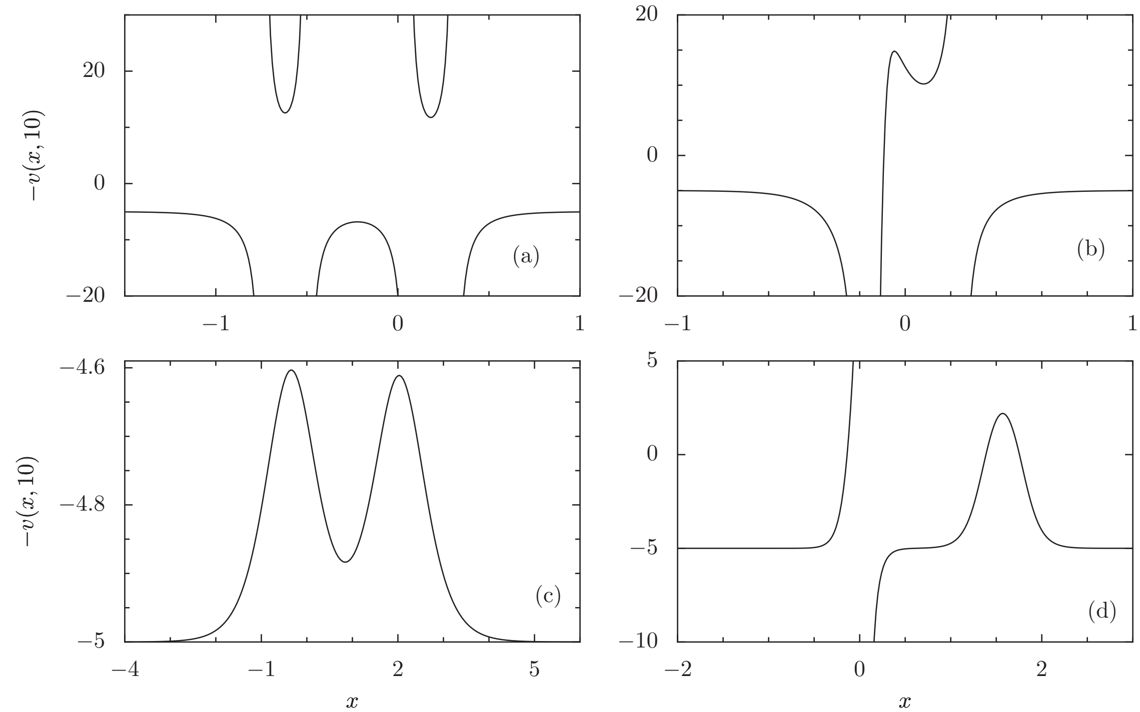

Figure 2: 2-soliton solutions for eqn. (2.21). $v_0 = 5$ and $t_{-2} = 10$. Parameters: (a) $\gamma_1 = -4,\ \gamma_2 = -3$; (b) $\gamma_1 = -4,\ \gamma_2 = -10$; (c) $\gamma_1 = 1.3,\ \gamma_2 = 1.5$; (d) $\gamma_1 = 4.8,\ \gamma_2 = 7$.

vacuum 2-soliton solutions.

5 Conclusions

We extended the mKdV hierarchy to include *negative even grade* equations, based on a graded infinite dimensional Lie algebra $\hat{sl}(2)$. This procedure systematically lead us to obtain new non-linear integrable equations, e.g. eqn. (2.21) which was previously obtained in [2]. Our method can also provide other higher order integro-diffential equations, like for example:

$$
\begin{aligned}
\partial_x \partial_{t_{-4}} \phi &= 4e^{-2\phi} d^{-1} \left[e^{2\phi} d^{-1} \left(e^{-2\phi} d^{-1} e^{2\phi} + e^{2\phi} d^{-1} e^{-2\phi} \right) \right] - \\
&- 4e^{2\phi} d^{-1} \left[e^{-2\phi} d^{-1} \left(e^{-2\phi} d^{-1} e^{2\phi} + e^{2\phi} d^{-1} e^{-2\phi} \right) \right]. \qquad (5.62)
\end{aligned}
$$

This subhierarchy of even grade equations are not solved by the usual dressing method, based on a trivial vacuum configuration. Nevertheless, we also extended the dressing method to incorporate a constant non trivial vacuum configuration v_0.

Remarkably, all these modifications lead us to obtain solutions for the whole negative even grade mKdV subhierarchy, in particular for eqn. (2.21). Our solutions for eqn. (2.21) does not appear in [2]. The introduction of the constant vacuum, v_0, showed that the simplest 1-soliton solution splits into three different classes, depending on the sign of the parameter γ_1 and its difference from v_0. The general form of the solutions agree with the trivial vacuum ones, but its behavior is modified by the presence of the v_0 parameter. The 1, 2 and 3 soliton solutions were explicit checked for eqn. (2.21). Moreover, the 1-soliton (4.56) with (4.55) and $m = 2$ was also verified to satisfy eqn. (5.62), using symbolic computational methods.

Acknowledgments. We thank CNPq for support.

6 Appendix - Matrix Elements

Consider the vertex operator for $\hat{sl}(2)$,

$$F(\gamma, v_0) = \sum_{n=-\infty}^{\infty} \left(\gamma^2 - v_0^2\right)^{-n} \left[h^{(n)} + \frac{v_0 - \gamma}{2\gamma}\delta_{n,0}\hat{c} + E_{\alpha}^{(n)}(\gamma + v_0)^{-1} - E_{-\alpha}^{(n+1)}(\gamma - v_0)^{-1}\right]. \tag{6.63}$$

In the highest weight representation $\{|\lambda_0>, |\lambda_1>\}$ we have the following action of $\hat{sl}(2)$ operators:

$$\begin{aligned} E_{\alpha}^{(0)}|\lambda_a> &= 0, \\ E_{\pm\alpha}^{(n)}|\lambda_a> &= 0, \quad n > 0 \\ h^{(n)}|\lambda_a> &= 0, \quad n > 0 \\ h^{(0)}|\lambda_a> &= \delta_{a1}|\lambda_a> \\ \hat{c}|\lambda_a> &= |\lambda_a> \end{aligned} \tag{6.64}$$

where $a = 0, 1$. Using the adjoint relations $\left(h^{(n)}\right)^{\dagger} = h^{(-n)}$, $\left(E_{\alpha}^{(n)}\right)^{\dagger} = E_{-\alpha}^{(-n)}$ and $\hat{c}^{\dagger} = \hat{c}$ we also know their actions on $<\lambda_a|$. From this, we have:

$$\begin{aligned} <\lambda_0|F(\gamma, v_0)|\lambda_0> &= \frac{(v_0 - \gamma)}{2\gamma} \equiv c^{-}, \\ <\lambda_1|F(\gamma, v_0)|\lambda_1> &= \frac{(v_0 + \gamma)}{2\gamma} \equiv c^{+}. \end{aligned} \tag{6.65}$$

In order to calculate $<\lambda_a|F(\gamma_1, v_0)F(\gamma_2, v_0)|\lambda_a>$, after distributing the products and keeping only non-trivial terms, we make use of the commutator rules to change the order. The double sum simplifies to a single sum, which can then be substituted for power series like $\sum_{n=0}^{\infty} x^n = 1/(1-x)$, $\sum_{n=1}^{\infty} x^n = x/(1-x)$ and $\sum_{n=1}^{\infty} nx^n = x/(1-x)^2$. The result is then given by:

$$<\lambda_a|F(\gamma_1, v_0)F(\gamma_2, v_0)|\lambda_a> = \delta_{a1} + \frac{2\left(\gamma_1^2 - v_0^2\right)\left(\gamma_2^2 - v_0^2\right)}{\left(\gamma_1^2 - \gamma_2^2\right)^2} + \frac{v_0 - \gamma_1}{2\gamma_1}\delta_{a1} +$$

$$+ \frac{v_0 - \gamma_2}{2\gamma_2}\delta_{a1} + \frac{(\gamma_1 - v_0)(\gamma_2 - v_0)}{4\gamma_1\gamma_2} - \frac{(\gamma_1 - v_0)(\gamma_2 + v_0)}{\gamma_1^2 - \gamma_2^2}\delta_{a1} + \frac{(\gamma_1 + v_0)(\gamma_2 - v_0)}{\gamma_1^2 - \gamma_2^2}\delta_{a1} -$$

$$-\left(\gamma_1 - v_0\right)\left(\gamma_2 + v_0\right)\frac{\gamma_2^2 - v_0^2}{\left(\gamma_1^2 - \gamma_2^2\right)^2} - \left(\gamma_1 + v_0\right)\left(\gamma_2 - v_0\right)\frac{\gamma_1^2 - v_0^2}{\left(\gamma_1^2 - \gamma_2^2\right)^2}. \quad (6.66)$$

This expression can be further simplified to

$$\begin{aligned} < \lambda_0|F\left(\gamma_1, v_0\right)F\left(\gamma_2, v_0\right)|\lambda_0 > &= \frac{\left(\gamma_1 - v_0\right)\left(\gamma_2 - v_0\right)}{4\gamma_1\gamma_2}\left(\frac{\gamma_1 - \gamma_2}{\gamma_1 + \gamma_2}\right)^2 = c_1^- c_2^- a_{12}, \\ < \lambda_1|F\left(\gamma_1, v_0\right)F\left(\gamma_2, v_0\right)|\lambda_1 > &= \frac{\left(\gamma_1 + v_0\right)\left(\gamma_2 + v_0\right)}{4\gamma_1\gamma_2}\left(\frac{\gamma_1 - \gamma_2}{\gamma_1 + \gamma_2}\right)^2 = c_1^+ c_2^+ a_{12}, \end{aligned} \quad (6.67)$$

where $c_i^\pm = c^\pm\left(\gamma_i\right)$, see (6.65), and we have defined:

$$a_{ij} = \left(\frac{\gamma_i - \gamma_j}{\gamma_i + \gamma_j}\right)^2. \quad (6.68)$$

Note that (6.67) $\rightarrow 0$ when $\gamma_2 \rightarrow \gamma_1$. This proves the nilpotency property of the vertex operator when evaluated within diagonal states $|\lambda_0>$ and $|\lambda_1>$.

A more tedious calculation shows that:

$$\begin{aligned} < \lambda_0|F\left(\gamma_1, v_0\right)F\left(\gamma_2, v_0\right)F\left(\gamma_3, v_0\right)|\lambda_0 > &= c_1^- c_2^- c_3^- a_{12}a_{13}a_{23}, \\ < \lambda_1|F\left(\gamma_1, v_0\right)F\left(\gamma_2, v_0\right)F\left(\gamma_3, v_0\right)|\lambda_1 > &= c_1^+ c_2^+ c_3^+ a_{12}a_{13}a_{23}. \end{aligned} \quad (6.69)$$

In general, using Wick theorem, it is possible to show that:

$$\begin{aligned} < \lambda_0|\prod_{i=1}^n F\left(\gamma_i, v_0\right)|\lambda_0 > &= \prod_{i=1}^n c_i^- \prod_{i,j=1,\, i<j}^n a_{ij}, \\ < \lambda_1|\prod_{i=1}^n F\left(\gamma_i, v_0\right)|\lambda_1 > &= \prod_{i=1}^n c_i^+ \prod_{i,j=1,\, i<j}^n a_{ij}. \end{aligned} \quad (6.70)$$

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