Measures of the Welfare Cost of Inflation: A Complete Ordering*

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Abstract

This paper presents three contributions to the literature on the welfare cost of inflation. First, it introduces a new sensible way of measuring this cost – that of a compensating variation in consumption or income, instead of the equivalent variation notion that has been extensively used in empirical and theoretical research during the past fifty years. We find this new measure to be interestingly related to the proxy measure of the shopping-time welfare cost of inflation introduced by Simonsen and Cysne (2001). Secondly, it discusses for which money-demand functions this and the shopping-time measure can be evaluated in an economically meaningful way. And, last but not least, it completely orders a comprehensive set of measures of the welfare cost of inflation for these money-demand specifications. All of our results are extended to an economy in which there are many types of monies present, and are illustrated with the log-log money-demand specification.

1 Introduction

The debate over whether and to what extent we should be concerned about inflation has very likely been around since the invention of inflation itself. Inflation, in its turn, is as old as money.[1] Nevertheless, human civilization would have to wait until the second half of the twentieth century, with the rise of modern economic science, to see rigorous and logical attempts at measuring inflation's burden on society.

According to Fischer (1981, p. 17), an anticipated 10% inflation rate yields a loss of 0.3% of the GNP. More recent figures (see, for instance, Lagos and Wright 2005, p. 478) are much more dramatic, going as high as 4.1%. If this value were uniformly distributed throughout society, it would mean about $1,000 a year for someone whose annual earnings were $24,000 – most definitely a considerable amount. But even if this cost were only 0.3%, it certainly could not be regarded as negligible – for the American economy, for example, it would represent about US$ 40 billion.

In general terms, the problem with inflation is that it taxes the holding of real balances, making people spend more of some other real resource instead, and obtain a lower utility level. At least three major approaches to measuring this welfare cost of inflation have been advocated: the welfare "triangle" (Bailey 1956), the comparison of steady states (Cooley and Hansen 1989) and the shopping time (Lucas 2000), the

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[1] The first-ever documented form of inflation being coin clippings in ancient Rome, some two thousand years ago.

first two being the most popular in applications. Depending on how money is introduced in our model, one or another measure becomes the most natural choice.

In the absence of distortionary taxes, Friedman (1969, pp. 33-4) showed that the optimal inflation rate is minus the real interest rate, that is, the rate corresponding to a zero nominal interest rate. That is because this rate equates the privately perceived cost of holding one additional dollar (the foregone nominal interest rate this dollar would yield if used to buy a government bond instead) to the social cost of providing it – idealistically, nothing at all. This environment in which the Friedman rule is valid is used as a benchmark in the first two measuring strategies mentioned above.

Bailey's welfare triangle, the area under the inverted money-demand curve, has been calculated in many different models. Barro (1972) finds a loss associated to a 10% annual inflation rate, in terms of gross national product (this will be the case for all the estimates shown below), of 2.2%.[2] According to Fischer's (1981) calculations for the U.S. economy, such an inflation rate would correspond to a loss of 0.3% of GNP, while Lucas (1981), using similar parameters, finds 0.45% (p. 43). The main difference is that Fischer considers high-powered money only, while Lucas uses M1. Aiyagari, Braun and Eckstein (1998), motivated by evidence that inflation boosts the relative size of the banking sector in the economy, use this size as their definition for the welfare cost of inflation, and show that it equals Bailey's measure (p. 1290). They estimate a loss equivalent to 2.0% of GNP (fig. 6, p. 1298).

The second approach to calculating the welfare cost of inflation is to address the question: *"By how much should we* raise *an individual's consumption level so that she would be as happy as if she lived in a world with an optimal inflation rate?"*. Assuming that government expenditures are financed by non-distortionary taxes only (which will be the case in this paper), and calling E_r the environment in which the nominal interest rate is r, we could restate this question as: *"How much would the equivalent variation in consumption be, as a function of r, when we move from E_r to E_0?"*.

In Cooley and Hansen's (1989) model, in which money enters via a cash-in-advance constraint, inflation makes people substitute leisure for consumption, lowering their output and utility level. The above question is answered for both the M0 and M1 definitions of money (with monthly and quarterly constraints, respectively), and the estimates found are 0.11% and 0.39% of GNP (table 2, p. 743). In İmrohoroğlu (1992), money is held as a form of insurance against unemployment, allowing a risk-averse agent to smooth his consumption schedule. She answers the same question with a 1.07% estimate (using M0), and also calculates the welfare triangle measure as being 0.41% (p. 88).

Dotsey and Ireland (1996) present a model in which inflation distorts the consumption decision in other ways than in Cooley and Hansen's model. Here, inflation also causes people to divert resources from goods production into the financial sector, and can affect the growth trajectory as well. For the welfare measure arising from the equivalent variation notion, they find estimates associated to a 10% inflation rate of 1.8% and 3.9% (table 1, p. 39), depending if the monetary aggregate considered is M0 or M1.[3] They also calculate the Bailey measure, for both money definitions (table 2, p. 41): 0.65% for M0 and 0.44% for M1.

For the shopping-time measure, Lucas (2000, p. 265) notes that the shopping-time variable in McCallum and Goodfriend's (1987) model may be naturally seen as a measure of the welfare cost of inflation itself. This variable represents the fraction of time spent transacting rather than producing, so that, in equilibrium, by economizing on the use of money because of inflation, the individual spends more time transacting, and produces and consumes less (as we shall see in section 3.1). His estimate for the welfare loss associated to a 10% inflation rate is 1.8% of the American GNP (see fig. 8, p. 266).[4]

[2] Making an average of the costs associated to a 0.5% and a 1% monthly inflation, which are 2.0% and 2.4% (values presented on table 2, p. 986).

[3] Since they normalize the welfare cost of inflation to equal zero when the inflation rate is zero, and not when the nominal interest rate is zero, we have to subtract the entry in the first column from the entry in the fourth column of that table.

[4] The relevant nominal interest rate is 13%, if we choose 3% as the long-term real interest rate (as Lucas does on page 252 of his paper).

The just-mentioned figure in Lucas's paper presents the shopping-time measure as lying always under the Bailey measure, and this is shown to be in fact the general case in Simonsen and Cysne (2001). Moreover, by looking at İmrohoroğlu's and Dotsey and Ireland's estimates above, one may conjecture that Bailey's measure is always lower than the equivalent variation measure, and we shall see in the present work that this also is true.

The reader should note that the usual "comparing-steady-states" measure seems to go in only one direction. There is a natural companion question to the one stated with the equivalent variation notion, that reads: *"How much would the* compensating *variation in consumption be, as a function of* r, *when we move from* E_r *to* E_0*?"*. In other words, *"By how much should we* decrease *an individual's consumption level so that she would be as happy as if she lived in a world with a positive nominal interest rate?"*. The answer to this question gives a measure of the welfare cost of inflation that is not at all traditional in the literature – although we should not claim that it is new.

In Williamson (1998) – the only paper we were able to track down that uses this compensation notion – the welfare loss associated to inflation is due to inefficiencies in the credit system. There it is said that "the welfare costs to each type of agent [risk-averse and risk-neutral] [...] are measured in a conventional way" (p. 561), and it seems that the author has Cooley and Hansen's measuring strategy in mind. But he goes on to define the welfare measure for these agents making use of the compensating variation notion, rather than Cooley and Hansen's equivalent variation notion. This can be seen as an additional reason (but probably not as important as the ones given in that paper) for the costs he estimates being "an order of magnitude lower than the costs calculated by Cooley and Hansen" (p. 563), since this paper is going to show that the welfare measure associated to the compensating variation notion is always smaller than the one associated to the equivalent variation notion.

Another question that might arise from looking at the estimates given above is: "After all, *what is* the monetary aggregate we should use to measure inflation and its burden on society?". Although we shall not address this question, we do give a sufficiently general framework so that one can pick his favorite aggregate and perform welfare calculations with it – or transport this methodology to his favorite monetary model. We've chosen to use a very simple monetary model, Sidrauski's (1967) money-in-the-utility-function model, since we want to focus on the relationship between these measures.

In this work, we shall arrive at various formulas for the measures of the welfare costs of inflation, or differential equations yielding them as solutions. Although our main ordering result depends on these formulas and differential equations only (plus a few natural restrictions on the money-demand functions), it would be meaningless to calculate a welfare measure for a money-demand specification that simply cannot emerge from the model at hand. Moreover, since we will also be making use of McCallum and Goodfriend's (1987) shopping-time model, and comparing welfare measures that appear from each model, it is important that the empirical money-demand function that the economist is willing to use to measure the welfare cost of inflation can really be rationalized by both models.

The paper is organized as follows. Section 2 introduces Sidrauski's and the shopping-time models for an economy in which cash is the only type of money available, as well as formulas for the aforementioned welfare measures and complete-ordering relations between them.[5] Section 3 repeats the work of its preceding section for an economy in which there are other types of monies available, for instance, bank deposits, and presents our main ordering proposition. It also studies the rationalization problem, in order for us to better delimit the scope of our ordering proposition. That is, we're interested in finding conditions on a money-demand specification so that it could be obtained through both of our models, so that all of our welfare measures are really meaningful. In section 4, all of our theoretical findings are applied to the traditional log-log money-demand specification. Section 5 summarizes and concludes the work.

[5] By "complete ordering", we only mean to emphasize the fact that all of these welfare-measure functions can be compared pairwise and do not cross or touch each other, unless identical.

2 Measuring the welfare cost of inflation

In this section, we analyze the case of an economy with only one type of money, cash. We present six different measures of the welfare cost of inflation, discuss how they're ordered, and apply the formulas obtained to the traditional log-log money-demand specification.

2.1 The shopping-time measure and its approximations

In Lucas (2000, p. 265), it is argued that the shopping-time variable s in the McCallum-Goodfriend (1987) model is a natural measure of the welfare cost of inflation, since it represents a waste of a real resource (time available) that wouldn't occur if we had an optimal inflation rate (or a zero nominal interest rate). Here we shall not present nor solve the model in detail – this is done in Simonsen and Cysne (2001, sec. 1). There it is shown that, in equilibrium, s is the solution to

$$\begin{cases} s'(r) = \frac{-rm'(r)}{1-s(r)+rm(r)}(1-s(r)) \\ s(0) = 0 \end{cases}, \tag{1}$$

where $r \in \mathbb{R}_{++}$ stands for the nominal interest rate and $m : \mathbb{R}_{++} \twoheadrightarrow \mathbb{R}_{++}$ is a money-demand function that arises from the model itself.[6]

Since the differential equation above doesn't have any obvious solution, it is natural to seek for approximations to it. By ignoring both $s(r)$'s on its right-hand side, for instance, we would obtain the solution $A : \mathbb{R}_+ \to \overline{\mathbb{R}}_+$ given by

$$A(r) := \int_0^r -\frac{\rho m'(\rho)}{1+\rho m(\rho)} d\rho. \tag{2}$$

Had we only ignored the $s(r)$ appearing in the denominator, the solution to the separable equation derived would be the function $1 - e^{-A}$.

Let $B : \mathbb{R}_+ \to \overline{\mathbb{R}}_+$ stand for Bailey's (1956) measure, that is,

$$B(r) := \int_0^r -\rho m'(\rho) d\rho. \tag{3}$$

In Simonsen and Cysne (2001, prop. 1), it is shown that $1-e^{-A} < s < A < B$, where we write "$f < g$" for "$f(r) < g(r), \forall r \in \mathbb{R}_{++}$", since all of these measures coincide only at 0. In the next subsection, we shall introduce two new measures of the welfare cost of inflation, and see how they relate to the preceding ones.

2.2 The Sidrauski model

We shall assume a forever-living, perfectly-foresighted, representative agent maximizing a time-separable constant-relative-risk-aversion utility function, the arguments of which are the flows of real consumption of a single non-monetary nonstorable good and of holdings of real cash balances, exactly as in Lucas (2000, sec. 3). For every $t \in [0, +\infty)$, let $B_t \in \mathbb{R}_+$, $M_t \in \mathbb{R}_+$, $H_t \in \mathbb{R}$, $Y_t \in \mathbb{R}_{++}$ and $C_t \in \mathbb{R}_{++}$ represent the nominal values of, respectively, holdings of government bonds and cash, a lump-sum tax (if negative, a transfer from the government to the individual), the product of the economy and consumption at instant t of time.

So the dynamic budget constraint faced by our representative agent is

$$\dot{B}_t + \dot{M}_t = Y_t - H_t - C_t + r_t B_t,$$

[6] We use the double-headed arrow to denote a surjective function.

where the dots mean time-derivatives and $r_t \in \mathbb{R}_{++}$ stands for the nominal interest rate bonds yield at time t (by definition, cash is a monetary asset always yielding a nominal interest rate of 0). Let's say she was endowed with $B_0, M_0 > 0$. Let $P_t \in \mathbb{R}_{++}$ be the (both expected and realized) price level and $\pi_t := \frac{\dot{P}_t}{P_t}$ be the inflation rate at time t. Lowercase variables are the real counterparts of the above nominal variables (that is, $b_t := \frac{B_t}{P_t}$ etc.), with the important exception that we shall call $z_t := \frac{M_t}{P_t}$ (m_t will stand for $\frac{M_t}{Y_t}$ instead; soon we'll see why this is convenient).

We can formally state our agent's problem ($\mathrm{P_S}$) as

$$\max_{c_t, z_t \geq 0} \int_0^{+\infty} e^{-\rho t} U(c_t, z_t) dt \tag{$\mathrm{P_S}$}$$

subject to

$$\dot{b}_t + \dot{z}_t = y_t - h_t - c_t + (r_t - \pi_t) b_t - \pi_t z_t, \forall t \in (0, +\infty),$$
$$b_0 \text{ and } z_0 \text{ given.}$$

Product growth is not being considered here, although the inclusion of a constant growth factor would not alter our solution to the model. Henceforth, time subscripts are ommited whenever they don't add to the comprehension of the text. Still following Lucas (2000), we make use of an instantaneous utility function $U : \mathbb{R}_{++} \times \mathbb{R}_{+} \to \mathbb{R}$ with the following functional form:

$$U(c, z) = \frac{1}{1-\sigma} \left(c \varphi \left(\frac{z}{c} \right) \right)^{1-\sigma},$$

where $\sigma > 0$, $\sigma \neq 1$ and $\varphi : \mathbb{R}_{+} \to \mathbb{R}_{+}$ is a twice-differentiable function, with $\varphi' > 0$ and $\varphi'' < 0$.

The expression $\varphi(m) - m\varphi'(m)$ will appear many times throughout this work, so it's in place to note right from the start that, for positive m, this expression is positive. Indeed, the strict concavity of φ gives $\varphi(0) - \varphi(m) < \varphi'(m)(0 - m)$, and since $\varphi(0) \geq 0$, we have $\varphi(m) - m\varphi'(m) > 0$.

In order to solve ($\mathrm{P_S}$), we plug the value of c taken from the dynamic budget constraint into the integrand of our agent's problem, so that ($\mathrm{P_S}$) becomes

$$\max_{b, z \geq 0} \int_0^{+\infty} e^{-\rho t} U(y - h + (r - \pi) b - \pi z - \dot{b} - \dot{z}, z) dt. \tag{$\mathrm{P_S}$}$$

Notice that U is strictly increasing in each of its variables and also strictly concave.[7] Therefore ($\mathrm{P_S}$) will have a unique solution. This enables us to dismiss corner solutions from now on and assume $b, z > 0$, since we will still be able to find a solution to this problem. Additionally, the concavity of U guarantees that ($\mathrm{P_S}$)'s first-order conditions really characterize its optimum. The transversality condition will be anodinous. The Euler equations, one with respect to b and the other with respect to z, are:

$$\begin{cases} (r - \pi) U_c = -\rho U + \frac{d}{dt}(-U_c) \\ -\pi U_c + U_z = -\rho U + \frac{d}{dt}(-U_c) \end{cases}.$$

Since the right-hand sides of these equations are the same, we at once obtain the intuitive relation

$$r = \frac{U_z}{U_c}. \tag{4}$$

[7] All we need to check is that, for $(c, z) \in \mathbb{R}^2_{++}$,
$U_c(c,z) = \left(c\varphi\left(\frac{z}{c}\right)\right)^{-\sigma} \left(\varphi\left(\frac{z}{c}\right) - \frac{z}{c}\varphi'\left(\frac{z}{c}\right)\right) > 0$,
$U_z(c,z) = \left(c\varphi\left(\frac{z}{c}\right)\right)^{-\sigma} \varphi'\left(\frac{z}{c}\right) > 0$,
$U_{cc}(c,z) = \left(c\varphi\left(\frac{z}{c}\right)\right)^{-\sigma-1} \left(-\sigma\left(\varphi\left(\frac{z}{c}\right) - \frac{z}{c}\varphi'\left(\frac{z}{c}\right)\right)^2 + \frac{z^2}{c^2}\varphi\left(\frac{z}{c}\right)\varphi''\left(\frac{z}{c}\right)\right) < 0$,
$U_{zz}(c,z) = \left(c\varphi\left(\frac{z}{c}\right)\right)^{-\sigma-1} \left(-\sigma\varphi'\left(\frac{z}{c}\right)^2 + \varphi\left(\frac{z}{c}\right)\varphi''\left(\frac{z}{c}\right)\right) < 0$,
$U_{cz}(c,z) = \left(c\varphi\left(\frac{z}{c}\right)\right)^{-\sigma-1} \left(\sigma\frac{z}{c}\varphi'\left(\frac{z}{c}\right)^2 - \varphi\left(\frac{z}{c}\right)\left(\sigma\varphi'\left(\frac{z}{c}\right) + \frac{z}{c}\varphi''\left(\frac{z}{c}\right)\right)\right)$, and
$U_{cc}(c,z)U_{zz}(c,z) - U_{cz}(c,z)^2 = \frac{-\sigma}{c^3}\left(c\varphi\left(\frac{z}{c}\right)\right)^{1-2\sigma}\varphi''\left(\frac{z}{c}\right) > 0$.

Let's define $m := \frac{z}{y}$ (which is positive, since $z > 0$). In equilibrium, we have $c = y$, so $m = \frac{z}{c}$. Then (4) can be rewritten as

$$r = \frac{\varphi'(m)}{\varphi(m) - m\varphi'(m)}, \tag{5}$$

which completely characterizes ($\mathrm{P_S}$)'s solution.

The reader may wish to note that if U instead had the functional form associated with a unitary relative risk aversion ($\sigma = 1$), that is, $U(c, z) = \log\left(c\varphi\left(\frac{z}{c}\right)\right)$, we would still obtain (5), so that all of our subsequent results would remain valid.[8] In fact, it is interesting to observe that (5) doesn't depend at all on σ (nor ρ).

A second thing to observe is that, particularly in the steady state solution, in which $\dot{c} = \dot{z} = 0$, we have $\frac{d}{dt}(-U_c) = -U_{cc}\dot{c} - U_{cz}\dot{z} = 0$. Since $U_c > 0$, the Euler equation relative to b gives us Fisher's classic equation, $r = \rho + \pi$. This justifies our taking the welfare cost of inflation as a function of the nominal interest rate, instead of inflation itself.

The equation above is equation 3.7 in Lucas's paper (which he derives using Bellman's Optimality Principle instead). It gives us r as a positive differentiable function of m, for which we shall write $r = \psi(m)$, where $\psi : \mathbb{R}_{++} \to \mathbb{R}_{++}$.[9] Since

$$\psi'(m) = \frac{\varphi(m)\varphi''(m)}{\left(\varphi\left(m\right) - m\varphi'\left(m\right)\right)^2} < 0,$$

ψ is strictly decreasing, therefore one-to-one.

From now on (until the end of this section), let's assume it is onto too. For this purpose, it suffices to make hypotheses on φ leading to $\lim_{m\to 0_+}\psi(m) = +\infty$ and $\lim_{m\to+\infty}\psi(m) = 0$ (from ψ's decreasingness, continuity and the Intermediate Value Theorem). Natural alternatives would be, for instance, $\varphi(0) = 0$ or $\lim_{m\to 0_+}\varphi'(m) = +\infty$, either one giving the former limit in the last sentence,[10] and $\lim_{m\to+\infty}\varphi'(m) = 0$, giving the latter.[11]

We shall call its inverse function $m : \mathbb{R}_{++} \to \mathbb{R}_{++}$ a "money-demand function". This function is strictly decreasing ($m'(r) = \frac{1}{\psi'(m(r))} < 0$) and surjective by construction. As a practical matter, since the economist does not know φ, he ends up using a money-demand function estimated by the applied scientist, leading us to our first rationalization problem.

2.3 The rationalization problem

What are the conditions a given money-demand specification $m : \mathbb{R}_{++} \to \mathbb{R}_{++}$ has to obey so that it can be rationalized by Sidrauski's model?

We already know that m has to be a differentiable and strictly decreasing function onto $\mathbb{R}_{++}$.[12] To see that these conditions are also sufficient, all we need to do is exhibit a φ consistent with this m. Let ψ denote m's inverse function. Note that equation (5) may be rewritten as

$$\varphi'(m) = \frac{\psi(m)}{1 + m\psi(m)}\varphi(m), \tag{6}$$

[8] In this case, we would have, for $(c, z) \in \mathbb{R}^2_{++}$, $U_c(c, z) = \left(c\varphi\left(\frac{z}{c}\right)\right)^{-1}\left(\varphi\left(\frac{z}{c}\right) - \frac{z}{c}\varphi'\left(\frac{z}{c}\right)\right)$ and $U_z(c, z) = \left(c\varphi\left(\frac{z}{c}\right)\right)^{-1}\varphi'\left(\frac{z}{c}\right)$.

[9] This is the same notation used by Lucas (2000, sec. 3).

[10] $\lim_{m\to 0_+}\psi\left(m\right) = \lim_{m\to 0_+}\frac{\varphi'(m)}{\varphi(m) - m\varphi'(m)} \geq \lim_{m\to 0_+}\frac{\varphi'(m)}{\varphi(m)}$. If $\lim_{m\to 0_+}\varphi'(m) = +\infty$, (A1) results immediately. If $\varphi(0) = 0$, since we may not have $\lim_{m\to 0_+}\varphi'(m) = 0$ (φ' has to be strictly decreasing and always positive), we again obtain $\lim_{m\to 0_+}\psi\left(m\right) = +\infty$.

[11] Letting $\Delta : \mathbb{R}_{++} \to \mathbb{R}_{++}$ be given by $\Delta(m) = \varphi(m) - m\varphi'(m)$, we have that Δ is strictly increasing, since $\Delta'(m) = -m\varphi''(m) > 0$. Therefore, $\lim_{m\to+\infty}\psi(m) \leq \frac{\lim_{m\to+\infty}\varphi'(m)}{\Delta(1)} = 0$.

[12] The surjectivity is a pretty strong condition, since it already excludes, for instance, the semi-log money-demand specification, $m(r) = Ke^{-\alpha r}$, from our analysis.

which is separable, and readily yields the general solution

$$\varphi(m) = Ce^{\int_1^m \frac{\psi(\mu)}{1+\mu\psi(\mu)}d\mu}, \tag{7}$$

for some constant $C > 0$. Bearing this in mind, take $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ given by $\varphi(m) = e^{\int_1^m \frac{\psi(\mu)}{1+\mu\psi(\mu)}d\mu}$. So $\varphi \geq 0$ and, for $m > 0$,

$$\varphi'(m) = \frac{\psi(m)}{1+m\psi(m)}\varphi(m) > 0$$

and

$$\varphi''(m) = \frac{\psi'(m)}{(1+m\psi(m))^2}\varphi(m) < 0.$$

Finally, since $\frac{\varphi'(m)}{\varphi(m)-m\varphi'(m)} = \frac{\frac{\psi(m)}{1+m\psi(m)}}{1-\frac{m\psi(m)}{1+m\psi(m)}} = \psi(m)$, we're done.

The reader should note that the assumption that ψ is onto $\mathbb{R}_{++}$ – that is, that m's domain is the whole $\mathbb{R}_{++}$ – is essential so that we can plug m into formulas (2) and (3). Beside this, through the transformation $\mu := m(\rho)$, (2) and (3) can be rewritten as

$$A(r) = \int_{m(r)}^{+\infty} \frac{\psi(\mu)}{1+\mu\psi(\mu)}d\mu$$

and

$$B(r) = \int_{m(r)}^{+\infty} \psi(\mu)d\mu,$$

where this last formula is the traditional one for the area under the inverted money-demand curve. Curiously enough, the integrand that appeared in the new expression for A is exactly the same as the one which appeared in the solution to the rationalization problem above.

Let $\varphi^* := \sup_{m>0}\varphi(m) = \lim_{m\to+\infty}\varphi(m)$. From (7), we then have $\varphi^* = \lim_{m\to+\infty} Ce^{\int_1^m \frac{\psi(\mu)}{1+\mu\psi(\mu)}d\mu} = Ce^{\int_1^{+\infty} \frac{\psi(\mu)}{1+\mu\psi(\mu)}d\mu}$, so that $\varphi^* < +\infty$ if and only if the integral $\int_1^{+\infty} \frac{\psi(\mu)}{1+\mu\psi(\mu)}d\mu$ converges. Therefore, in either case ($\varphi^* < +\infty$ or $\varphi^* = +\infty$), we may write

$$\frac{\varphi(m(r))}{\varphi^*} = \frac{Ce^{\int_1^{m(r)} \frac{\psi(\mu)}{1+\mu\psi(\mu)}d\mu}}{Ce^{\int_1^{+\infty} \frac{\psi(\mu)}{1+\mu\psi(\mu)}d\mu}} = e^{\int_{+\infty}^{m(r)} \frac{\psi(\mu)}{1+\mu\psi(\mu)}d\mu} = e^{-A(r)}. \tag{8}$$

This is a very unexpected way in which the welfare measure A appears, if we regard its original role in Simonsen and Cysne (2001). There, it was introduced in order to approximate the shopping-time welfare cost of inflation s, as we saw in the previous subsection. In the present work, it has already reappeared in the rationalization of a given money-demand specification by our model, through equation (8) (which shall appear once more in section 3, in a multidimensional format). In the next subsection, we shall see how A also relates in an *exact* manner with the measure of the welfare cost of inflation to be introduced in this paper (the one based on a compensating variation notion).

2.4 The welfare cost of inflation in Sidrauski's model: two different approaches

As argued before, if we're interested in steady-state equilibria, then it is perfectly valid to assume our welfare cost of inflation to be actually a function of the nominal interest rate. If $w : \mathbb{R}_+ \to \mathbb{R}_+$ deserves to be regarded as a measure of the welfare cost of inflation, it should be strictly increasing, making it consistent with Friedman's (1969) rule for the optimal inflation rate. Our common sense also tells us that a sensible normalization would be $w(0) = 0$. All this should be valid for any measure of the welfare cost of inflation – and it surely will be valid for the measure introduced here.

In the Sidrauski framework, Lucas (2000, p. 257) "define[s] the welfare cost $w(r)$ of a nominal rate r to be the percentage income compensation needed to leave the household indifferent between r and 0".

Obviously, there are two different ways of interpreting this definition. The first one, used in that paper, is by using the concept of an *equivalent* variation in income: the percentage raise in income necessary to make people as well off *as if* they would be if the nominal interest rate were to fall to zero. That is,

$$U((1+\overline{w}(r))y, m(r)y) = \lim_{j\to 0_+} U(y, m(j)y). \tag{9}$$

In our framework, (9) gives

$$(1+\overline{w}(r))\varphi\left(\frac{m(r)}{1+\overline{w}(r)}\right) = \lim_{j\to 0_+} \varphi\left(m\left(j\right)\right) = \varphi^*. \tag{10}$$

Differentiating with respect to r and dividing through by $\varphi'\left(\frac{m(r)}{1+\overline{w}(r)}\right)$, we get

$$\overline{w}'(r)\frac{\varphi\left(\frac{m(r)}{1+\overline{w}(r)}\right)}{\varphi'\left(\frac{m(r)}{1+\overline{w}(r)}\right)} + m'\left(r\right) - \overline{w}'(r)\frac{m\left(r\right)}{1+\overline{w}(r)} = 0.$$

But (6) gives us

$$\frac{\varphi\left(\frac{m(r)}{1+\overline{w}(r)}\right)}{\varphi'\left(\frac{m(r)}{1+\overline{w}(r)}\right)} = \frac{1}{\psi\left(\frac{m(r)}{1+\overline{w}(r)}\right)} + \frac{m(r)}{1+\overline{w}(r)},$$

so

$$\overline{w}'(r) = -\psi\left(\frac{m(r)}{1+\overline{w}(r)}\right)m'(r). \tag{11}$$

This is equation 3.11 in Lucas's paper, which, together with the condition $\overline{w}(0) = 0$, enables us to find $\overline{w}$. We've repeated his math here since we will imitate it in section 3.4, where we generalize the present analysis to an economy in which there are many types of monies available.

We now turn our attention to another natural way of interpreting Lucas's definition for the welfare cost of inflation in Sidrauski's model: the one of a *compensating* variation in income.

That is, we will regard as the welfare cost of inflation the percentage fall in people's income that will make them as well off as if they would be, had no decrease in the nominal interest rate taken place:

$$U(y, m(r)y) = \lim_{j\to 0_+} U((1-\underline{w}(r))y, m(j)y). \tag{12}$$

We shall see below that, in accordance with our intuition gained for the partial-equilibrium context (see, for instance, Varian (2002, chap. 14)), the welfare cost of inflation associated to the compensating variation notion is smaller than the one associated to the equivalent variation notion, and the "surplus" of the money-consumer – Bailey's measure – passes somewhere in between (Proposition 1). A possible objection that could be raised to this second approach to the welfare cost of inflation is that we do not usually see such an interest rate cutoff in the real world. But then again, this objection is really immaterial, since it would also serve as an objection to the first approach. Even if the first approach makes it more explicit than the second one that this decrease in the interest rate is not really going to take place, the point is – it doesn't need to. The only thing we are trying to do when we measure the welfare cost of inflation is to somehow compare a theoretical world that has a nonoptimal inflation rate (which incidentally reminds us of our real imperfect world) to another theoretical world (or a limiting theoretical world) with an optimal inflation rate (which is *not* intended to resemble the real world). This is the only way to do it, and it is done equally well by the two different approaches, the one related to the notion of an equivalent and the one related to the notion of a compensating variation in income.

In our model, definition (12) gives

$$\frac{\varphi(m(r))}{1-\underline{w}(r)} = \lim_{j\to 0_+} \varphi\left(\frac{m(j)}{1-\underline{w}(r)}\right). \tag{13}$$

Now, since, by construction, $\underline{w}(r) \in [0,1)$, we have $\varphi\left(\frac{m(j)}{1-\underline{w}(r))}\right) \geq \varphi(m(j))$. Taking limits, we get $\lim_{j\to 0_+} \varphi\left(\frac{m(j)}{1-\underline{w}(r))}\right) \geq \lim_{j\to 0_+} \varphi(m(j)) = \lim_{m\to+\infty} \varphi(m) = \varphi^*$, so $\lim_{j\to 0_+} \varphi\left(\frac{m(j)}{1-\underline{w}(r))}\right) = \varphi^*$, and (13) ends up giving the very simple formula for the welfare cost of inflation $\underline{w}$:

$$\underline{w}(r) = \frac{\varphi^* - \varphi(m(r))}{\varphi^*}. \tag{14}$$

Note how $\frac{\varphi^* - \varphi \circ m}{\varphi^*}$ is a very natural measure of the loss of utility as a consequence of the economy not being completely satiated with real balances – which corresponds to the case with the lowest possible nominal interest rate, 0. That is, a measure of the welfare cost of inflation. So the previous developments can be understood as providing a formalization for this intuitive measure.

Taking (8) into account, we achieve the surprising relation

$$\underline{w}(r) = 1 - e^{-A(r)}. \tag{15}$$

This is the result promised at the end of the preceding subsection. In Simonsen and Cysne (2001), the measure $1-e^{-A}$ was shown to approximate the measure of the welfare cost of inflation in the shopping-time model, s. Here, we've seen how a different model, the money-in-the-utility-function model, can provide a sensible explanation to this measure too.

Comparing (15) to (11), we may observe two advantages $\underline{w}$ presents in relation to $\overline{w}$: formal elegance and computational ease. We no longer have to solve a possibly very difficult differential equation – although we do still need to evaluate a possibly very difficult integral. To any extent, this is exactly the same integral one would evaluate if trying to approximate the shopping-time measure.

The following proposition corroborates our intuition that, associated with a decrease in the cost of holding money (r), the equivalent variation in income ($\overline{w}$) is no smaller than the money-consumer surplus (B), which in turn is no smaller than the compensating variation in income ($\underline{w}$).

Proposition 1 *Let $m : \mathbb{R}_{++} \twoheadrightarrow \mathbb{R}_{++}$ be a differentiable and strictly decreasing money-demand function, so that we can calculate for it s, A, B, $\overline{w}$, and $\underline{w}$ (using (1), (2), (3), (9) and (12)). Then we have the following inequality chain: $\underline{w} = 1 - e^{-A} < s < A < B < \overline{w}$.*

Proof. The equality has just been shown. The first, second and third inequalities have already been shown in Simonsen and Cysne (2001, prop. 1), and the reader may want to notice that their proof draws only on the strict decreasingness of m, not on other possible characteristics enjoined by money-demand functions arising from the shopping-time model. As to the fourth inequality, (11) gives us, for $\rho \in (0, r]$, $\overline{w}(\rho) > -\psi(m(\rho))m'(\rho) = -\rho m'(\rho)$ (remember that $m' < 0$ and that ψ is a strictly decreasing function), so all that is left to do is integrate both sides of this last inequality. ■

One should notice that, although the thesis of this proposition already is the complete ordering promised in the introduction and the title of this paper, we may be calculating welfare measures for money-demand specifications that simply do not arise from the relevant model. For instance, the hypothesis made only guarantees that this specification can emerge from the Sidrauski model, but we do not know, as for now, if it could also emerge from the McCallum-Goodfriend model. Therefore, perhaps the measure s above is simply meaningless.

In the next section, we shall generalize the preceding analysis to an economy in which there are many types of monies available. Then the problem of determining which money-demands are compatible with both the Sidrauski and the shopping-time models will be addressed directly in this multidimensional framework.

3 The case of many types of monies

In the real world, people may choose to keep some of their money in banks, so that they are able to earn interest on it, besides being able to buy stuff with it. For non-arbitrage reasons, these bank deposits must pay a nominal interest rate not as good as the one paid by bonds (which have reduced liquidity), but also not as bad as the one paid by cash (which has total liquidity), zero. Bearing that in mind, we shall extend the basic only-one-type-of-money framework in section 2 to a framework with n types of monies available, introduce the measures $\underline{w}$ and $\overline{w}$ for this context and study which money-demand specifications can be rationalized by our model. Our basic strategy will be to try to imitate the analysis done in the previous section.

For now, we only make a few general remarks. The first is just notational: $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{R}^n_+$ will represent the vector of real quantities of each type of money being demanded, relative to the output of the economy (where m_1 is chosen to be m, real currency per output). Each m_i yields a nominal interest rate of r_i, and $\mathbf{r} := (r_1, \ldots, r_n) \in \mathbb{R}^n_+$ (with $r_1 = 0$, by definition). We shall write $\mathbf{u} := (r, r - r_2, \ldots, r - r_n) \in \mathbb{R}^n_{++}$ for the vector of opportunity costs of holding money instead of government bonds.

The second is that, in this general context, a money-demand function will not be a function taking $\mathbf{r}$ into $\mathbf{m}$. In the unidimensional case, the first-order conditions of the money-in-the-utility-function model yielded a function ψ taking m into r, and the simple imposition that this function was onto $\mathbb{R}^n_{++}$ was enough in order for us to have its inverse function, which we called the "money-demand function", well defined. In the multidimensional case, the first-order conditions of the money-in-the-utility-function model, as well as of the shopping-time model, will give a function $\boldsymbol{\psi}$ taking $\mathbf{m}$ into $\mathbf{u}$ (and not $\mathbf{r}$). But, in order to obtain its invertibility, it will not suffice to ask for its surjectivity, since its injectivity will not be guaranteed in the first place (although it will be in the case $n = 2$, we shall develop this point in the appendix).

The third is that, for this reason, our measures of the welfare cost of inflation in many-types-of-monies economies will actually be evaluated at $\mathbf{m}$, rather than $\mathbf{u}$. This extends what we've done before because, in that particular situation, the function ψ taking m into r was invertible.

3.1 The shopping-time measure and its approximations

The multidimensional McCallum-Goodfriend model is introduced and solved in Cysne (2003). There, it is assumed a transacting-technology constraint of the form

$$c_t = G(\mathbf{m}_t)\,\phi(s_t)\,, \forall t \in [0, +\infty),$$

where c and s stand for the real level of consumption and the portion of time spent transacting rather than producing. The money aggregator function $G : \mathbb{R}^n_+ \to \mathbb{R}_+$ is a twice-differentiable 1-homogeneous concave function such that $G_{x_i} > 0$ and $G_{x_i x_i} < 0$ for all $i \in \{1, \ldots, n\}$,[13] and $\phi : [0, 1] \to \mathbb{R}_+$ is a twice-differentiable function such that $\phi(0) = 0$, $\phi' > 0$ and $\phi'^2 - \phi\phi'' > 0$.[14] This model's first-order conditions yield the equilibrium relations (Cysne 2003, p. 224):

$$\begin{cases} u_i = \frac{G_{x_i}(\mathbf{m})\phi(s)}{G(\mathbf{m})\phi'(s)}, \forall i \in \{1, \ldots, n\} \\ G(\mathbf{m})\,\phi(s) = 1 - s \end{cases}.$$

[13] If $n = 1$, G would have to be linear, whence $G'' = 0$. Therefore, our analysis in this section is restricted to the case $n > 1$. Even so, it yields exactly the same results as the $n = 1$ framework analyzed in the last section.

[14] We have intentionally made a minor departure from the model introduced in Cysne (2003): we are not supposing $\phi'' \leq 0$, but only something weaker.

The hypothesis $\phi'' \leq 0$ isn't intrinsic to McCallum and Goodfriend's original model (the only related restriction in that paper is equation 2 on page 776, which just gives $\phi' > 0$ in our setting), nor is it needed or does it interfere in any manner in the solution derived in Cysne (2003).

The assumption in the way we made it (which is equivalent to $\left(\frac{\phi}{\phi'}\right)' > 0$) is important so that the log-log money-demand specification can be rationalized by this model, as we shall see in subsection 4.4.

This last equation, $G\phi(s) = 1 - s$, shows us that, in equilibrium, we necessarily have $s > 0$. More than this, it can be seen to give us s as a function of G alone. For this purpose, we only need to check that the function $H : (0, 1] \to \mathbb{R}_{++}$ defined by $H(s) = \frac{1-s}{\phi(s)}$ is invertible. In fact, since $H'(s) = \frac{-\phi(s)-(1-s)\phi'(s)}{\phi(s)^2} < 0$, $H(1) = 0$ and $\lim_{s\to 0_+} H(s) = +\infty$, we're through. By writing τ for its inverse function, the preceding set of equations becomes a money-demand specification of n variables and n equations:

$$\psi_i(\mathbf{m}) = \frac{\phi\left(\tau\left(G\left(\mathbf{m}\right)\right)\right)}{G(\mathbf{m})\phi'\left(\tau\left(G\left(\mathbf{m}\right)\right)\right)} G_{x_i}(\mathbf{m}), \forall i \in \{1, \ldots, n\}, \tag{16}$$

where ψ_i is the i-th component of the function $\boldsymbol{\psi}$ taking $\mathbf{m}$ into $\mathbf{u}$. We shall return to this expression in section 3.3, in order to compare money-demand functions that may arise from this model with those that may arise from Sidrauski's model. But we can already observe an interesting property about the function $\boldsymbol{\psi}$: along rays starting at the origin, each ψ_i can be seen to be strictly decreasing. In fact, if $k > 1$, we get, by G's strict increasingness in each variable, $G(k\mathbf{m}) > G(\mathbf{m})$. Therefore, since τ is strictly decreasing (like H), $\tau(G(k\mathbf{m})) < \tau(G(\mathbf{m}))$, and $\left(\frac{\phi}{\phi'}\right)(\tau(G(k\mathbf{m}))) < \left(\frac{\phi}{\phi'}\right)(\tau(G(\mathbf{m})))$. From G's homogeneity, we also get $G_{x_i}(k\mathbf{m}) = G_{x_i}(\mathbf{m})$.[15] Therefore $\psi_i(k\mathbf{m}) = \frac{G_{x_i}(k\mathbf{m})}{G(k\mathbf{m})}\left(\frac{\phi}{\phi'}\right)(\tau(G(k\mathbf{m}))) < \frac{G_{x_i}(\mathbf{m})}{G(\mathbf{m})}\left(\frac{\phi}{\phi'}\right)(\tau(G(\mathbf{m}))) = \psi_i(\mathbf{m})$, as we wished.

The multidimensional analog of (1) was derived in Cysne (2003, eq. 14):

$$s_{x_i}(\mathbf{m}) = -\frac{\psi_i(\mathbf{m})}{1 - s(\mathbf{m}) + \boldsymbol{\psi}(\mathbf{m}) \cdot \mathbf{m}}(1 - s(\mathbf{m})). \tag{17}$$

As argued in Bruce (1977, appendix) and Cysne (2003, p. 225), line integrals are an appropriate tool for measuring the welfare cost of inflation in economies with many types of monies. Let's consider a C^1 path $\boldsymbol{\chi} : (0, 1] \to \mathbb{R}^n_{++}$ and the following 1-forms in $\mathbb{R}^n_{++}$:

$$dA := -\frac{1}{1 + \mathbf{u} \cdot \mathbf{m}} \mathbf{u} \cdot d\mathbf{m} \tag{18}$$

and

$$dB := -\mathbf{u} \cdot d\mathbf{m}. \tag{19}$$

The line integrals $A(\boldsymbol{\chi}) := \int_{\boldsymbol{\chi}} dA$ and $B(\boldsymbol{\chi}) := \int_{\boldsymbol{\chi}} dB$ will extend for the present framework the notions, respectively, of Simonsen and Cysne's (2001) proxy measure and of Bailey's (1956) measure.[16] In Cysne (2003, prop. 1), it is shown that, if $\boldsymbol{\psi}$ arises from the shopping-time model, these integrals are path-independent.

When measuring the welfare cost of inflation, we are specifically interested in paths $\boldsymbol{\chi}$ such that $\lim_{\lambda\to 0_+} \boldsymbol{\chi}(\lambda) = (+\infty, \ldots, +\infty) =: +\infty$. This is because that's how we get the lowest possible values for the u_i's (since each ψ_i is strictly decreasing along rays starting at the origin), and this is the benchmark used for measuring the welfare cost of inflation ($\mathbf{u} \to \mathbf{0} \Rightarrow \mathbf{r} \to \mathbf{0}$). So we are replacing the initial condition introduced in the beginning of section 2.4 ($w(0) = 0$) with this one: $w(+\infty) = 0$, for any measure w of the welfare cost of inflation. Because of the path-independences of A and B, we shall from now on write $A(\mathbf{m})$ instead of $A(\boldsymbol{\chi})$, and the same for B, where $\mathbf{m} = \boldsymbol{\chi}(1)$.

In proposition 2 of Cysne (2003), it is demonstrated that, exactly as happens in the unidimensional case, we have $1 - e^{-A} < s < A < B$. In the next subsection, we shall introduce the multidimensional analogs of $\underline{w}$ and $\overline{w}$, and see how they relate to the above welfare measures.

3.2 The extended Sidrauski model

Let $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n_+$ be the vector of real quantities demanded for each type of money (we shall think of x_1 as being z, real currency), so that $\mathbf{m} = \frac{1}{y}\mathbf{x}$. Our representative agent's instantaneous utility

[15] Any partial derivative of a g-homogeneous function is a $(g-1)$-homogeneous function.

[16] These are, in order, the additive inverses of the Divisia indices $DE(\boldsymbol{\chi})$ and $DG(\boldsymbol{\chi})$ presented in Cysne (2003).

will now have the form

$$U(c,\mathbf{x}) = \frac{1}{1-\sigma}\left(c\varphi\left(\frac{G(\mathbf{x})}{c}\right)\right)^{1-\sigma},$$

where σ and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ are exactly as in section 2.2, and G is exactly as in the previous subsection. Her maximization problem will be:

$$\max_{c_t>0,\mathbf{x}_t\geq 0} \int_0^{+\infty} e^{-\rho t} U(c_t, \mathbf{x}_t) dt \tag{$\mathrm{P}_{\mathrm{S}^n}$}$$

subject to

$$\dot{b}_t + \mathbf{1}\cdot\dot{\mathbf{x}}_t = y_t - h_t - c_t + (r_t - \pi_t) b_t + (\mathbf{r}_t - \pi_t \mathbf{1})\cdot \mathbf{x}_t, \forall t \in (0,+\infty),$$
$$b_0 > 0 \text{ and } \mathbf{x}_0 > 0 \text{ given.}$$

where we write $\mathbf{1}$ for the vector $(1,\ldots,1)\in\mathbb{R}^n$ and $\cdot$ for the canonical inner product of $\mathbb{R}^n$, and all the non-bold letters have the same meaning as in the model introduced in section 2.2. Considering only regular solutions and substituting for c as we've done before, we may rewrite our maximization problem as

$$\max_{b>0,\mathbf{x}\gg\mathbf{0}} \int_0^{+\infty} e^{-\rho t} U(y - h + (r-\pi)b + (\mathbf{r}-\pi\mathbf{1})\cdot\mathbf{x} - \dot{b} - \mathbf{1}\cdot\dot{\mathbf{x}}, \mathbf{x}) dt. \tag{$\mathrm{P}_{\mathrm{S}^n}$}$$

Its Euler equations are

$$\begin{cases} (r-\pi)U_c = -\rho U + \frac{d}{dt}(-U_c) \\ (r_i - \pi)U_c + U_{x_i} = -\rho U + \frac{d}{dt}(-U_c), \forall i \in \{1,\ldots,n\} \end{cases},$$

which really correspond to the optimum of ($\mathrm{P}_{\mathrm{S}^n}$), by U's concavity.[17] As before, these equations immediately give

$$r - r_i = \frac{U_{x_i}}{U_c}, \forall i \in \{1,\ldots,n\}, \tag{20}$$

and since $U_c, U_{x_i} > 0$,[18] we have $r - r_i > 0$.

Let $\mathbf{m} = \frac{1}{y}\mathbf{x} \in \mathbb{R}^n_{++}$. From the homogeneity of G, we know that $\frac{G(\mathbf{x})}{y} = G(\mathbf{m})$ and $G_{x_i}(\mathbf{x}) = G_{x_i}(\mathbf{m})$. In equilibrium, we have $c = y$, so that (20) gives

$$\psi_i(\mathbf{m}) = \frac{\varphi'(G(\mathbf{m}))}{\varphi(G(\mathbf{m})) - G(\mathbf{m})\varphi'(G(\mathbf{m}))} G_{x_i}(\mathbf{m}), \forall i \in \{1,\ldots,n\}. \tag{21}$$

This equation is analogous to (5), giving us a differentiable function $\boldsymbol{\psi} : \mathbb{R}^n_{++} \to \mathbb{R}^n_{++}$ just like the function ψ of section 2.2. But now, in contrast with what we've done there, we will not work with its inverse function, because we're not assured of its existence in the first place. This extends what we've done before because, in that particular situation, the function ψ taking m into r was invertible. In the theoretical development ahead, we don't make any assumptions leading to the invertibility of $\boldsymbol{\psi}$. Nevertheless, in the appendix, you can see two natural instances in which a corresponding function taking $\mathbf{u}$ into $\mathbf{m}$ indeed exists – followed by an example where such a function would not exist at all.

[17] For now, take V to be the function U of section 2 (of only two variables), so that $U(c,\mathbf{x}) = V(c, G(\mathbf{x}))$. Given $c, d \in \mathbb{R}_{++}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n_+$, we have

$U(d,\mathbf{y}) - U(c,\mathbf{x}) = V(d, G(\mathbf{y})) - V(c, G(\mathbf{x})) \leq D_c V_{(c,G(\mathbf{x}))}(d-c) + D_z V_{(c,G(\mathbf{x}))}(G(\mathbf{y}) - G(\mathbf{x})) \leq D_c V_{(c,G(\mathbf{x}))}(d-c) + D_z V_{(c,G(\mathbf{x}))} DG_{\mathbf{x}}(\mathbf{y}-\mathbf{x}) = DU_{(c,\mathbf{x})}(d-c, \mathbf{y}-\mathbf{x})$, where we've used the concavity of G and the fact that $D_z V \geq 0$.

[18] For $\mathbf{x} \in \mathbb{R}^n_{++}$, since $\nabla G \gg \mathbf{0}$ and $G \geq 0$, we have $G(\mathbf{x}) > 0$. So

$U_c(c,\mathbf{x}) = \left(c\varphi\left(\frac{G(\mathbf{x})}{c}\right)\right)^{-\sigma}\left(\varphi\left(\frac{G(\mathbf{x})}{c}\right) - \frac{G(\mathbf{x})}{c}\varphi'\left(\frac{G(\mathbf{x})}{c}\right)\right) > 0$, and

$U_{x_i}(c,\mathbf{x}) = \left(c\varphi\left(\frac{G(\mathbf{x})}{c}\right)\right)^{-\sigma}\varphi'\left(\frac{G(\mathbf{x})}{c}\right) G_{x_i}(\mathbf{x}) > 0.$

3.3 The rationalization problem

What are the conditions a given money-demand specification $\boldsymbol{\psi} : \mathbb{R}^n_{++} \to \mathbb{R}^n_{++}$ has to obey so that it can be rationalized by the extended Sidrauski model?

From (21), we see that it is necessary for $\boldsymbol{\psi}$ to be a function taking the form

$$\boldsymbol{\psi}(\mathbf{m}) = F(G(\mathbf{m}))\nabla G(\mathbf{m}), \tag{22}$$

where G has the aforementioned properties and $F : \mathbb{R}_{++} \to \mathbb{R}_{++}$ is a differentiable function with $F' < 0$ (the proof of this last fact is formally identical to the proof that ψ from section 2.2 was such that $\psi' < 0$). In order to see that this is also sufficient, we proceed as in section 2.3: we exhibit a $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\varphi' > 0$ and $\varphi'' < 0$, such that $F(G) = \frac{\varphi'(G)}{\varphi(G) - G\varphi'(G)}$. Take φ given by

$$\varphi(G) = e^{\int_1^G \frac{F(\tilde{G})}{1+\tilde{G}F(\tilde{G})} d\tilde{G}},$$

so that

$$\varphi'(G) = \frac{F(G)}{1 + GF(G)}\varphi(G) > 0,$$

and

$$\begin{aligned}\varphi''(G) &= \frac{F'(G)(1 + GF(G)) - F(G)(F(G) + GF'(G))}{(1 + GF(G))^2}\varphi(G) + \left(\frac{F(G)}{1 + GF(G)}\right)^2 \varphi(G) \\ &= \frac{F'(G)}{(1 + GF(G))^2}\varphi(G) < 0.\end{aligned}$$

Moreover, $\frac{\varphi'(G)}{\varphi(G) - G\varphi'(G)} = \frac{\frac{F(G)}{1+GF(G)}\varphi(G)}{\varphi(G) - \frac{GF(G)}{1+GF(G)}\varphi(G)} = F(G)$, as we wanted.

We are now ready to introduce our equivalence-type result between the two models, McCallum and Goodfriend's and Sidrauski's. Our proof is based on a different strategy than the similar result obtained by Feenstra (1986, prop. 1). His proof, although not done explicitly for the shopping-time model, is based on a formal equivalence of the maximization problems themselves, while ours is based on the necessary and sufficient first-order conditions. Although very elegant, his proof doesn't apply to our case.[19]

Actually, since we will see a counterexample for the converse of this lemma at the end of the next section, this is *not* an equivalence result, but an embedding result only. In fact, while (21) gives us $\frac{u_i}{G_{x_i}}$ as a strictly decreasing function of G, (16) gives us $\frac{u_i}{G_{x_i}}$ as a decreasing function of G (from $\left(\frac{\phi}{\phi'}\right)$'s increasingness and τ's decreasingness) *divided by* G, and this imposition is clearly stronger than the former.

Lemma 1 *Money-demand specifications which are rationalizable by McCallum and Goodfriend's model are also rationalizable by Sidrauski's model.*

Proof. By comparing (16) with (22), and from the solution to the rationalization problem in section 3.3, it is enough to show that the function $F : \mathbb{R}_{++} \to \mathbb{R}_{++}$ given by $F(G) = \frac{\phi(\tau(G))}{G\phi'(\tau(G))}$ is differentiable and $F' < 0$. In section 3.1, we've calculated H', and the reader may note that it is continuous. So the Inverse Function Theorem applied to function H guarantees that the function τ is also C^1. Therefore F

[19] The reader may observe that the first of his assumptions 2(c) on page 281 ($W_{xx} \leq 0$, where W is defined in his equation 16') would yield the concavity of our ϕ – an imposition that eliminates the possibility of the log-log money-demand specification being compatible with the shopping-time model (details below, in subsection 4.4). In fact, the application of his strategy to the McCallum-Goodfriend model, in the notation we've been using, would give $W_{xx}(x, m) = \frac{m\phi''(s)}{(1+m\phi'(s))^3}$, where s is such that $m\phi(s) + s = x$.

is differentiable. We also obtain, by that theorem, $\tau'(G) = \frac{1}{H'(\tau(G))} < 0$, so that

$$\begin{aligned} F'(G) &= \frac{\phi'(\tau(G))^2 \tau'(G) G - \phi(\tau(G)) \left[\phi'(\tau(G)) + G\phi''(\tau(G)) \tau'(G)\right]}{\left(G\phi'(\tau(G))\right)^2} \\ &= \frac{G\tau'(G) \left[\phi'(\tau(G))^2 - \phi(\tau(G)) \phi''(\tau(G))\right] - \phi(\tau(G)) \phi'(\tau(G))}{\left(G\phi'(\tau(G))\right)^2} \\ &\leq \frac{-\phi(\tau(G)) \phi'(\tau(G))}{\left(G\phi'(\tau(G))\right)^2} < 0, \end{aligned}$$

and we're finished. ∎

Now we can proceed a little further, and try to recognize what a suitable G would be such that $\boldsymbol{\psi} = (F \circ G) \nabla G$, in case this piece of information wasn't provided by the econometrician. That is, we can imagine that the only thing we have been told is that $\boldsymbol{\psi}$ *can* be put in that form – but we do not know *how*. Using (21) and Euler's formula for homogeneous functions, we get

$$\boldsymbol{\psi}(\mathbf{m}) \cdot \mathbf{m} = \frac{G(\mathbf{m})\, \varphi'(G(\mathbf{m}))}{\varphi(G(\mathbf{m})) - G(\mathbf{m})\, \varphi'(G(\mathbf{m}))}, \tag{23}$$

which, together with (21), gives the following system of partial differential equations:

$$\begin{cases} G_{x_1}(\mathbf{m}) = \frac{G(\mathbf{m})}{\boldsymbol{\psi}(\mathbf{m}) \cdot \mathbf{m}} u_1(\mathbf{m}) \\ \qquad \vdots \\ G_{x_n}(\mathbf{m}) = \frac{G(\mathbf{m})}{\boldsymbol{\psi}(\mathbf{m}) \cdot \mathbf{m}} u_n(\mathbf{m}) \end{cases}. \tag{24}$$

Since G is positive for $\mathbf{m} \in \mathbb{R}^n_{++}$, we can conveniently rewrite this system as

$$\begin{cases} (\log G)_{x_1}(\mathbf{m}) = \frac{1}{\boldsymbol{\psi}(\mathbf{m}) \cdot \mathbf{m}} u_1(\mathbf{m}) \\ \qquad \vdots \\ (\log G)_{x_n}(\mathbf{m}) = \frac{1}{\boldsymbol{\psi}(\mathbf{m}) \cdot \mathbf{m}} u_n(\mathbf{m}) \end{cases}, \tag{25}$$

which yields the solution

$$G^*(\mathbf{m}) = De^{\int_\Gamma \frac{\boldsymbol{\psi}(\boldsymbol{\mu}) \cdot d\boldsymbol{\mu}}{\boldsymbol{\psi}(\boldsymbol{\mu}) \cdot \boldsymbol{\mu}}},\ D > 0, \tag{26}$$

where $\boldsymbol{\Gamma} : [0,1] \to \mathbb{R}^n_{++}$ is a piecewise-C^1 path such that $\boldsymbol{\Gamma}(0) = \mathbf{1}$ and $\boldsymbol{\Gamma}(1) = \mathbf{m}$. Evidently, in order for us to write (26), the integral that appears in this expression has to be really well-defined – that is, path-independent. Since $\log \circ G : \mathbb{R}^n_{++} \to \mathbb{R}$ is an obvious potential function for the vector field taking $\mathbf{m} \in \mathbb{R}^n_{++}$ into $\frac{1}{\boldsymbol{\psi}(\mathbf{m}) \cdot \mathbf{m}} \boldsymbol{\psi}(\mathbf{m}) \in \mathbb{R}^n$, we have no problem.

Therefore, just like the solution for φ, the solution for G is also unique up to a multiplicative constant. You may also want to notice how (26) extends the case $n = 1$, where G would be just the identity function.

Also, in the case of a money-demand function that we happen to know can be rationalized by the extended McCallum-Goodfriend model, its money aggregator function G has to be, up to a multiplicative constant, exactly the same as if we would try to rationalize this demand by the extended Sidrauski model. This is because (16) also gives (24), exactly in the same way (21) does.

As with the money-demand functions originating from the shopping-time model, an interesting thing to observe about each ψ_i is that it has to be strictly decreasing along rays starting at the origin. In fact, if $k > 1$, since G is strictly increasing in each variable, we get $G(k\mathbf{m}) > G(\mathbf{m})$. Looking at (22), since F is strictly decreasing, we get $F(G(k\mathbf{m})) < F(G(\mathbf{m}))$. But $\nabla G(k\mathbf{m}) = \nabla G(\mathbf{m})$, so $\boldsymbol{\psi}(k\mathbf{m}) < \boldsymbol{\psi}(\mathbf{m})$, as we wished.

Another striking fact about the structure of these demands is that, for every $i, j \in \{1, \ldots, n\}$, $\frac{\psi_j(\mathbf{m})}{\psi_i(\mathbf{m})} = \frac{G_{x_j}(\mathbf{m})}{G_{x_i}(\mathbf{m})}$. Now since both G_{x_i} and G_{x_j} are homogeneous functions of the same degree, $\frac{G_{x_j}}{G_{x_i}}$ is a 0-homogeneous function – that is, a constant along each ray starting at the origin (but excluding this point). So, given u_1, for instance, $u_2, \ldots, u_n$ should be completely determined by the values they take on S^{n-1}_{++}, the intersection of the positive orthant of $\mathbb{R}^n$ with the $(n-1)$-dimensional sphere. By the lemma above, this must automatically hold for money-demand functions arising from the shopping-time model too.

3.4 The welfare cost of inflation in the extended Sidrauski model: two different approaches

The lemma of the preceding subsection shows that the maximal domain of application of our complete-ordering proposition is the set of money-demand specifications which are rationalizable by the extended shopping-time model. Therefore, we do not have to check if A and B as defined in section 3.1 would continue to be path-independent if calculated for any money-demand function compatible with the extended money-in-the-utility-function model, not only those compatible with the extended shopping-time model – although this is very easy to do.[20]

Motivated by the path-independences of A and B (the other two measures to be introduced in this subsection will not be defined as line integrals), the path $\boldsymbol{\chi}$ we take from now on not only has the properties $\lim_{\lambda \to 0_+} \boldsymbol{\chi}(\lambda) = +\infty$ and $\boldsymbol{\chi}(1) = \mathbf{m}$, but also $\dot{\boldsymbol{\chi}} \ll \mathbf{0}, \forall i \in \{1, \ldots, n\}$. This will be important in order for us to obtain our ordering result.

Let's now write down the equations defining the measures $\overline{w}$ and $\underline{w}$ of the welfare cost of inflation:[21]

$$\begin{cases} U((1+\overline{w}(\mathbf{m}))y, yG(\mathbf{m})) = \lim\limits_{\lambda \to 0_+} U(y, yG(\boldsymbol{\chi}(\lambda))), \text{ or} \\ (1+\overline{w}(\mathbf{m}))\varphi\left(\frac{G(\mathbf{m})}{(1+\overline{w}(\mathbf{m}))}\right) = \lim\limits_{\lambda \to 0_+} \varphi\left(G\left(\boldsymbol{\chi}\left(\lambda\right)\right)\right) = \varphi^*; \end{cases} \tag{27}$$

$$\begin{cases} U(y, yG(\mathbf{m})) = \lim\limits_{\lambda \to 0_+} U((1-\underline{w}(\mathbf{m}))y, yG(\boldsymbol{\chi}(\lambda))), \text{ or} \\ \varphi(G(\mathbf{m})) = \lim\limits_{\lambda \to 0_+} \left[(1-\underline{w}(\mathbf{m}))\varphi\left(\frac{G(\boldsymbol{\chi}(\lambda))}{1-\underline{w}(\mathbf{m})}\right)\right] = (1-\underline{w}(\mathbf{m}))\varphi^*. \end{cases} \tag{28}$$

Partially differentiating the first of these equations with respect to x_i, and dividing through by $\varphi'\left(G\left(\frac{1}{1+\overline{w}(\mathbf{m})}\mathbf{m}\right)\right)$, we obtain

$$\overline{w}_{x_i}(\mathbf{m}) \frac{\varphi\left(G\left(\frac{1}{1+\overline{w}(\mathbf{m})}\mathbf{m}\right)\right)}{\varphi'\left(G\left(\frac{1}{1+\overline{w}(\mathbf{m})}\mathbf{m}\right)\right)} + \left[G_{x_i}(\mathbf{m}) - \overline{w}_{x_i}(\mathbf{m})\, G\left(\frac{1}{1+\overline{w}(\mathbf{m})}\mathbf{m}\right)\right] = 0.$$

But (21) gives us

$$\frac{\varphi\left(G\left(\frac{1}{1+\overline{w}(\mathbf{m})}\mathbf{m}\right)\right)}{\varphi'\left(G\left(\frac{1}{1+\overline{w}(\mathbf{m})}\mathbf{m}\right)\right)} = \frac{G_{x_i}(\mathbf{m})}{\psi_i\left(\frac{1}{1+\overline{w}(\mathbf{m})}\mathbf{m}\right)} + G\left(\frac{1}{1+\overline{w}(\mathbf{m})}\mathbf{m}\right),$$

so we get the expression

$$\overline{w}_{x_i}(\mathbf{m}) = -\psi_i\left(\frac{1}{1+\overline{w}(\mathbf{m})}\mathbf{m}\right), \tag{29}$$

[20] The interested reader may want to notice that $-\log \circ \varphi \circ G$ is the relevant potential function for the first measure. As for the second, since $\mathbb{R}^n_{++}$ is simply connected, all that is needed is to notice that the form dB is closed, that is, $\frac{\partial}{\partial x_j}(-\psi_i) = \frac{\partial}{\partial x_i}(-\psi_j), \forall i, j \in \{1, \ldots, n\}$. This follows immediately from (22).

[21] Where, as before, $\varphi^* := \sup_{G>0} \varphi(G) = \lim_{\lambda \to 0_+} \varphi(G(\boldsymbol{\chi}(\lambda)))$, and the equilibrium relation $c = y$ is valid.

analogous to (11). From ψ_i's strict decreasingness along rays starting at the origin, we have $\overline{w}_{x_i}(\mathbf{m}) < -\psi_i(\mathbf{m})$. Therefore

$$\begin{aligned} \overline{w}(\mathbf{m}) &= \int_0^1 \frac{d}{d\lambda}\overline{w}(\boldsymbol{\chi}(\lambda))\, d\lambda = \int_0^1 \left[-\boldsymbol{\psi}\left(\frac{1}{1+\overline{w}(\boldsymbol{\chi}(\lambda))}\boldsymbol{\chi}(\lambda)\right)\cdot\nabla\boldsymbol{\chi}(\lambda)\right] d\lambda \\ &> \int_0^1 \left[-\boldsymbol{\psi}(\boldsymbol{\chi}(\lambda))\cdot\nabla\boldsymbol{\chi}(\lambda)\right] d\lambda = \int_{\boldsymbol{\chi}} -\boldsymbol{\psi}(\boldsymbol{\mu})\cdot d\boldsymbol{\mu} = B(\mathbf{m}), \end{aligned} \tag{30}$$

where the first equality follows from the Fundamental Theorem of Calculus and the condition $\overline{w}(+\infty) = 0$. Notice also that the strict decreasingness of $\boldsymbol{\chi}$ in every direction was essential in obtaining the inequality above.

Let's now turn our attention to $\underline{w}$. We have already seen in (28) that $\underline{w}(\mathbf{m}) = \frac{\varphi^* - \varphi(G(\mathbf{m}))}{\varphi^*}$. On the other hand, writing $dG(\mathbf{m}) = \nabla G(\mathbf{m}) \cdot d\mathbf{m}$ and using the fact that $\lim_{\lambda\to 0_+} G(\boldsymbol{\chi}(\lambda)) = +\infty$,[22] (18) gives

$$A(\mathbf{m}) = \int_{\boldsymbol{\chi}} -\frac{\varphi'(G(\boldsymbol{\mu}))}{\varphi(G(\boldsymbol{\mu}))} dG(\boldsymbol{\mu}) = \int_{+\infty}^{G(\mathbf{m})} -\frac{\varphi'(\tilde{G})}{\varphi(\tilde{G})} d\tilde{G} = \int_{\varphi^*}^{\varphi(G(\mathbf{m}))} -\frac{d\tilde{\varphi}}{\tilde{\varphi}} = \log\left(\frac{\varphi^*}{\varphi(G(\mathbf{m}))}\right),$$

so

$$\frac{\varphi(G(\mathbf{m}))}{\varphi^*} = e^{-A(\mathbf{m})}. \tag{31}$$

Therefore we've obtained a perfect analog to (15):

$$\underline{w}(\mathbf{m}) = 1 - e^{-A(\mathbf{m})}. \tag{32}$$

The measures $\underline{w}$ and $\overline{w}$ were introduced for Sidrauski's model, and shown to relate in a precise way with other measures. We're now ready to state and prove our main ordering result, extending Proposition 1 to an economy with many types of monies, and with the necessary and sufficient hypothesis to make the proposition economically meaningful.

Proposition 2 *Given a money-demand function arising from the extended McCallum-Goodfriend model, we have for it:* $\underline{w} = 1 - e^{-A} < s < A < B < \overline{w}$.

Proof. On the one hand, in Cysne (2003, p. 236) it is shown that $1 - e^{-A} < s < A < B$ for this money-demand function. On the other, the lemma tells us that this money-demand function is also compatible with Sidrauski's model. For such a demand function, the equality of the thesis has already been proved (see (32)), as well as the rightmost inequality (see (30)). Now it's only a matter of gluing these inequalities and this equality together. ■

4 An example: the log-log money-demand specification

In this section, we propose to solve four exercises involving the traditional log-log money-demand function. For the only-one-type-of-money framework, we find expressions for all the measures considered in section 3 for this function, and then estimate the difference between the lowest and highest measure. For the many-types-of-monies framework, we again find expressions for all the measures considered in sections 4 and 5, and then find necessary and sufficient conditions on the parameters of this function so that it can be rationalized by our models.

[22] Since $G(\mathbf{1}) > 0$ and G is 1-homogeneous.

4.1 Formulas for the unidimensional welfare measures

Let's say an estimated unidimensional log-log money demand specification,

$$m(r) = Kr^{-\alpha},$$

where $K > 0$, $\alpha > 0$ and $\alpha \neq 1$, has been given to us by the econometrician. How can we calculate the different measures of the welfare cost of inflation associated to a nominal interest rate of r?

First of all, it's good to know that s has already been calculated in the literature (see Cysne 2005), and is given in an implicit manner by the formula

$$(1 - s(r))\left(1 - (1 - s(r))^{-1/\alpha}\right) + \frac{K}{1-\alpha} r^{1-\alpha} = 0. \tag{33}$$

Both Bailey's measure and the proxy measure A are straightforward:

$$B(r) = \int_0^r -\rho(-\alpha K \rho^{-\alpha-1}) d\rho = \frac{\alpha K}{1-\alpha} r^{1-\alpha} \tag{34}$$

and

$$\begin{aligned} A(r) &= \int_0^r \frac{-\rho(-\alpha K \rho^{-\alpha-1})}{1+\rho K \rho^{-\alpha}} d\rho = \alpha \int_0^r \frac{K\rho^{-\alpha}}{1+K\rho^{1-\alpha}} d\rho \\ &= \frac{\alpha}{1-\alpha} \int_1^{1+Kr^{1-\alpha}} \frac{du}{u} = \frac{\alpha}{1-\alpha} \log\left(1 + Kr^{1-\alpha}\right). \end{aligned} \tag{35}$$

Both measures show that the value of α given to us should be in the $(0,1)$ range. That's not a problem, since α is usually found to be close to 0.5 in empirical work.

Now for the welfare measure associated to the equivalent variation notion. Since $\psi(m) = \left(\frac{K}{m}\right)^{1/\alpha}$, (11) takes the form

$$\overline{w}'(r) = -\left(\frac{K(1+\overline{w}(r))}{Kr^{-\alpha}}\right)^{1/\alpha} (-\alpha K r^{-\alpha-1}) = \alpha K (1+\overline{w}(r))^{1/\alpha} r^{-\alpha},$$

which – very luckily – is a separable equation. So $\int_0^{\overline{w}(r)} (1+\nu)^{-1/\alpha} d\nu = \int_0^r \alpha K \rho^{-\alpha} d\rho$, making

$$\frac{(1+\overline{w}(r))^{\frac{\alpha-1}{\alpha}} - 1}{\frac{\alpha-1}{\alpha}} = \frac{\alpha K}{1-\alpha} r^{1-\alpha},$$

or

$$\overline{w}(r) = -1 + \left(1 - Kr^{1-\alpha}\right)^{\frac{\alpha}{\alpha-1}}. \tag{36}$$

You may want to notice that in order for $\overline{w}$ to be real, it's important that $r \in [0, K^{\frac{1}{\alpha-1}}]$.

Finally, for the welfare measure associated to the compensating variation notion, we simply have

$$\underline{w}(r) = 1 - e^{-A(r)} = 1 - \left(1 + Kr^{1-\alpha}\right)^{\frac{\alpha}{\alpha-1}}. \tag{37}$$

Notice that we must have $\alpha < 1$ so that $\underline{w}$ is positive (and every other measure as well, by Proposition 1).

In figure 1, we plot these five measures for the estimated money-demand function $m(r) = 0.05r^{-0.5}$.[23] Note how s and A are undistinguishable to the naked eye.

[23] These parameters were calibrated as to fit the American economy. See Lucas (2000, pp. 258-9).

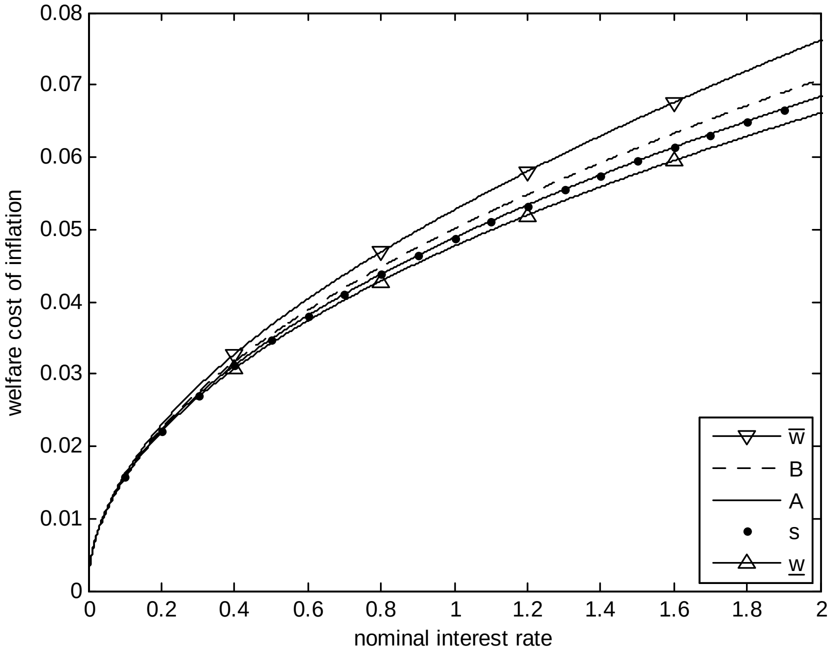

Figure 1: Different measures of the welfare cost of inflation for the money-demand specification $m(r) = 0.05r^{-0.5}$.

Another way of finding $\underline{w}$ would be to first ask ourselves what is the φ underlying the money-demand handed out to us, and then use (8). Looking at equation (7), we first calculate the integral that appears there:

$$\begin{aligned}\int_1^m \frac{\left(\frac{K}{\mu}\right)^{1/\alpha}}{1+\mu\left(\frac{K}{\mu}\right)^{1/\alpha}}d\mu &= \int_1^m \frac{K^{1/\alpha}}{\mu^{1/\alpha}+K^{1/\alpha}\mu}d\mu = \int_1^m \frac{d\mu}{\mu} - \int_1^m \frac{\mu^{\frac{1}{\alpha}-2}}{K^{1/\alpha}+\mu^{\frac{1}{\alpha}-1}}d\mu \\ &= \log m - \frac{\alpha}{1-\alpha}\int_{K^{1/\alpha}+1}^{K^{1/\alpha}+m^{\frac{1}{\alpha}-1}} \frac{du}{u} = \log\left(m\Bigg/\left(\frac{K^{1/\alpha}+m^{\frac{1}{\alpha}-1}}{K^{1/\alpha}+1}\right)^{\frac{\alpha}{1-\alpha}}\right).\end{aligned}$$

Therefore that equation gives us

$$\varphi(m) = \frac{Cm}{\left(K^{1/\alpha}+m^{\frac{1-\alpha}{\alpha}}\right)^{\frac{\alpha}{1-\alpha}}}, \tag{38}$$

for a positive constant C. So $\varphi^* = C$, and

$$\begin{aligned}\underline{w}(r) &= 1-\frac{\varphi(m(r))}{\varphi^*} = 1-\frac{m(r)}{\left(K^{1/\alpha}+m(r)^{\frac{1-\alpha}{\alpha}}\right)^{\frac{\alpha}{1-\alpha}}} \\ &= 1-\frac{Kr^{-\alpha}}{\left(K^{\frac{1-\alpha}{\alpha}}K+K^{\frac{1-\alpha}{\alpha}}r^{\alpha-1}\right)^{\frac{\alpha}{1-\alpha}}} = 1-\frac{1}{(Kr^{1-\alpha}+1)^{\frac{\alpha}{1-\alpha}}},\end{aligned}$$

which coincides with the previous result.

4.2 The relative measuring difference

A natural question that might arise by looking at Figure 1 is: *"How much larger is $\overline{w}$ in relation to $\underline{w}$?"*. For reasonable interest rates, not much indeed. Let's take, for instance, the above log-log case. From (36) and (37), we get

$$\Delta(r) := \frac{\overline{w}(r) - \underline{w}(r)}{\underline{w}(r)} = \begin{cases} -1 + \frac{-1+\left(1-Kr^{1-\alpha}\right)^{\frac{\alpha}{\alpha-1}}}{1-\left(1+Kr^{1-\alpha}\right)^{\frac{\alpha}{\alpha-1}}}, & \text{if } r > 0 \\ 0, & \text{if } r = 0 \end{cases}.$$

Write $f(x) = \begin{cases} -1 + \frac{-1+(1-x)^{\frac{\alpha}{\alpha-1}}}{1-(1+x)^{\frac{\alpha}{\alpha-1}}}, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \end{cases}$, so that $f(Kr^{1-\alpha}) = \Delta(r)$. So we obtain (with the recommended aid of a mathematics software) the following third-order Maclaurin expansion for f:

$$f(x) = 0 + \frac{1}{1-\alpha}x + \frac{1}{2(1-\alpha)^2}x^2 + \frac{-2\alpha^2 + 5\alpha - 5}{12(1-\alpha)^3}x^3 + O(x^4),$$

where $O(x^4)$ means a function whose absolute value is less than a constant times x^4, for small enough values of x (we're only interested in reasonable interest rates, which correspond to low x's).

Now write $g(x) = \frac{1}{\alpha}\left(-1 + (1-x)^{\frac{\alpha}{\alpha-1}}\right)$, where $x \geq 0$, as a guess for an approximation to f (insights gained by looking at their graphs, see Figure 2). That is, g is defined so that $g\left(Kr^{1-\alpha}\right) = \frac{1}{\alpha}\overline{w}(r)$. Its third-order Maclaurin expansion is

$$g(x) = 0 + \frac{1}{1-\alpha}x + \frac{1}{2(1-\alpha)^2}x^2 + \frac{2-\alpha}{6(1-\alpha)^3}x^3 + O(x^4).$$

Therefore, for small x, we have $f(x) = g(x) + O(x^3)$. Putting $x = Kr^{1-\alpha}$, this can be rewritten as

$$\begin{aligned} \Delta(r) &= \frac{1}{\alpha}\overline{w}(r) + O(r^{3-3\alpha}) \\ &= \frac{1}{\alpha}\left[-1 + \left(1 - Kr^{1-\alpha}\right)^{\frac{\alpha}{\alpha-1}}\right] + O(r^{3-3\alpha}), \end{aligned} \tag{39}$$

which is a good approximation to this relative difference formula. Figure 2 gives an even better idea of how good this approximation is.

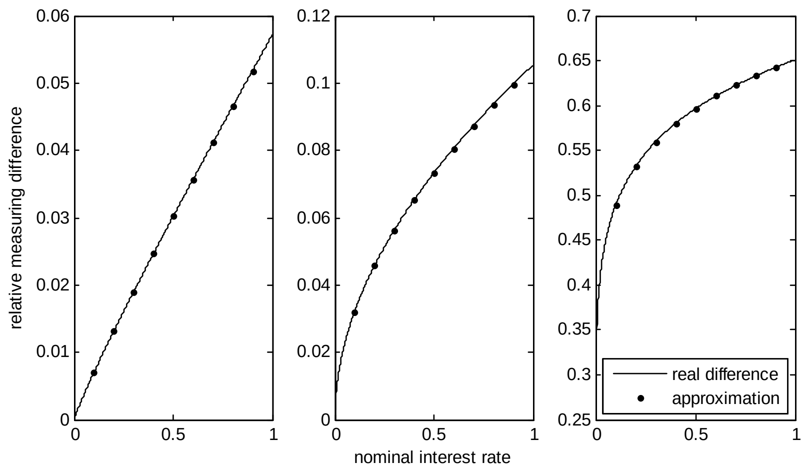

Figure 2: The real relative measuring difference Δ and its approximation, $\frac{1}{\alpha}\overline{w}$, for the specifications $m(r) = 0.05r^{-0.1}$, $m(r) = 0.05r^{-0.5}$ and $m(r) = 0.05r^{-0.9}$.

Note that, for practical purposes, we can have an idea of how relevant this difference is by entering $\alpha = 0.5$, $K = 0.05$ and $r = 15\%$ into (39), which yields $\Delta(r) \approx 4.0\%$. This value of the nominal interest rate would account for and inflation of 10% plus a long-term real interest rate of 5% (being conservative). So, for instance, when someone reports having found a welfare loss of 3% of national product associated with a 10% inflation rate (a high estimate, if compared to most of the estimates presented in the introduction), regardless of which particular measure was chosen, we know that such an estimate could vary at most between 2.88% and 3.12% – a very precise confidence interval. So one can be at ease about which measure to take, when considering low-inflation countries. On the other hand, consider a country where the annual inflation rate has reached 400% (in Brazil, for instance, inflation reached 1783% in 1989). In this case, for the same parameters, the relative measuring difference Δ reaches 22% (considering $r = 4$, since the long-term real interest rate would become negligible), which is to say that one has to be really careful about which measuring strategy is being used, in the case of hyperinflations.

4.3 Formulas for the multidimensional welfare measures

Let's now extend the log-log money-demand specification to the multidimensional case. It is natural enough to propose an extension of the form

$$\boldsymbol{\psi}(\mathbf{m}) = \left(\frac{K}{G(\mathbf{m})}\right)^{1/\alpha} \nabla G(\mathbf{m}),$$

where $K > 0$ and $\alpha \in (0,1)$. The reader should notice that this is the demand that follows from (38) and (21). It is also worth noticing that, in the usual case where G is a weighted geometric mean, $G(m_1, \ldots, m_n) = \prod_{i=1}^{n} m_i^{\beta_i}$ (where $\beta_i \geq 0$, $\forall i \in \{1, \ldots, n\}$ and $\sum_{i=1}^{n} \beta_i = 1$), we would obtain the system of equations

$$\begin{cases} \psi_1(\mathbf{m}) = \frac{K^{1/\alpha}\beta_1}{m_1} \prod_{j=1}^{n} m_j^{\left(1-\frac{1}{\alpha}\right)\beta_j} \\ \vdots \\ \psi_n(\mathbf{m}) = \frac{K^{1/\alpha}\beta_n}{m_n} \prod_{j=1}^{n} m_j^{\left(1-\frac{1}{\alpha}\right)\beta_j} \end{cases},$$

which, inverted,[24] gives a demand in the format

$$\begin{cases} m_1(\mathbf{u}) = L_1 \prod_{j=1}^{n} u_j^{\alpha_{1j}} \\ \vdots \\ m_n(\mathbf{u}) = L_n \prod_{j=1}^{n} u_j^{\alpha_{nj}} \end{cases},$$

where α_{ij} is the demand elasticity of type i of money relative to the opportunity cost of holding type j.

Again, B and A are straightforward. Taking $\boldsymbol{\chi}$ as in section 3.1, we have:

$$B(\mathbf{m}) = \int_{\boldsymbol{\chi}} -\boldsymbol{\psi}(\boldsymbol{\mu}) \cdot d\boldsymbol{\mu} = -K^{1/\alpha} \int_{\boldsymbol{\chi}} \frac{\nabla G(\mathbf{m}) \cdot d\mathbf{m}}{G(\mathbf{m})^{1/\alpha}} = -K^{1/\alpha} \left. \frac{G(\mathbf{m})^{1-\frac{1}{\alpha}}}{1-\frac{1}{\alpha}} \right|_{+\infty}^{\mathbf{m}} = \frac{\alpha K^{1/\alpha}}{1-\alpha} G(\mathbf{m})^{1-\frac{1}{\alpha}}, \quad (40)$$

and

$$A(\mathbf{m}) = \int_{\boldsymbol{\chi}} \frac{-\boldsymbol{\psi}(\boldsymbol{\mu}) \cdot d\boldsymbol{\mu}}{1 + \boldsymbol{\psi}(\boldsymbol{\mu}) \cdot \boldsymbol{\mu}} = -\int_{\boldsymbol{\chi}} \frac{K^{1/\alpha} dG(\mathbf{m})}{G(\mathbf{m})^{1/\alpha} + K^{1/\alpha} G(\mathbf{m})}.$$

Notice how this is exactly the same integrand that appeared in our second approach to finding $\underline{w}$ in the unidimensional case. With that knowledge behind us, we can immediately write:

$$\begin{aligned} A(\mathbf{m}) &= \lim_{G^* \to +\infty} \left[\int_{G(\mathbf{m})}^{G^*} \frac{d\tilde{G}}{\tilde{G}} - \frac{\alpha}{1-\alpha} \int_{K^{1/\alpha} + G(\mathbf{m})^{\frac{1}{\alpha}-1}}^{K^{1/\alpha} + (G^*)^{\frac{1}{\alpha}-1}} \frac{d\tilde{G}}{\tilde{G}} \right] \\ &= \lim_{G^* \to +\infty} \log \left[\frac{G^*}{\left(K^{1/\alpha} + G^{*\frac{1}{\alpha}-1}\right)^{\frac{\alpha}{1-\alpha}}} \frac{\left(K^{1/\alpha} + G(\mathbf{m})^{\frac{1}{\alpha}-1}\right)^{\frac{\alpha}{1-\alpha}}}{G(\mathbf{m})} \right] \\ &= \log \left(\frac{\left(K^{1/\alpha} + G(\mathbf{m})^{\frac{1}{\alpha}-1}\right)^{\frac{\alpha}{1-\alpha}}}{G(\mathbf{m})} \right) = \frac{\alpha}{1-\alpha} \log \left(1 + K^{1/\alpha} G(\mathbf{m})^{1-\frac{1}{\alpha}}\right). \end{aligned} \quad (41)$$

[24] The very name of this kind of demand gives us a hint of how to invert this system of equations: apply the logarithm function on each side. This yields a linear system.

Here we have made implicit use of the obvious fact that, since G is a positive, increasing in each variable and 1-homogeneous function, it is unbounded: $\lim_{\mathbf{m}\to+\infty} G(\mathbf{m}) = +\infty$.

Then (41) and (32) give at once

$$\underline{w}(\mathbf{m}) = 1 - \left(1 + K^{1/\alpha} G(\mathbf{m})^{1-\frac{1}{\alpha}}\right)^{-\frac{\alpha}{1-\alpha}}. \tag{42}$$

Now for $\overline{w}$. Equation (29) takes, in this case, the form

$$\overline{w}_{x_i}(\mathbf{m}) = -(1 + \overline{w}(\mathbf{m}))^{1/\alpha} \left(\frac{K}{G(\mathbf{m})}\right)^{1/\alpha} G_{x_i}(\mathbf{m}),$$

which, as in the unidimensional case, gives

$$\int_0^{\overline{w}(\mathbf{m})} \frac{d\nu}{(1+\nu)^{1/\alpha}} = -K^{1/\alpha} \int_{\boldsymbol{\chi}} \frac{\nabla G(\mathbf{m}) \cdot d\mathbf{m}}{G(\mathbf{m})^{1/\alpha}} = B(\mathbf{m}),$$

so that

$$\frac{1}{1-\frac{1}{\alpha}} \left((1 + \overline{w}(\mathbf{m}))^{1-\frac{1}{\alpha}} - 1\right) = \frac{\alpha K^{1/\alpha}}{1-\alpha} G(\mathbf{m})^{1-\frac{1}{\alpha}},$$

and

$$\overline{w}(\mathbf{m}) = -1 + \left(1 - K^{1/\alpha} G(\mathbf{m})^{1-\frac{1}{\alpha}}\right)^{\frac{\alpha}{\alpha-1}}. \tag{43}$$

Finally, for s, the easiest thing to do is just imitate (33) in the obvious way, and verify that it satisfies (17). That is, take s given implicitly by equation

$$D(\mathbf{m}, s(\mathbf{m})) = 0, \tag{44}$$

where

$$D(\mathbf{m}, s) := \frac{K^{1/\alpha}}{1-\alpha} G(\mathbf{m})^{1-\frac{1}{\alpha}} + (1-s)\left(1 - (1-s)^{-1/\alpha}\right).$$

The Implicit Function Theorem gives

$$\begin{aligned} s_{x_i}(\mathbf{m}) &= -\frac{\left(\frac{\partial D}{\partial x_i}\right)(\mathbf{m}, s(\mathbf{m}))}{\left(\frac{\partial D}{\partial s}\right)(\mathbf{m}, s(\mathbf{m}))} = \frac{\frac{1}{\alpha} K^{1/\alpha} G(\mathbf{m})^{-1/\alpha} G_{x_i}(\mathbf{m})}{-1 + \left(1 - \frac{1}{\alpha}\right)(1 - s(\mathbf{m}))^{-1/\alpha}} \\ &= \frac{\psi_i(\mathbf{m})}{-1 + (1-\alpha)\left(1 - (1 - s(\mathbf{m}))^{-1/\alpha}\right)}, \end{aligned}$$

whence

$$-\frac{1}{s_{x_i}(\mathbf{m})} = \frac{1}{\psi_i(\mathbf{m})} - \frac{(1-\alpha)\left(1 - (1 - s(\mathbf{m}))^{-1/\alpha}\right)}{K^{1/\alpha} G(\mathbf{m})^{-1/\alpha} G_{x_i}(\mathbf{m})}.$$

Substituting from (44), we get

$$-\frac{1}{s_{x_i}(\mathbf{m})} = \frac{1}{\psi_i(\mathbf{m})} + \frac{K^{1/\alpha} G(\mathbf{m})^{1-\frac{1}{\alpha}}}{K^{1/\alpha} G(\mathbf{m})^{-1/\alpha} G_{x_i}(\mathbf{m})(1 - s(\mathbf{m}))} = \frac{1}{\psi_i(\mathbf{m})} + \frac{G(\mathbf{m})}{G_{x_i}(\mathbf{m})(1 - s(\mathbf{m}))},$$

an expression equivalent to (17). The reader may wish to check that these formulas really extend the unidimensional ones.

All these measures are plotted in Figure 3, where $n = 2$, $K = 0.05$ and $\alpha = 0.5$. In Figure 3a, G is taken so that $G(m, x) = m^{0.5} x^{0.5}$, and in Figure 3b, $G(m, x) = m^{0.7} x^{0.3}$. The ordering of the surfaces is the one given in Proposition 2. Again, as in the unidimensional case, we see that A approximates s so well that it is almost impossible to visualize them separately. The money-demand functions specified under the figures result from the inversion strategy presented in the previous footnote.

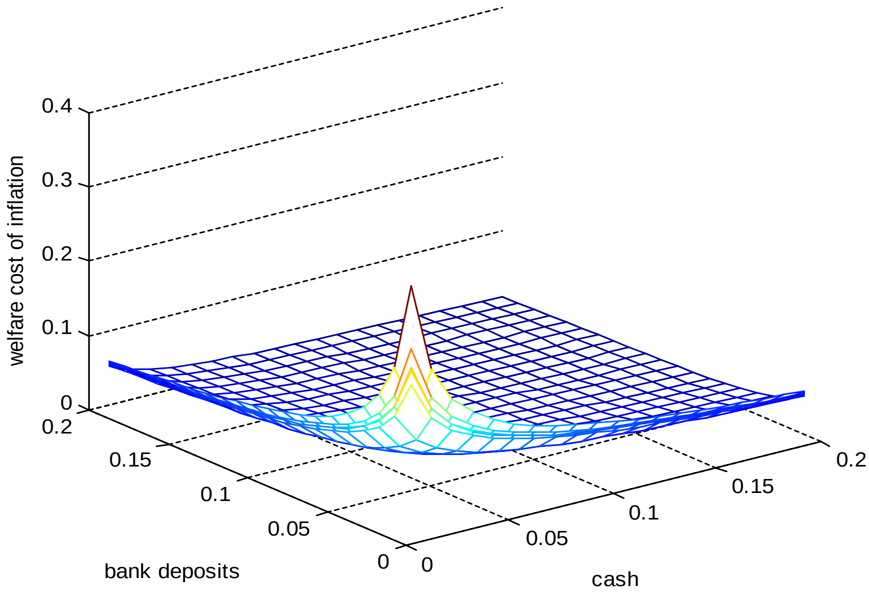

Figure 3a: Different measures of the welfare cost of inflation for the money-demand specification

$$\begin{cases} m = 0.025 r^{-0.75}(r - r_x)^{0.25} \\ x = 0.025 r^{0.25}(r - r_x)^{-0.75} \end{cases} .$$

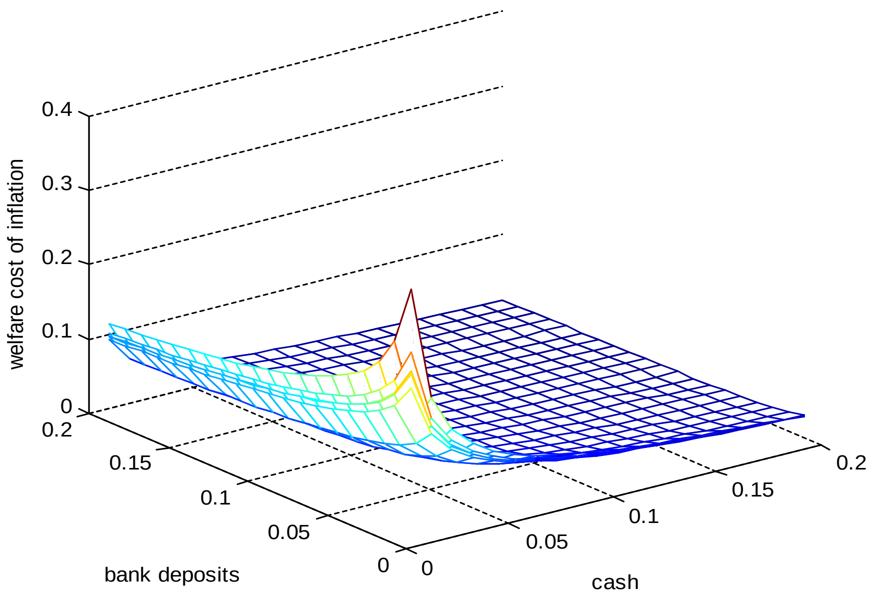

Figure 3b: Different measures of the welfare cost of inflation for the money-demand specification

$$\begin{cases} m = \frac{0.05 \times 0.7^{0.65}}{0.3^{0.15}} r^{-0.65} r_x^{0.15} \\ x = \frac{0.05 \times 0.3^{0.85}}{0.7^{0.35}} r^{0.35} r_x^{-0.85} \end{cases} .$$

4.4 Obtaining this demand from our models

We now solve our last exercise concerning the log-log money-demand function: can it originate from the shopping-time model? We will see that the answer is in the affirmative (so that it could have originated from the money-in-the-utility-function model as well, given the lemma in the previous section).

Looking at the proof of Lemma 1, a natural starting point would seem to be: write $\frac{\phi(\tau(G))}{G\phi'(\tau(G))} = F(G) = \left(\frac{K}{G}\right)^{1/\alpha}$. Remembering that $G = \frac{1-s}{\phi(s)}$ in equilibrium, we obtain the differential equation

$$\phi'(s) = \frac{(1-s)^{\frac{1}{\alpha}-1}}{K^{1/\alpha}} \phi(s)^{2-\frac{1}{\alpha}}. \tag{45}$$

This is a Bernoulli equation, and we have to find out if its solution satisfying $\phi(0) = 0$ also satisfies the other properties in our version of McCallum and Goodfriend's model. In order to solve it, write $y = \phi^{\frac{1}{\alpha}-1}$, so that

$$y'(s) = \left(\frac{1}{\alpha} - 1\right) \phi(s)^{\frac{1}{\alpha}-2} \phi'(s) = \left(\frac{1}{\alpha} - 1\right) \frac{(1-s)^{\frac{1}{\alpha}-1}}{K^{1/\alpha}}.$$

Therefore $y(s) = \frac{\alpha-1}{K^{1/\alpha}} (1-s)^{1/\alpha} + C$ for some constant C. Since $y(0) = 0$ ($\alpha \neq 1$), we get $C = -\frac{\alpha-1}{K^{1/\alpha}}$, so that

$$\phi(s) = \left[\frac{1-\alpha}{K^{1/\alpha}} \left(1 - (1-s)^{1/\alpha}\right)\right]^{\frac{\alpha}{1-\alpha}}.$$

For $s \in (0,1)$, (45) gives $\phi'(s) > 0$. After some tedious calculations for ϕ' and ϕ'' (or, with the aid of a mathematics software), we get

$$\phi''(s) = -\frac{K^{-\frac{1}{1-\alpha}}}{\alpha} (1-s)^{-2+\frac{1}{\alpha}} \left[(1-\alpha)\left(1 - (1-s)^{1/\alpha}\right)\right]^{\frac{3\alpha-2}{1-\alpha}} \left[(1-\alpha)^2 - \alpha^2 (1-s)^{1/\alpha}\right].$$

Although it is impossible to tell the sign of this expression,[25] we have

$$\begin{gathered} \phi'(s)^2 - \phi(s)\phi''(s) = \tfrac{(1-\alpha)K^{-\frac{2}{1-\alpha}}}{\alpha} (1-s)^{-2+\frac{1}{\alpha}} \times \\ \left[(1-\alpha)\left(1-(1-s)^{1/\alpha}\right)\right]^{\frac{4\alpha-2}{1-\alpha}} \left[1 - \alpha\left(1-(1-s)^{1/\alpha}\right)\right] \geq 0, \end{gathered}$$

as we wished. We have shown that the log-log money-demand specification can emerge from the extended shopping-time model. Therefore, by Lemma 1, we see that it can also emerge from the extended money-in-the-utility-function model.

Let's now leave aside for a moment the assumptions we've made on α and look for the minimum ones we should make so that the log-log specification is rationalized by each model. Let's only assume $\alpha > 0$, so that $\boldsymbol{\psi}$ keeps its negative relation with $\mathbf{m}$. By simply looking at the above expression for ϕ, we see that $\alpha < 1$ is really needed if this money-demand function is to be compatible with McCallum and Goodfriend's model, since ϕ must be a real function.[26] But since $\left(\frac{K}{G}\right)^{1/\alpha}$ is a strictly decreasing function of G for any $\alpha > 0$, our result in section 3.3 tells us that the log-log money-demand function *could* be achieved from Sidrauski's model even if $\alpha \geq 1$ – although our object of investigation, the expressions for the welfare cost of inflation derived in sections 4.1 and 4.3, would be meaningless.

5 Conclusion

The present work has introduced a new measure of the welfare cost of inflation in Sidrauski's (1967) money-in-the-utility-function model, and shown how it relates to the other measures of the welfare cost of inflation present in the literature. We've extended the traditional framework of only one type of money available in the economy to one in which there is an arbitrary fixed number of different types of monies available, and extended all our welfare measures to this more general framework.

[25] For instance, for $\alpha = 0.6$ and $K = 0.1$, we have $\phi''(0.4) \approx -8.3$, but $\phi''(0.3) \approx 54.2$.

[26] The only dismissed case in the above analysis was $\alpha = 1$. In this case, (45) would read $\phi'(s) = \frac{1}{K^{1/\alpha}}\phi(s)$, which, together with $\phi(0) = 0$, gives $\phi = 0$, a contradiction with $\phi' > 0$.

We've obtained a full characterization of the money-demand specifications which are rationalizable by this model, and shown that money-demand functions rationalizable by McCallum and Goodfriend's (1987) shopping-time model are also rationalizable by Sidrauski's. We've obtained a complete ordering of these measures, extending the ordering given in Cysne (2003, pp. 228 and 236). We illustrated our results with the well-known log-log money-demand specification, and found that, for reasonable empirical parameters, the highest possible relative measuring difference must be very low in our model, considering low inflation countries. In a hyperinflationary scenario, the case is totally different, and this relative measuring difference must be quite large, so it is very important to specify what measuring strategy is being used.

A Appendix

Here we shall study the invertibility of the function $\boldsymbol{\psi}$ defined in (21) (which, as seen in Lemma 1 and in the last paragraph of section 4, is more general than (16)). That is, given $\mathbf{u} \in \mathbb{R}^n_{++}$, we will try to find $\mathbf{m} \in \mathbb{R}^n_{++}$ such that $\boldsymbol{\psi}(\mathbf{m}) = \mathbf{u}$. We will only analyze its injectivity, since its surjectivity is easy to obtain, by simply putting, as in the unidimensional case, $\varphi(0) = 0$ and $\lim_{m \to +\infty} \varphi'(m) = 0$, for instance, so that each $\psi_i(\mathbf{m})$ tends to $+\infty$ when $\mathbf{m}$ is approaching the origin and to 0 when $\mathbf{m}$ is walking away from it. We divide our analysis into two cases.

Case 1 $n = 2$.

In this case, it is possible to show that $\boldsymbol{\psi}$ is indeed one-to-one. Let's say $\boldsymbol{\psi}(m, x) = \boldsymbol{\psi}(m^*, x^*)$. From (21), we see that it is necessary that $\frac{G_x(m,x)}{G_m(m,x)} = \frac{\psi_2(m,x)}{\psi_1(m,x)} = \frac{\psi_2(m^*,x^*)}{\psi_1(m^*,x^*)} = \frac{G_x(m^*,x^*)}{G_m(m^*,x^*)}$. But from G's homogeneity we also have $G_m(m^*, x^*) = G_m\left(m, \frac{m}{m^*}x^*\right)$ and $G_x(m^*, x^*) = G_x\left(m, \frac{m}{m^*}x^*\right)$, so that $\frac{G_x(m,x)}{G_m(m,x)} = \frac{G_x\left(m,\frac{m}{m^*}x^*\right)}{G_m\left(m,\frac{m}{m^*}x^*\right)}$. We already know that we cannot have $\frac{m}{m^*}x^* = x$, since this would mean that (m, x) and (m^*, x^*) are on the same ray starting at the origin, and we already know that $\boldsymbol{\psi}$ restricted to these rays is injective. Let's define the function $f : \mathbb{R}_{++} \to \mathbb{R}$ by $f(h) = \frac{G_x(m,h)}{G_m(m,h)}$. For any $h \in \mathbb{R}_{++}$, we have $f'(h) = \frac{G_{xx}(m,h)G_m(m,h) - G_x(m,h)G_{mx}(m,h)}{G_m(m,h)^2}$. Let's analyze the sign of this derivative.

We know that G_m and G_x are positive. Since G_x is 0-homogeneous, Euler's formula for homogeneous functions gives, for any $(m, h) \in \mathbb{R}^2_{++}$, $mG_{xm}(m, h) + hG_{xx}(m, h) = 0G_x(m, h) = 0$. Now, since $G_{xx} < 0$, this means $G_{mx} = G_{xm} > 0$. Therefore $f'(h) < 0$, and f is one-to-one. This contradicts the facts that $\frac{m}{m^*}x^* \neq x$ and $f\left(\frac{m}{m^*}x^*\right) = f(x)$.

Case 2 $n > 2$.

Here $\boldsymbol{\psi}$'s injectivity is no longer guaranteed. Take, for instance, $\varphi(\cdot) = \sqrt{\cdot}$, so that (21) can be rewritten as

$$\psi_i(\mathbf{m}) = \frac{1}{G(\mathbf{m})} G_{x_i}(\mathbf{m}), \forall i \in \{1, \ldots, n\}.$$

Let's say G was a weighted geometric mean, say $G(m_1, \ldots, m_n) = \prod_{i=1}^n m_i^{\beta_i}$, with $\beta_i \geq 0, \forall i \in \{1, \ldots, n\}$, and $\sum_{i=1}^n \beta_i = 1$. Then the equation above could be rewritten as

$$\psi_i(\mathbf{m}) = \frac{\beta_i}{m_i}, \forall i \in \{1, \ldots, n\},$$

and $\boldsymbol{\psi}$ would evidently be invertible.

But what if G weren't a geometric mean? For example, take $n = 3$ and G such that $G(m, x, y) = m^{0.5}(x + y)^{0.5}$. Then $G_m(m, x, y) = 0.5\left(\frac{x+y}{m}\right)^{0.5}$, and $G_x(m, x, y) = 0.5\left(\frac{m}{x+y}\right)^{0.5} = G_y(m, x, y)$. Therefore $\boldsymbol{\psi}(1, 2, 2) = \frac{1}{2}(1, 0.25, 0.25) = \boldsymbol{\psi}(1, 1, 3)$, and $\boldsymbol{\psi}$ would not be invertible.

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