Conselho Nacional de Desenvolvimento Científico e Tecnológico
Associação Instituto Nacional de Matemática Pura e Aplicada

KNIGHTIAN ASSET BUBBLES IN GENERAL EQUILIBRIUM

Candidato: Rodrigo Novinski Nunes

Orientador: Aloisio Araujo

Co-orientador: Mário Páscoa

Tese apresentada para obtenção do Título de Doutor em Ciências

Rio de Janeiro

Abril, 2007

ii)

Ao professor Carlos Augusto Sholl Isnard (in memoriam).

INTRODUCTION

We wrote this thesis having as the main goal the study of speculative bubbles existence in spite of asset positive supply with and a finite present value of asset returns and total endowments.

In economies with precautionary infinite lived agents, Arrow-Debreu equilibrium prices may fail to be summable sequences as the utility functions are no longer Mackey continuous. Arrow-Debreu allocations supported by non-summable prices can be implemented sequentially with a speculative bubble in the prices of the assets that complete the markets. Actually, in this context, transversality conditions become quite different, allowing for speculative bubbles in the prices of assets in positive net supply, even when the present value of aggregate wealth is finite.

This analysis is presented in Chapter 1 for the case that each agent has Knightian impatience, the sequential economy is deterministic and the asset is fiat money. An example of sequential equilibrium is given where fiat money has a bubble component in his price.

The Chapter 2 treats the extension to stochastic frameworks. It is supposed that exists a probability measure over the possible states of nature. The results that were shown in Chapter 1 are generalized.

At last, we close this work with some remarks on existence of speculative bubbles that are induced by more general forms of non-Mackey-continuous and some relations between Knightian impatience and Overllaping Generations.

CONTENTS

1. KNIGHTIAN BUBBLES AND FIAT MONEY: DETERMINISTIC CASE

In this chapter agents have a precautionary attitude with regard to the consumption stream along their infinite life. The resulting precautionary demand for assets creates speculative bubbles, even when assets' net supply is positive (as in the case of fiat money) and the present value of aggregate wealth is finite, contrary to what had been established under standard assumptions on preferences.

Although bubbles are an important phenomenon on the real world economies, their occurrence in a GE framework did not seem to be robust. Santos and Woodford's paper [25], for example, show that, when agents are impatient enough and traditional borrowing constraints are imposed in order to avoid Ponzi Schemes (i.e., the possibility of postpone indefinitely the debt payments by insure a new debt at every date) if the present value of total endowments is finite, then speculative bubbles for assets in positive net supply, such as fiat money, are ruled out. Bewley [7] had already given an example in which fiat money had a positive price at equilibrium, but the present value of wealth was infinite in this example.

We deviate from the standard time-separable preferences by adding a term concerned with the infimum of the utilities at all dates. Equivalently, agents could be using several discount factors and tend to pick, for each consumption plan, the factor that discounts the future in the most severe way. This formulation reminds us, in a deterministic setting, the well-known models of Knightian uncertainty (see Schmeidler [27] and Gilboa [15]). For this reason we refer to our approach as Knightian impatience. An important consequence of this modification is that agents' preferences are no longer Mackey-continuous.

Absence of Mackey-continuity in a two-dates model allows for Arrow-Debreu equilibrium prices which are not countably additive. We show that these A-D equilibrium allocations with non-summable prices can be implemented as equilibrium of sequential economies where an asset, possibly in positive net supply, is being used to complete markets, date by date.

Under Knightian impatience, the transversality conditions that hold at solutions of the sequential optimization problem are now quite different from standard ones. Following the traditional transversality condition approach, we impose on every portfolio a constraint that mimics the transversality condition that the optimal portfolio must satisfy. Under suitable borrowing requirements (that mimical constraint itself or explicit requirements that imply it), we can implement an AD-equilibrium with non-summable prices as a sequential equilibrium with a bubble component in the price of the asset that completes sequentially the markets. This asset may be fiat money, in positive net supply, and its positive price is not due to binding borrowing constraints, but rather to the bubble induced by the non-summable component of the underlying Arrow-Debreu prices. Intuitively, fiat money has a positive value because agents want to hold it for precautionary reasons, since they are particularly concerned with the worst outcomes.

1.1 A Guiding Example: Outline

Consider a deterministic infinite horizon economy with a single commodity and two agents whose preferences depart from the standard time separable utilities since agents are particularly worried about the worst possible outcome:

$$U^i(x) = \sum_{s\geq 1} \delta^{s-1} u^i\left(x_s\right) + \beta \inf_{s\geq 1} u^i(x_s)$$

with $\delta \in (0,1)$, $\beta \in [0,\infty)$ and $u^i(x) = \log x$ for $i = 1,2$.

This precautionary behavior is actually equivalent to a maxmin attitude, in the sense that the consumer looks for a consumption plan that maximizes the worst discounted time separable utility, within a class of discount factors (not necessarily of the exponential form) having δ^{s-1} as lower bound at each date. It is as if the consumer were unsure about the discount factor that should be used and, therefore, uses the severest one. More precisely (as shown in the next section),

$$\sum_{s\geq 1} \delta^{s-1} u^i\left(x_s\right) + \beta \inf_{s\geq 1} u^i(x_s) = \inf_{(\sigma_s)_{s\geq 1}\in A} \sum_{s} \sigma_s u^i\left(x_s\right)$$

where A is the set of all real sequences $(\sigma_s)_{s\geq 1}$ such that $\sum_{s\geq 1} \sigma_s = \frac{1}{1-\delta} + \beta$ [1] and $\sigma_s \geq \delta^{s-1}$ for

[1] So the weight sum of discount factors σ_s must be equal to sum of the bounds δ^{s-1} and the extra weight β.

every date s. Hence, by analogy with the Knightian uncertainty literature, we reinterpret the above precautionary behavior in terms of imprecise impatience (or flexible discount factor) and refer to it as Knightian impatience.

There are endowment shocks that agents try to get rid of. Let $\omega^1 = (\frac{s+1}{s} + \varphi_s)_{s\geq 1}$ and $\omega^2 = (\frac{s+1}{s} - \varphi_s)_{s\geq 1}$ where φ_s is $1/2$ when s is even and $-1/4$ when s is odd.

Let us start by finding an Arrow-Debreu equilibrium. The market clearing consumption plans $\overline{x}^1 = \overline{x}^2 = (\frac{s+1}{s})_{s\geq 1}$ can be shown to be optimal (see section 1.3) for the Arrow-Debreu single budget constraint when $\delta < 3/8$, β is suitable and prices π are given by

$$\pi x \equiv \sum_s \delta^{s-1} u'(\overline{x}_s)\, x_s + \beta u'(\inf \overline{x}) b(x) = \sum_s \delta^{s-1} \frac{s}{s+1} x_s + \beta b(x)$$

where b is a linear functional (and continuous for the norm topology of the space of bounded sequences) such that $b((x_1, ..., x_s, ...)) = \lim_n \sum_{s=1}^n \frac{x_s}{n}$ when this limit exists.

Let us see next (section 1.4) that this Arrow-Debreu equilibrium can be implemented sequentially in a monetary equilibrium. We introduce an infinite-lived asset that pays no dividends and may be in positive net supply, fiat money. At each date, agents trade an amount $z_s \in \mathbb{R}$ of money at a price $q_s \geq 0$. Taking the single consumption good to serve as numeraire at each date, the budget constraints that replace the Arrow-Debreu constraint are

$$x_s + q_s z_s \leq w_s + q_s z_{s-1} \forall s \geq 1$$

where $z_0 \geq 0$ are endowments of money at the initial date.

It is clear that some borrowing constraints have to be added to avoid Ponzi schemes. At a solution for which those constraints are not binding, the usual Euler equations must hold:

$$q_s u'(x_s^*) = \delta q_{s+1} u'(x_{s+1}^*)$$

However, the transversality condition is now quite different, requiring only

$$\lim_s \delta^{s-1} u'(x_s^*) q_s z_s^* \in [\beta \liminf_s u'(x_s^*) q_s (z_{s-1}^* - z_s^*), \beta \limsup_s u'(x_s^*) q_s (z_{s-1}^* - z_s^*)]$$

which becomes the usual one when $\beta = 0$. That is, contrary to the standard case, for $\beta > 0$, agents can be lenders or borrowers at infinity. It is the precautionary behavior, trying to avoid a bad outcome at distant dates, that leads agents not to clear the asset position as time goes to infinity.

Following the usual transversality approach, we impose the above transversality condition (which must hold at an optimal solution) on every admissible portfolio plan and look for borrowing constraints that imply those conditions.

For a suitable choice of those borrowing constraints, the above Arrow-Debreu consumption plans can be implemented in an equilibrium of the sequential economy with a positive price of money at every date (see section 1.5). More specifically, money prices can be obtained from the Euler equations $p_s q_s = p_{s+1} q_{s+1}$, where the deflator process is given by $p_s = \delta^{s-1} u'(x_s) = \delta^{s-1} \frac{s}{s+1}$ (i.e., the summable component of A-D equilibrium price), by setting $q_s = 1/p_s$. Money holdings are given by $z_s^i = \sum_{t \leq s} (\omega_t^i - x_t)/q_t + z_0^i$, that is, $z_s^i = \sum_{t \leq s} ((-1)^{i+1} \varphi_t \delta^{t-1} t/(t+1)) + z_0^i$, for $i = 1, 2$. Clearly, when $z_0^1, z_0^2 > (1-\delta)^{-1}$ short-sales are not made in equilibrium (inside money is not created).

Moreover, as the shadow prices of the borrowing constraints are zero, the fundamental value of money is zero and the positive price of money must be due to a speculative bubble (see the section 1.6). This bubble occurs under a positive net supply of money and finite present values of aggregate endowments (as endowments are bounded sequences and the deflator process $(p_s)_{s \geq 1}$ is summable), which was known to be impossible under complete markets and time separable preferences, as shown by Santos and Woodford [25] (or even under incomplete markets, time-state separable preferences and endowments uniformly bounded away from zero).

As agents are no longer required not to be lenders at infinity under a common deflator process ($\lim_s p_s q_s z_s^* \neq 0$), the transversality condition becomes compatible with a positive price at infinity ($\lim_s p_s q_s > 0$) for an asset in positive net supply. This limiting price of the asset, as time goes to infinity, is precisely the bubble. This example will be studied in more detail (and refered to as example 1) as we move along this chapter presenting our results.

1.2 Precautionary Behavior and Knightian Impatience

We model a deterministic economy with countably many dates. Agents have infinite life and care specifically about the worst future outcome. More precisely, agents have a utility function that allows for precautionary behavior (when $\beta > 0$) as follows:

$$U(x) = \sum_{s\geq 1} \delta^{s-1} u\left(x_s\right) + \beta \inf_{s\geq 1} u(x_s) \tag{1.1}$$

with $\delta \in (0,1)$, $\beta \in [0,\infty)$, where the function $u : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{-\infty\}$ is: i) above-bounded, ii) increasing, iii) concave and iv) of class C^1 at $(0,\infty)$ and satisfies v) the Inada Conditions, i.e. $\lim_{x\uparrow\infty} u'(x) = 0$ and $\lim_{x\downarrow 0} u'(x) = \infty$.

If $\delta + \beta < 1$ (and so $\delta^{s-1} + \beta < 1\ \forall s$), the agent has several ways of discount the future. For instance, for a date $\overline{s}$, let $y = (y_s)_{s\in\mathbb{N}}$ be a consumption plan such that $u(y_{\overline{s}}) = \inf_s u(y_s)$. Then $U(y) = \sum_{s\neq\overline{s}} \delta^{s-1} u\left(y_s\right) + (\delta^{\overline{s}-1} + \beta) u(y_{\overline{s}})$, i.e., the date $\overline{s}$ has been discounted by the rate $(\delta^{\overline{s}-1} + \beta)^{1/\overline{s}-1}$ (the one within the possible that less depreciates the consumption welfare at this date), whereas all the other dates by the rate δ (the most severe one). Thus the degree of impatience for an arbitrary date can change according to his relative level of consumption.

Let us make two important remarks about the above utility function. First, it fails the usual impatience requirement of Mackey continuity in the space of bounded sequences. In fact, it is Mackey-u.s.c., but it is not Mackey-l.s.c. since for $z_n = c\chi_{\mathbb{N}} - \frac{c}{2}\chi_{E_n}$ with $c > 0$ and $E_n = \{n, n+1, \ldots\}$ we have $z_n \overset{\text{Mackey}}{\to} c\chi_{\mathbb{N}}$ [2], but it is not true that $U(z_n) \to U(c\chi_{\mathbb{N}})$ (as $u(c/2) = \inf u(z_n) \nrightarrow \inf u(c\chi_{\mathbb{N}}) = u(c)$).

Secondly, the above utility function can be reinterpreted as the minimal separable utility when the discount factors have a certain lower bound at each date. In this reinterpretation consumers are not sure how to discount future events and they end up maximizing the worst discounted utility, over a certain set of possible discount factors. Let us be more precise.

The preferences represented by (1.1) are a particular case of

$$\alpha \int_S u \circ x d\mu + (1-\alpha) \inf u \circ x \quad \text{with} \quad \alpha \in (0,1] \tag{1.2}$$

In fact, we can write the summation at (1.1) as $\frac{1}{1-\delta}\int_S u\circ x d\mu$ where μ is the probability measure on subsets of $S = \mathbb{N}$ defined by $\mu_s = \mu(\{s\}) = \delta^{s-1}(1-\delta)$. Now, multiplying the utility function by $\frac{1}{\beta + \frac{1}{1-\delta}}$ we obtain (1.2) by setting $\alpha = \frac{1}{1+\beta(1-\delta)}$. In this form, preferences can be easily related to **Schmeidler's [27]** work on the Choquet integral and non-additive expected utility. Given

[2] See the appendix of Bewley [6].

an arbitrary set S and $\mathcal{S}$, a σ algebra of subsets of S, we say that a set-function $\nu : \mathcal{S} \to \mathbb{R}$ is a *capacity* if $\nu(\emptyset) = 0$, and $\nu(A) \leq \nu(B)$ whenever $A \subseteq B$. A capacity ν is said to be *convex* when $\nu(A \cup B) + \nu(A \cap B) \geq \nu(A) + \nu(B)$ $\forall A, B \in \mathcal{S}$.

Now, the utility representation that we considered above can be shown to be the infimum over all integrals of $u \circ x$ with respect to all probability measures that dominate a convex capacity ν obtained by making a linear distortion of μ with coefficient $\alpha \in (0, 1]$ (i.e., taking $\nu(A) = \alpha\mu(A)$ for $A \subsetneq S$ and $\nu(S) = 1$) [3]. In fact, let $M(S)$ be the set of probability measures on $\mathcal{S}$, then [4]

$$\alpha \int_S u \circ x \, d\mu + (1 - \alpha) \inf_S u \circ x = \inf_{\substack{\eta \in M(S) \\ \eta \geq \nu}} \int_S u \circ x \, d\eta \tag{1.3}$$

The representation of preferences as a minimal integral over a set of beliefs was suggested first by **Schmeidler [27]** in his pioneering work on Knightian uncertainty. The right-hand side of (1.3) is actually a particular case of the Choquet integral, proposed by Schmeidler, where the minimum is taken over charges η - finitely additive set functions - that dominate the convex capacity ν and such that $\eta(S) = 1$. In our case, the minimum over dominating charges coincides with the infimum over dominating probability measures (see appendix). That this Choquet integral coincides with the left-hand side of (3) is shown, for instance, in Hodges and Lehmann [18].

Notice that the minimal integral over the beliefs set tries to represent a precautionary or pessimistic behavior. The minimization solution η^* is selected so that it puts more weight on sets where the utility reaches its lowest values. When each agent has a "collection" of beliefs about nature states, this approach tries to capture the concept of Knigthian uncertainty. In a deterministic setting, each agent may be unsure about the discount factor and, therefore, by analogy, we say that preferences given by (1) or (2) represent agents' Knightian impatience. **Gilboa [15]** had already remarked that a possible explanation for the use of the Choquet integral as a representation of deterministic preferences may be that the agent dislikes great wobbles in his consumption level along time and is actually concerned with the worst future outcome.

[3] This is the called ϵ-contamination capacity. Some applications can be found in Berger and M. Berliner [5] and in Boratynska [9].

[4] A proof is given in the Appendix A

1.3 Arrow-Debreu Economies with Non-Summable Prices

In this section we address the Arrow-Debreu equilibria of the infinite-dimensional economy with the precautionary preferences defined above. Consumption plans are nonnegative bounded sequences, that is, elements in ℓ^∞_+. For an endowment vector $\omega \in \ell^\infty_+$ and prices π given by a linear functional on ℓ^∞, we define the Arrow-Debreu budget set as the set

$$B_{AD}(\pi, \omega) = \{x \in \ell^\infty_+ : \pi(x - \omega) \leq 0\}$$

Consumer's problem is defined as

$$\begin{aligned} \max \quad & U(x) \\ s.t. \quad & x \in B_{AD}(\pi, \omega) \end{aligned}$$

Suppose that there exists a number finite I of consumers each one characterized by $(\alpha, u^i, \mu, \omega^i)$. A couple $(\pi, (\overline{x}^i)_{i=1}^I)$ is said to be an Arrow-Debreu Equilibrium when

- $\overline{x}^i$ is a solution of the problem of consumer i for (π, ω^i);
- Markets clear: $\sum_{i=1}^I (\overline{x}^i - \omega^i) = 0$.

Let us recall Bewley's [6] result on existence of Arrow-Debreu equilibrium. Endowments are such that $\omega^i \gg 0 \quad \forall i$ and preferences are assumed to be strongly monotone, $\|.\|_\infty$-l.s.c. and such that have convex upper-contour sets. There exist equilibrium prices in ℓ^1 if preferences are Mackey continuous, but if only the Mackey upper semi-continuity holds equilibrium prices may be in the dual of ℓ^∞, which is the space of bounded finitely-additive set functions, also known as charges, denoted by ba.[5]

Since U^i is not Mackey-l.h.c., sometimes we have $\pi \notin \ell^1$. Indeed, we will show sufficient conditions for this to happen, but first we need to define the Banach limit functional LIM as a continuous linear mapping from ℓ^∞ into $\mathbb{R}$ such that $\text{LIM}(x) = \lim_n x_n$ when this limit exists. Clearly, this functional is not uniquely defined and $\text{LIM}(x) \in [\liminf(x), \limsup(x)]$ (See Dunford and Schwartz [13]).

[5] Recall that set function η is said to be finitely-additive set if $\eta(\emptyset) = 0, \eta(\cup_{i=1}^n A_i) = \sum_{i=1}^n \eta(A_i), \forall \{A_i\}$ where $A_i \cap A_j = \emptyset \ \forall i \neq j\}$.

Proposition 1: *Let y be a consumption plan in ℓ^∞_+ such that $\underline{y} := \inf_n y_n > 0$. If $\lim_n y_n = \underline{y}$ and it is not attained, then y is maximal for U in $B_{AD}(\pi, \omega)$ when*

(i) $\pi \in ba$ is given by

$$\pi x = \alpha \int x\, u'(y) d\mu + (1-\alpha) u'(\underline{y}) \mathrm{LIM}(x)$$

and (ii) ω is such that $\pi\omega = \pi y$. □

Remark: Intuition from the finite-dimensional case: given $x = (x_1, x_2, ...) \in \ell^\infty_+$ let $x^n \in \mathbb{R}^n$ be the n-truncation $(x_1, ..., x_n)$ of x and the function $u^n : \mathbb{R}^n_+ \to \mathbb{R}_+$ defined by $u^n(x^n) = \alpha \sum_{i=1}^n u(x_i)\mu_i + (1-\alpha) \min_{i=1}^n u(x_i)$ the n-truncation of u. Now, once u^i is increasing, $\sum_{i=1}^n u(y_i) + \min_{i=1}^n u(y_i) = \sum_{i=1}^n u(y_i) + u(\min y_k)$. Hence, if y is the utility maximizing bundle, then the price vector π will be a scalar multiple of $\alpha \nabla u^n(y) \Box \mu + (1-\alpha) \nabla u^n(y) \Box \delta_{\min y_k}$ where $\delta_{\min y_k}$ is the Dirac probability measure with unit mass at k_0 such that $y_{k_0} = \min y_k$. As n goes to infinity, if $\inf y_s$ is not attained then this expression tends to the one given in the proposition.

Proof: Let $x \in \ell^\infty_+$ such that $\pi x \leq \pi y$. We will show that $Ux - Uy \leq \pi x - \pi y$ and so y is maximal at $\{x \in \ell^\infty : \pi(x-y) \leq 0\}$.

Since u is concave, it is true that $u(z) - u(w) \leq u'(w)(z-w) \quad \forall z, w \in \mathbb{R}_+$, then

$$\int (u \circ x - u \circ y)\, d\mu \leq \int (u' \circ y)\,(x-y)\, d\mu \tag{1.4}$$

on the other side

$$\begin{aligned} \inf u \circ x - u(y_n) &\leq u(x_m) - u(y_n) \\ &\leq u'(y_n)(x_m - y_n) \end{aligned}$$

making $n \to +\infty$ and using that $u \in C^1(0, +\infty)$, we get

$$\inf u \circ x - \inf u \circ y \leq u'(\underline{y})(x_m - \underline{y})$$

making $m \to +\infty$ we get

$$\begin{aligned} \inf u \circ x - \inf u \circ y &\leq u'(\underline{y}) \left(\liminf x - \underline{y}\right) \\ &\leq u'(\underline{y}) \mathrm{LIM}(x) - u'(\underline{y}) \mathrm{LIM}(y) \end{aligned} \tag{1.5}$$

by (1.4) and (1.5), we get $Ux - Uy \leq \pi x - \pi y$. ■

Remark: (Gilles-LeRoy) By the Yosida-Hewitt Decomposition Theorem [6], every $\pi \in ba_+$ can be written as $\pi = p + \xi$ where p is a non-negative countably additive set function and

[6] See Bhaskara Rao and Bhaskara Rao [8].

ξ is a non-negative pure charge[7]. In the previous proposition, $(1-\alpha)u'(\underline{y})\mathrm{LIM}(\cdot)$ is the pure charge component. Gilles and LeRoy [17], suggested that the pure charge component of an AD-Equilibrium price system could be interpreted as a speculative bubble. However, these authors did not develop a sequential general equilibrium model and failed to relate this pure charge component to price bubbles in the assets that serve to complete the markets. Our goal will be to show that the Arrow-Debreu price can be implemented as an equilibrium of a deterministic sequential economy with an asset completing the markets in such a way that the countably additive component of the Arrow-Debreu price induces the deflator process whereas the pure charge component induces the asset price bubble.

We close this section with two examples of Arrow-Debreu equilibria with non-summable prices.

Example 1 Consider the example outlined in section 2. Let $u^1(x) = \log x, \quad \omega^1 = (\frac{s+1}{s} + \varphi_s)_{s\geq 1}$ and $u^2(x) = \log x, \quad \omega^2 = (\frac{s+1}{s} - \varphi_s)_{s\geq 1}$ where φ_s is $1/2$ when s is even and $-1/4$ when s is odd.

The candidates to Arrow-Debreu equilibrium consumption are $\overline{x}^1 = \overline{x}^2 = (\frac{s+1}{s})_{s\geq 1}$. Markets clear. Let us check individual optimality.

$\overline{x}^i$ is maximal (by Proposition (1)) for price system π and income $\pi\overline{x}^i$ when π is the functional defined by

$$\pi x = \sum_s \delta^{s-1} \frac{s}{s+1} x_s + \beta b(x)$$

where b is a particular Banach limit such that $b((x_1, .., x_s)) = \lim_n \sum_{s=1}^n \frac{x_s}{n}$ when this limit exists[8].

It remains to show that $\pi\omega^1 = \pi\overline{x}^i = \pi\omega^2$ or equivalently that $\pi\overline{x}^1 + \pi\varphi = \pi\overline{x}^1 - \pi\varphi$. So we need to show that $\pi\varphi = 0$ for the functional π defined above. That is, we want to show that

$$\sum_s \delta^{s-1} \frac{s}{s+1} \varphi_s + \beta b(\varphi) = 0 \tag{1.6}$$

[7] A charge $\rho \geq 0$ is a *pure charge* when $[\lambda \in ca_+, \ \rho \geq \lambda \Rightarrow \lambda \equiv 0]$.

[8] See Bhaskara Rao and Bhaskara Rao [8].

This Banach limit is uniquely defined on the set $\mathcal{B} := \{x \in \ell^\infty : \exists \lim_{n\to\infty} \frac{\sum_{j=1}^n x_j}{n}\}$. We start by verifying that $\varphi \in \mathcal{B}$. Let $S_n = \frac{\sum_{j=1}^n x_n}{n}$. We will show that $\lim_n S_n = 1/8$. In fact, $S_{2n} = \frac{n(-1/4+1/2)}{2n} = 1/8$ and that $S_{2n+1} = \frac{n(-1/4+1/2)-1/4}{2n+1} = \frac{-1/4+1/2}{2+1/n} + \frac{-1/4}{2n+1}$ converges to $1/8$ when $n \to \infty$. So equation (1.6) becomes

$$\sum_{s\geq 1} \delta^{s-1} \frac{s}{s+1} \varphi_s + \frac{\beta}{8} = 0 \tag{1.7}$$

It suffices to check that the first term is negative so that β can be chosen in order to make this equation hold. The first term can be rewritten as

$$\sum_{s=1}^{\infty} (\delta^{2s-1} \frac{2s}{2s+1} \varphi_{2s} + \delta^{2(s-1)} \frac{2s-1}{2s} \varphi_{2s-1}) = \sum_{s=1}^{\infty} \frac{\delta^{2s-1}}{2} \frac{2s}{2s+1} (1 - \frac{4s^2-1}{8\delta s^2})$$

which is negative when $\delta < \frac{1}{2}(1 - \frac{1}{4s^2})$ for every s and this holds if $\delta < 3/8$.

So $(\overline{x}^1, \overline{x}^2)$ is an AD-Equilibrium, for $\delta < 3/8$. This is an example of an Arrow-Debreu equilibrium whose prices are not in ℓ^1, as contemplated by Bewley's [6] existence theorem dispensing Mackey lower semicontinuity of preferences.

Example 2 The economy has two consumers whose preferences and endowments are given by

$u^1(x) = x^{1/2}, \quad \omega^1 = \frac{n+8}{n}$ and $\delta = 1/2$,

$u^2(x) = \log x, \quad \omega^2 = \frac{1}{4}(\frac{n+8}{n})^2$ and $\delta = 1/2$.

Let the consumption bundles be $y^1 = \omega^2$ and $y^2 = \omega^1$. We have market clearing. Let us construct the price functionals induced by marginal utilities (given by Proposition 2), which are the same for the two agents. In fact, $\frac{du^1}{dz}(y^1_n) = \frac{du^2}{dz}(y^2_n) = \frac{n}{n+8}$ and $\frac{du^1}{dz}(\underline{y}^1) = \frac{du^2}{dz}(\underline{y}^2) = 1$. Hence, for any bundle x, we have

$$\pi x = \alpha \sum_{n\geq 1} \frac{n}{n+8} \frac{1}{2^{n-1}}\, x_n + (1-\alpha)\, b(x)$$

i.e., π does not depend on i.

Finally, for a suitable choice of α we have that $\pi\omega^1 = \pi y^1$ (and, therefore, $\pi\omega^2 = \pi y^2$). In fact, $\pi\omega^1 = 1$ and $\pi y^1 = \frac{1}{4}[\alpha \sum_{n\geq 1} \frac{n+8}{n} \frac{1}{2^{n-1}} + (1-\alpha)]$. So $\alpha = \frac{3}{\sum_{n\geq 1} \frac{n+8}{n} \frac{1}{2^{n-1}} - 1}$ and we want show that $\sum_{n\geq 1}\frac{n+8}{n}\frac{1}{2^{n-1}} > 4$. If suffices to notice that the first term in the series is already greater than 4. Then, (π, y^1, y^2) is an Arrow-Debreu Equilibrium.

1.4 Optimality in a Sequential Economy

Let us define now a sequential economy whose equilibria can be put in one-to-one correspondence with the above Arrow-Debreu equilibria, under certain borrowing constraints. In this sequential economy agents can transfer income across different dates through a financial structure. More specifically, it will be enough to introduce an infinite-lived asset that pays no dividends and may be in positive net supply, fiat money. That is, agents can trade at every date t an amount $z_t \in \mathbb{R}$ of an asset that does not pay any kind of remuneration beyond the market price. Denoting by $q_s \geq 0$ the asset price at time s and taking the single consumption good to serve as numeraire at each date, the budget contraints added to optimization problem of the agent, in place of B_{AD}, are

$$x_s + q_s z_s \leq w_s + q_s z_{s-1} \forall s \geq 1 \tag{1.8}$$

where the portfolio $z = (z_s)_{s\geq 0}$ is such that holds the inicial condition $z_0 = \overline{z}_0 \geq 0$

Now, if $q_s = 0$ at some date s , then we have also $q_s = 0$ at every s subsequent (by a non-arbitrage condition that is necessary for existence of solution on agent's problem) and, therefore, one can not transfer income across time. In this case, since there is only one good at each date, the consumption solution would be $x^*_s = \omega_s$ and nothing interesting would happen. That is, the asset would not play any role and the market becomes incomplete. Consider now the opposite case, $q_s > 0$ and recall that money does not give any pay-off. If the borrowing constraints, which will be introduced below with the purpose of preventing Ponzi schemes, are not binding, then the associated shadow prices are zero and, therefore, the fundamental value is zero (see section (1.6) for a more precise discussion). Hence, the positive price of money has to be interpreted as a speculative bubble that can have real effects since the consumption opportunities are increased. Hence, until the end of section (1.4) we will consider only the relevant case where $q_s > 0\ \forall s$.

It is well-known that the budget restrictions (1.8) are not sufficient in order to guarantee the existence of a constrained maximal solution due to possibility of implementing Ponzi Schemes. Then, we need to impose borrowing constraints. The tip about which one we must choose will be given by the following analysis.

Suppose that we already know that the Property P over z ensures that there exists a solution for

$$\begin{array}{ll} \max & \sum_{s\geq 1} \delta^{s-1} u\left(x_s\right) + \beta \inf_s u(x_s) \\ s.t. & (x_s)_{s\in\mathbb{N}} \in \left\{x \in \ell^\infty_+ : \exists z \in \mathbb{R}^\infty \text{ such that (1.8) and } P \text{ are valid} \right\} \end{array} \tag{1.9}$$

Since there is one good in each period s, the decision of values for z_{s-1} and z_s determines the consumption $x_s(z) := \omega_s + q_s\left(z_{s-1} - z_s\right)$, i.e., the variable of choice becomes the portfolio $z = (z_s)_{s\in\mathbb{N}}$. So, problem (1.9) can be restated in the form of a typical problem of dynamic programming:

$$\begin{array}{ll} \max & \sum_{s\geq 1} \delta^{s-1} u \circ x_s(z) + \beta \inf_s u \circ x_s(z) \\ s.t. & 0 \leq \omega_s + q_s(z_{s-1} - z_s) \forall s \\ & (P) \text{ is valid for } z \\ & (\omega_s + q_s(z_{s-1} - z_s))_{s\in\mathbb{N}} \in \ell^\infty \\ & z_0 = \overline{z}_0 \geq 0 \end{array} \tag{1.10}$$

Let $x(z)$ be the sequence $(x_s(z))_{s\in\mathbb{N}}$ and $B_P(q,\omega)$ be the set

$$\left\{x \in \ell^\infty_+ : \exists z \in \mathbb{R}^\infty \text{ such that (1.8) and } P \text{ are valid} \right\}.$$

Definition: $v \in \mathbb{R}^\infty$ is said to be a *P-admissible direction at $z \in B_P(q,\omega)$* when $\exists\, \varepsilon_0 > 0$ such that $x \circ (z + vt) \in B_P(q,\omega)\ \forall t \in (-\varepsilon_0, \varepsilon_0)$. □

Let z^* be optimal on (1.10). Let v be a P-admissible direction at z^*. If there exists the limit

$$D(U \circ x \circ z)(z^*, v) = \lim_{t\to 0} \frac{U \circ x(z^* + tv) - U \circ x(z^*)}{t},$$

it is clear that it must be equal to zero.

Given a sequence x in ℓ^∞, we say that x is *positive* - which will be denoted by $x > 0$ - when $x_s > 0\ \forall s \geq 0$ and we say that x is *bounded away from 0* - which will be denoted by $x \gg 0$ - when $\exists h > 0$ such that $x_s \geq h\ \forall s$.

Since u satisfies Inada's Condition a suitable choice of Property P would guarantee that $x(z^*) > 0$. In this case, $u'(x_s(z^*))$ would be always well-defined. So, suppose for the moment that this is true.

Definition: Let z be a real sequence. z is said to be a *P-feasible portfolio* when $z_0 = \overline{z}_0 \geq 0$, $x(z) \in \ell^\infty_+$ and P is valid for z. □

1.4.1 Necessary Conditions for Optimality

Euler Equation

Proposition 2: *Let z^* be optimal for problem (1.10) and $x^* := x(z^*)$ Suppose that i) $x^* > 0$, and ii)* $\inf x^*$ *is not attained.*
Let us denote by v_s the sequence that has all components equal to zero except for the $s^{\underline{th}}$ which is 1. *If v_s is a P-admissible direction, then it is true that*

$$q_s u'(x_s^*) = \delta q_{s+1} u'(x_{s+1}^*) \qquad (1.11) \quad \square$$

Remark: Equation (1.11) is called the Euler's Equation, at date s, associated to the solution z^* of problem (1.10) - (EE) for short. This is a necessary condition to optimality that implies absence of arbitrage. In fact, letting $\lambda_s = \delta^{s-1} u'(x_s^*) > 0$ as date s deflator, we get $\lambda_s q_s = \lambda_{s+1} q_{s+1}$.

Remark: When $\beta = 0$, (EE) holds if (i), (ii) and (iii) can be replaced by the weaker condition $u'(x_s^*), u'(x_{s+1}^*) > 0$ [9].

Proof: : Continuity of u implies that $\inf_s u(x_s^*)$ is not attained either. For t with $|t|$ small enough it is true that

$$\inf_\sigma u(x_\sigma(z^* + tv_s)) = \inf_\sigma u(x_\sigma(z^*))$$

thus, we have

$$\begin{aligned} 0 = D(U \circ x \circ z)(z^*, v_s) = \; & \lim_{t\to 0} \tfrac{1}{t} \, [\, \delta^{s-1}(u(x_s^* - tq_s z_s^*) - u(x_s^*)) + \\ & +\delta^s (u(x_{s+1}^* + tq_{s+1} z_s^*) - u(x_{s+1}^*)) \,] \\ = \; & -\delta^{s-1} q_s u'(x_s^*) + \delta^s q_{s+1} u'(x_{s+1}^*) \end{aligned}$$

and so we are done. ∎

Transversality Condition

Proposition 3: *Let z^* be optimal for problem (1.10) such that i) $x^* := x(z^*) \gg 0$, ii)* $\inf x^*$ *is not attained and iii)* $\lim_s x_s^* = \inf_s x_s^*$.

[9] See, for instance, Stokey and Lucas [29].

Suppose that (EE) holds for every s at a portfolio z^* which is also a P-admissible direction at z^*. Then

$$\lim_s \delta^{s-1} u'(x_s^*) q_s z_s^* \in [\beta \liminf_s u'(x_s^*) q_s (z_{s-1}^* - z_s^*), \beta \limsup_s u'(x_s^*) q_s (z_{s-1}^* - z_s^*)] \tag{1.12}$$

If ω converges then $\lim_s \delta^{s-1} u'(x_s^*) q_s z_s^* = \beta \lim_s u'(x_s^*) q_s (z_{s-1}^* - z_s^*)$ □

Remark: : According to a previous remark, $\lambda_s = \delta^{s-1} u'(x_s^*)$ can be interpreted as a deflator of money price q. It is clear that when $\beta = 0$, we get $\lim_s \lambda_s q_s z_s^* = 0$, which is the traditional transversality condition. However, in general, the transversality condition - (βTC) for short - that we derived does not require that at an optimal portfolio z^* the present value of portfolio positions must be asymptotically zero.

Proof: Let y the sequence defined by $y_s = z_{s-1}^* - z_s^* \quad \forall s \geq 1$. We will estimate $\lim_{t \uparrow 0}$ and $\lim_{t \downarrow 0}$ of $\frac{1}{t}[U \circ x(z^* + tz^*) - U \circ x(z^*)]$. Since

$$\begin{aligned} \lim_{t \to 0} \tfrac{1}{t} \textstyle\sum_{s \geq 1} \delta^{s-1} [u(x_s^* + t q_s y_s) - u(x_s^*)] &= \textstyle\sum_{s \geq 1} \delta^{s-1} \lim_{t \uparrow 0} \tfrac{1}{t} [u(x_s^* + t q_s y_s) - u(x_s^*)] \\ &= \textstyle\sum_{s \geq 1} \delta^{s-1} u'(x_s^*) q_s y_s \\ &= - \lim_s \delta^{s-1} u'(x_s^*) q_s z_s^* \end{aligned}$$

we get

$$\begin{aligned} &\lim_{t \downarrow 0} \tfrac{1}{t} \{ \textstyle\sum_{s \geq 1} \delta^{s-1} [u(x_s^* + t q_s y_s) - u(x_s^*)] + \beta [\inf_s u(x_s^* + t q_s y_s) - \inf_s] u(x_s^*) \} = \\ &\quad = - \lim_s \delta^{s-1} u'(x_s^*) q_s z_s^* \ + \ \beta \lim_{t \downarrow 0} \tfrac{1}{t} [\inf_s u(x_s^* + t q_s y_s) - \inf_s u(x_s^*)] \end{aligned} \tag{1.13}$$

and all that is lacking to estimate is $\lim_{t \downarrow 0} \frac{1}{t}[\inf_s u(x_s^* + t q_s y_s) - \inf_s u(x_s^*)]$.

Affirmation:

$$(i) \lim_{t \to 0^-} \frac{1}{t} [\inf_s u(x_s^* + t q_s y_s) - \inf_s u(x_s^*) = \limsup_s u'(x_s^*) q_s y_s]$$

$$(ii) \lim_{t \to 0^+} \frac{1}{t} [\inf_s u(x_s^* + t q_s y_s) - \inf_s u(x_s^*) = \liminf_s u'(x_s^*) q_s y_s]$$

(the proof of this affirmation is given in Appendix (B.3))

Let $t < 0$. By optimality of z^* we get $\frac{1}{t}[U \circ x(z^* + tz^*) - U \circ x(z^*)] \geq 0$. Thus, the affirmation together with equation (1.13) imply $\lim_s \delta^{s-1} u'(x_s^*) q_s z_s^* \leq \beta \limsup_s u'(x_s^*) q_s (z_{s-1}^* - z_s^*)$. The demonstration of the other inequality is analogous.

When ω converges, $q_s(z_{s-1}^* - z_s^*) = x_s^* - \omega_s \to \inf x_s^* - \underline{\omega}$ and the interval in condition (11) collapses into $\beta \lim_s u'(x_s^*) q_s (z_{s-1}^* - z_s^*)$. ■

1.4.2 Sufficient Conditions for Optimality

We already know that (EE) is true at every optimal solution for which the direction v_s is admissible for every s. Now we investigate what is the other condition that should be added in order to ensure that a P-feasible portfolio will indeed be a solution.

Lemma 4: *Suppose that for all z P-feasible portfolio we have*

$$\begin{aligned}&\beta \limsup_s u'(x_s(\widetilde{z}))q_s(z_{s-1}-z_s) - \lim_s \delta^{s-1}u'(x_s(\widetilde{z}))q_s z_s \leq \\ &\leq \beta \liminf_s u'(x_s(\widetilde{z}))q_s(\widetilde{z}_{s-1}-\widetilde{z}_s) - \lim_s \delta^{s-1}u'(x_s(\widetilde{z}))q_s \widetilde{z}_s\end{aligned} \tag{1.14}$$

If ω^i converges and $\widetilde{z}$ satisfies (EE) at every date, then it is a solution for the problem (1.10). □

Proof: Now, let $\widetilde{z} \in B_P(q,\omega)$ be such that $x(\widetilde{z}) \gg 0$ and $\inf_s x_s(\widetilde{z})$ is not attained. Suppose that (1.11) is true for every period. Let z be a P-feasible portfolio. So

$$\begin{aligned}U \circ x \circ \widetilde{z} - U \circ x \circ z = \quad &\textstyle\sum_s \delta^{s-1}[u(x_s(\widetilde{z})) - u(x_s(z))] + \\ &+\beta[\inf_s u(x_s(\widetilde{z})) - \inf_s u(x_s(z))]\end{aligned}$$

If we denote the series difference by $D_1 = \sum_s \delta^{s-1}[u(x_s(\widetilde{z})) - u(x_s(z))]$ it can be shown that $D_1 \geq \lim_s \delta^{s-1}u'(x_s(\widetilde{z}))q_s z_s - \lim_s \delta^{s-1}u'(x_s(\widetilde{z}))q_s\widetilde{z}_s$ (see the Appendix (B.2)).

Let $D_2 = \beta[\inf_s u(x_s(\widetilde{z})) - \inf_s u(x_s(z))]$, which is bounded from below, as shown by the next lemma:

Affirmation:

$$\beta[\inf_s u(x_s(\widetilde{z})) - \inf_s u(x_s(z))] \geq \beta\underline{\lim}_s u'(\widetilde{x}_s)q_s(\widetilde{z}_{s-1}-\widetilde{z}_s) + \beta\underline{\lim}_s u'(\widetilde{x}_s)q_s(z_s - z_{s-1})$$

(the proof is given in Appendix B.4).

So,

$$\begin{aligned}D_1 + D_2 \geq \quad &\lim_S \delta^{s-1}u'(x_s(\widetilde{z}))q_s z_s - \lim_s \delta^{s-1}u'(x_s(\widetilde{z}))q_s\widetilde{z}_s + \\ &+\beta \liminf_s u'(x_s(\widetilde{z}))q_s(\widetilde{z}_{s-1}-\widetilde{z}_s) - \beta\limsup_s u'(x_s(\widetilde{z}))q_s(z_{s-1}-z_s)\end{aligned}$$

and by condition (1.14) we get $D_1 + D_2 \geq 0$. Since the limit $\lim_s u'(x_s(\widetilde{z}))q_s(\widetilde{z}_{s-1}-\widetilde{z}_s)$ exists, (1.14) trivially holds for $\widetilde{z}$. ■

Proposition 5: *Under the same hypothesis of Lemma (4), if the property P is such that*

$$z \text{ is } (P)\text{−feasible } \Rightarrow \lim_s \delta^{s-1} u'(x_s(\tilde{z}))q_s z_s + \beta \underline{\lim}_s u'(\tilde{x}_s)q_s(z_s - z_{s-1}) \geq 0 \tag{1.15}$$

then (EE) at every date and (βTC) become sufficient for optimality of z^*. □

Proof: By the Lemma (4) and the second part of proposition (1.12), (βTC) reduces to the claimed condition. ■

1.4.3 Borrowing Constraints

Next, we discuss different types of borrowing constraints that rule out Ponzi schemes, or equivalently, serve as forms for Property P considered above. Let us start by recalling the constraints that have been imposed in **the standard case where** $\beta = 0$. Notice first that, in this case, for positive asset prices, a plan z^* satisfying Euler and transversality becomes optimal within all plans z satisfying $\lim_s z_s \geq 0$. In fact, by Lemma (4), it suffices to show that $\lim \lambda_s q_s z_s \geq \lim \lambda_s q_s z^*_s = 0$, where, by (EE), $\lambda_s q_s$ is a positive constant.

For $\beta = 0$ and in models without default, these constraints have been of four kinds: explicit upper bounds on borrowing, implicit bounds on borrowing, transversality requirements and constraints relating borrowing to future wealth.

($P = M$): Explicit upper bound on borrowing: a bound $M > 0$ imposed on short-sales that requests $q_s z_s \geq -M$ for all portfolios that could be chosen by agent. Taking limit at this inequality we get $\lim_s z_s \geq \lim_s \frac{1}{q_s} M$. Denoting $\varphi = q_1 u'(x^*_1)$, (EE) implies $q_s = \varphi / u'(x^*_s)\delta^{s-2}$. If $x^* \gg 0$, then, as u' is continuous in $(0, \infty)$, results that the sequence $(u'(x^*_s))_{s \geq 1}$ is bounded and, therefore, $\lim_s q_s = +\infty$. Then, $\lim_s z_s \geq 0$.

($P = DC$): Implicit bound on borrowing: it is required that $q_s z_s \in \ell^\infty$. Under the assumption we have been making that $\inf_s x^*_s > 0$, we have $(u'(x^*_s))_s \in \ell^\infty$, implying that $\lambda \in \ell^1$ and, therefore, $\lambda q z \in \ell^1$. So $\lim_s \lambda_s q_s z_s = 0$.

($P = TC$): Transversality requirement: condition (1.12) that must hold for a suitable optimal is now imposed on all portfolios that can be chosen by agent. So, say that z is a TC-feasible portfolio means $\lim_s \delta^{s-1} u'(x^*_s) q_s z_s = 0$ where $x^* = x(z^*)$ for z^* optimal. In our setting this conditon is equivalent to $\lim_s z_s = 0$ and so it is clear that (1.15) is true for $P = TC$.

$(P = W)$: Credit dependent on future wealth: require portfolio values $q_s z_s$ to be bounded from below by the ability to repay out of your deflated future wealth $-\frac{1}{\lambda_s}\sum_{t>s}\lambda_t\omega_t$. Now $\lambda\omega \in \ell^1$ and, therefore, $\lim \lambda_s q_s z_s \geq 0$.

Under Knightian impatience, for $\beta > 0$, to rule out Ponzi schemes, we will propose new borrowing constraints that force all portfolios to mimic the new transversality condition that the optimal portfolios must satisfy.

We start by presenting some forms for property-P which rule out Ponzi schemes when the endowments sequence converges.

Let us now choose a constraint (P) on short-sales such that (1.15) be true. We suggest three kinds of constraints that impose (1.15):

$(P = \beta TC)$: The first will be called β-Transversality requirement. Following the traditional transversality condition approach we will impose (1.12) (which we know that a suitable z^* must satisfy if it is optimal) to every portfolio that can be chosen by the agent, i.e., z βTC-feasible requires $\lim_s \lambda^s q_s z_s = \beta \limsup_s \frac{\lambda_s}{\delta^{s-1}} q_s(z_{s-1} - z_s)$, for a given deflator process $\lambda > 0$ and so (1.15) is true for $P = \beta TC$ provided that $\delta^{s-1}u'(x_s^*) = \lambda_s$.

Remark: When $P = \beta TC$, Proposition (5) implies that (EE) is sufficient condition to get z^* optimal.

$(P = \beta DC)$: The second is a condition governing short-sale variations: given $\bar{s} \geq 1$ one requests that

$$z_s \geq \gamma_s z_{s-1} \ \forall s \geq \bar{s} \tag{1.16}$$

where γ_s are exogenous parameters holding $\gamma_s = 1 - \delta^{s-1}/(\beta + \delta^{s-1}) \quad \forall s \geq \bar{s}$.

So z βDC-feasible portfolio implies

$\beta u'(x_s^*)q_s(z_s - z_{s-1}) + \delta^{s-1}u'(x_s^*)q_s z_{s-1} = u'(x_s^*)q_s[\beta(z_s - z_{s-1}) + \delta^{s-1}z_s] \geq 0 \quad \forall s \geq \bar{s}$,

therefore, $\lim_s \delta^{s-1}u'(x_s^*)q_s z_s + \beta \underline{\lim}_s u'(x_s^*)q_s(z_s - z_{s-1}) \geq 0$.

Actually, we can relax $P = \beta DC$ to accommodate an additional negative term on the right-hand side that goes to zero as s tends to infinity. This allows for short-sales even when long positions were taken before.

$(P = \beta W)$. The third is a variant of the wealth-conditional borrowing constraint and requires

$$\lambda_s q_s z_s \geq -\sum_{t>s} \lambda_t \omega_t + \beta \limsup \lambda_s \delta^{-(s-1)} q_s (z_{s-1} - z_s). \tag{1.17}$$

It is immediate to see that this constraint implies condition (1.15) since $\lambda\omega \in \ell^1$.

We close this subsection with some remarks on the meaning of the requirements of P-admissibility of directions v_s and z^*.

Let us see first what does it mean to require v_s to be a P-admissible direction in the there possible configurations proposed for property P. Let z^* be a P-feasible portfolio.

v_s P-**admissible:**

i) $P = \beta TC$: Defining $z^{(s,t)} = z^* + tv_s$ we have

$$\lim_s \delta^{s-1} u'(x_s^*) q_s z_s^{(s,t)} - \beta \lim_s u'(x_s^*) q_s (z_{s-1}^{(s,t)} - z_s^{(s,t)}) = \lim_s \delta^{s-1} u'(x_s^*) q_s z_s^* - \beta \lim_s u'(x_s^*) q_s (z_{s-1}^* - z_s^*) = 0$$

so, v_s is every βTC-admissible;

ii) $P = \beta DC$: Suppose that (1.16) is never binding. For $s \geq 0$ let $h_s = z_s^* - \gamma_{s-1} z_{s-1}^*$ and $h_{s+1} = z_{s+1}^* - \gamma_s z_s^*$. Defining

$$h_s(t) = h_s + t \quad , \quad h_{s+1}(t) = h_{s+1} - tv_s$$

is true that $(h_s(0), h_{s+1}(0)) > 0$ and by a continuity argument v_s is an βDC-admissible direction;

iii) $P = \beta W$: If (1.17) is not binding at date s, then it it is clear that the direction v_s is βW-admissible.

We will see now what does it mean to require z^* to be a P-admissible direction in the first two cases proposed above for property P.

z^* P-**admissible:**

i) $P = \beta TC$: Defining $z^t = (1+t)z^*$ we have
$\lim_s \delta^{s-1} u'(x_s^*) q_s z_s^t - \beta \lim_s u'(x_s^*) q_s (z_{s-1}^t - z_s^t) =$
$(1+t) \lim_s \delta^{s-1} u'(x_s^*) q_s z_s^* - (1+t)\beta \lim_s u'(x_s^*) q_s (z_{s-1}^* - z_s^*) = 0$
so, v_s is every βTC-admissible;

ii) $P = \beta DC$: for $t > -1$, $z_s^* \geq \gamma_s z_{s-1}^*$ $\forall s \geq \bar{s}$ implies $(1+t)z_s^* \geq v_s(1+t) z_{s-1}^*$ $\forall s \geq \bar{s}$, and so z^* is ever βDC-admissible;

1.5 Sequential Implementation of Arrow-Debreu Equilibria

Let $\mathcal{U}$ be the set of real functions $u : \mathbb{R}_+ \to \mathbb{R}$ such that hold all the requirements in section (1.2).

Given $(\delta, \beta, u^i, \omega^i, \overline{z}_0^i)_{i=1}^I \in (0,1) \times \mathbb{R}_+ \times (\mathcal{U} \times \ell_+^\infty \times \mathbb{R}_+)^I$, a couple $(q, (\overline{z}^i)_{i=1}^I)$ is said to be a Sequential P-Equilibrium when

1. $\overline{z}^i$ is U^i-maximal in $B_P(q, \omega^i, \overline{z}_0^i) \quad \forall i$
2. $\sum_{i=1}^I \overline{z}_s^i = \sum_{i=1}^I \overline{z}_0^i \geq 0 \quad \forall s$
3. $\sum_{i=1}^I x(\overline{z}^i) = \sum_{i=1}^I \omega^i$

Remark: It is clear that (2) implies (3) and also that, when $q > 0$, (3) implies (2) in the above definition.

Let us see whether an Arrow-Debreu equilibrium can be implemented as a sequential P-equilibrium.

Recall that, by the Yosida-Hewitt decomposition theorem, any Arrow-Debreu equilibrium price can be decomposed into a countably additive component $p \in \ell_{++}^1$ and a non-negative pure charge τ. The asset that will allow for the sequential implementation may be in zero or positive net supply and its prices will be given by a vector $q^p > 0$ such that

$$q_s^p = 1/p_s \, \forall s \geq 1 \tag{1.18}$$

and hence q^p satisfies $p_s q_s^p = p_{s+1} q_{s+1}^p \quad \forall s \geq 1$.

Let x be a consumption plan in $B_{AD}(p, \omega^i)$ and $z_0 \geq 0$. The asset positions that implement this consumption plan are given by the vector $z^x \in \mathbb{R}^\infty$ satisfying

$$z_s^x = (\omega_s^i - x_s)/q_s^p + z_{s-1}^x \quad \forall s \geq 1 \tag{1.19}$$

that is, $x - \omega = q\Box(z_{-1}^x - z^x)$.

The following theorem establishes the sequential implementation of A-D equilibria and, conversely, that sequential equilibria can be seen as A-D equilibria, under a condition on the asymptotic behavior of portfolios. To present this condition we need to introduce a functional $\widetilde{b}$ defined on ℓ^∞ which coincides with the Banach limit b on $\mathcal{B} = \{x \in \ell^\infty : \exists \lim_{n\to\infty} \frac{\sum_{j=1}^n x_j}{n}\}$ and is given by $\limsup x$ for $x \notin \mathcal{B}$. Therefore, as discussed in section 3, the values of $\widetilde{b}$ are uniquely determined.

Theorem 6: *Given $\beta > 0, \delta \in (0,1)$, $(u_i, \omega^i, z_0^i)_{i=1}^I \in (\mathcal{U} \times \mathcal{B} \times \mathbb{R}_+)^I$. Suppose that Property P implies that for a certain deflator process $\lambda > 0$ we have*

$$\lim_s \lambda_s q_s(z_s - z_0^i) \geq \beta \lim_s \frac{\lambda_s}{\delta^{s-1}} \widetilde{b}(q\square(z_{-1} - z)). \tag{1.20}$$

i) If $(q, (\overline{z}^i)_{i=1}^I)$ is a Sequential P-Equilibrium with $q > 0$ such that $\inf_s x_s(\overline{z}^i) > 0$ is not attained, $\lim_s x_s(\overline{z}^i) = \inf_s x_s(\overline{z}^i) \quad \forall i$ and, at $x(\overline{z}^i)$, v_s is a P-admissible direction, for every s, then $(\pi, (\overline{x}^i)_{i=1}^I))$ is an Arrow-Debreu Equilibrium with π defined by

$$\pi x = \sum_{s\in\mathbb{N}} \delta^{s-1} u_i'(\overline{x}_s^i) x_s + \beta u_i'(\inf \overline{x}^i) b(x) \tag{1.21}$$

ii) If $(\pi, (\overline{x}^i)_{i=1}^I)$ is an Arrow-Debreu Equilibrium with π given by equation (1.21) and such that $\inf_s x_s^i > 0$ is not attained and $\lim_s x_s^i = \inf_s x_s^i \quad \forall i$.

then $(q^p, (\overline{z}^i)_{i=1}^I)$ is a Sequential P-Equilibrium with $p = (\delta^{s-1} u_i'(\overline{x}_s^i))_{s\in\mathbb{N}}$ and $\overline{z}^i = z^{\overline{x}^i}$ (by (1.19)). □

Proof: (i): Market clearing in the A-D economy follows from market clearing in the sequential economy. Proposition 1 implies that for the price functional given by $\pi^i x = \sum_{s\in\mathbb{N}} \delta^{s-1}(u_i)'(\overline{x}_s^i)x_s + \beta u_i'(\inf \overline{x}) b(x)$, $\overline{x}^i$ is optimal in $B_{AD}(\pi^i, \overline{x}^i)$. We need proof that $\pi^i \overline{x}^i \leq \pi^i \omega^i$ and that π^i can be replaced by a price functional π that does not depend on i.

First, notice that for any budget feasible portfolio plan z we have $z_1 = \frac{1}{q_1}(\omega_1^i - x(z)_1) + z_0^i$ and $z_s = \sum_{t\leq s} \frac{1}{q_t}(\omega_t^i - x(z)_t) + z_0^i$. So

$$\lim_s z_s - z_0^i = \sum_{t=1}^{\infty} \frac{1}{q_t}(\omega_t^i - x(z)_t) \tag{1.22}$$

By individual optimality in the sequential economy, (EE) holds at every date s and, therefore, $\lambda_s = \delta^{s-1}u_i'(\overline{x}_s^i)$. Then, $\pi^i(\omega^i - \overline{x}^i) = \sum_s \delta^{s-1}u_i'(\overline{x}_s^i)(\omega^i - \overline{x}_s^i) + \beta u_i'(\inf \overline{x}^i)b(\omega^i - \overline{x}^i) = \sum_s \lambda_s(\omega^i - \overline{x}_s^i) + \beta \lim_s \frac{\lambda_s}{\delta^{s-1}}\widetilde{b}(q\Box(z_{-1} - z)) = \lim_s \lambda_s q_s(z_s - z_0) + \beta \lim_s \frac{\lambda_s}{\delta^{s-1}}\widetilde{b}(q\Box(z_{-1} - z)) \geq 0$ by the assumption of this proposition.

By (EE) we have $\frac{u_i'(\overline{x}_s^i)}{u_i'(\overline{x}_1^i)} = \delta^{s-2}\frac{q_s}{q_1}$. Normalizing $\pi = \pi^i/u_i'(\overline{x}_1^i)$ we obtain a price functional π that does not depend on i.

(ii) For asset prices and positions constructed from the A-D equilibrium, as explained in the beginning of this section, we have, for any $s \geq 1$, adding the individual budget constraints of date s, $\sum_i q_s z_s^i = \sum_i(\omega_s^i - \overline{x}_s^i) + \sum_i q_s z_{s-1}^i$, where the first sum on the right-hand side is zero. Hence, for $q_s > 0$, we get $\sum_i z_s^i = \sum_i z_{s-1}^i = ... = \sum_i z_0^i$. That is, asset markets clear. Commodity markets clearing follows from A-D market clearing.

Let us show that the portfolios induced by the A-D equilibrium satisfy condition 1.20. By equations (1.18) and (1.19) we get again (1.22), that joined with $\pi(\omega^i - \overline{x}^i) = 0$ implies

$$\lim_s \overline{z}_s^i - z_0^i = \beta \lim_s \frac{\lambda_s}{\delta^{s-1}}\widetilde{b}(q\Box(\overline{z}_{-1}^i - \overline{z}^i))$$

where the value of the functional $\widetilde{b}$ that appears on the right-hand side can be replaced by the value of the Banach limit b since $q\Box(\overline{z}_{-1}^i - \overline{z}^i) = \overline{x}^i - \omega^i$, where $\overline{x}^i$ converges and $\omega^i \in \mathcal{B}$.

Let us show next that, for every i, the sequential budget set $B_p(q^p, \omega^i, z_0^i)$ is contained in the A-D budget set at (π, ω^i). Let z be P-feasible. By equation (1.22), we get $\pi(\omega_i - x(z)) = \lim_s(z_s - z_0) + \beta \lim_s \frac{p_s}{\delta s-1} b(q^p\Box(z_{-1}^x - z^x))$ and the right hand side is non-negative by assumption.■

Remark: It is immediate to see that in a sequential economy whose asset is in zero net supply, 1.20 assumed in the theorem is satisfied under borrowing constraints of the form $P = \beta TC$ or $P = \beta DC$ or $P = \beta W$.

Let us illustrate part (ii) of the above theorem in the context of the two examples that were presented in Section 3.

Example 1 (cont.)

Recall that agents' preferences and endowments are $u^1(x) = \log x$, $\omega^1 = (\frac{s+1}{s} + \varphi_s)_{s\geq 1}$, $u^2(x) = \log x$, $\omega^2 = (\frac{s+1}{s} - \varphi_s)_{s\geq 1}$ where φ_s is 1/2 when s is even and $-1/4$ when s is odd. In section 3 we found an A-D equilibrium consisting of the allocation $\overline{x}^1 = \overline{x}^2 = (\frac{s+1}{s})_{s\geq 1}$ and the price functional π given by

$$\pi x = \sum_s \delta^{s-1} \frac{s}{s+1} x_s + \beta b(x).$$

To see that it can be implemented sequentially, we apply Theorem 2 (ii). We should find a borrowing constraint that (a) implies condition (1.20) and (b) is verified by the portfolios induced, through (1.19), by $\overline{x}$.

Consider a variant of constraint $P = \beta W$ that requires [10]

$$\lambda_s q_s (z_s - z_0) \geq -\sum_{t>s} \lambda_t \omega_t^i + \beta \widetilde{b}(x - \omega^i) \lim_s \frac{\lambda_s}{\delta^{s-1}}$$

This constraint implies condition (1.20). In the context of this example, the requirement becomes

$$z_s - z_0 \geq -\sum_{t>s} \lambda_t \omega_t^i + \beta \widetilde{b}(x - \omega^i)$$

Let us show, for agent 1, that this borrowing constraint is verified by the portfolios induced by $\overline{x}$ through (1.19). The proof in the case of agent 2 is almost the same.

By equations (1.18) and (1.19) (applied recursively) we have

$$\sum_{t \leq s} p_t \varphi_t \geq -\sum_{t>s} p_t (\overline{x}_t^1 + \varphi_t) + \beta b(-\varphi)$$

By equation (1.7) this inequality reduces to $0 \geq -\sum_{t>s} p_t \overline{x}_t^1$ and, therefore, we are done.

Example 2 (cont.)

Recall that

$u^1(x) = x^{1/2}, \quad \omega^1 = \frac{n+8}{n}, \quad \delta = 1/2;$

$u^2(x) = \log x, \quad \omega^2 = \frac{1}{4}(\frac{n+8}{n})^2, \quad \delta = 1/2.$

We constructed before an A-D equilibrium given by $\overline{x}^1 = \omega^2$ and $\overline{x}^2 = \omega^1$ together with the

[10] We also could to impose a variant of the borrowing constraint βDC: $\lambda_{s_k} q_{s_k} z_{s_k} \geq \phi_{s_k} \lambda_{s_{k-1}} q_{s_{k-1}} z_{s_{k-1}} + \phi_{s_k} \lambda_{s_{k-1} \omega^i_{s_{k-1}}} - \xi_{s_k}$, where (ϕ_{s_k}) and (ξ_{s_k}) are properly chosen and hold $\phi_{s_k} \to 1, \quad \xi_{s_k} \to 0$. For more details, see the Appendix $(C.1)$.

price functional

$$\pi x = \alpha \sum_{n \geq 1} \frac{n}{n+8} \frac{1}{2^{n-1}}\, x_n + (1-\alpha)\, b(x)$$

Let us implement this A-D equilibrium as a sequential equilibrium with a single asset, money, in zero net supply. subject to borrowing constraints of the form $P = \beta TC$. We know that condition 1.20 assumed in the theorem holds. Then, the βTC-sequential equilibrium has the above consumption plan, spot prices given by $p_n = \alpha \frac{n}{n+8} \frac{1}{2^{n-1}}$, asset prices $q_n = 1/p_n$ and portfolios $z_n^i = \frac{1}{q_n}(\omega_n^i - x_n^i) + z_{n-1}^i$ where $z_0^i = 0$ and, again, $\pi(\overline{x}^i - \omega^i)$ implies that condition (1.20) holds for $\overline{x}^i$.

1.6 Bubbles in the Price of Money

In the two examples above of sequential equilibria, fiat money has a positive price. We show next this positive price is just a speculative bubble. Let us start by defining the fundamental value of money.

In general, if the direction v_s is not P-admissible at the optimal plan, Euler conditions would take the form $\lambda_s q_s = \lambda_{s+1} q_{s+1} + \gamma_s$ where $\gamma_s \geq 0$ is the shadow price of the borrowing constraint at date s. Applying these Euler inequalities recursively we get

$\lambda_s q_s = \sum_{s \leq t \leq T} \gamma_t + \lambda_{T+1} q_{T+1}$

Dividing by $\lambda_s > 0$ we get

$q_s = \frac{1}{\lambda_s} \sum_{t \geq s} \gamma_t + \frac{1}{\lambda_s} \lim_T \lambda_T q_T$

Since dividends are null, the fundamental value of fiat money is just the series $\frac{1}{\lambda_s} \sum_{t \geq s} \gamma_t$. The difference between the price of money and the fundamental value, $\frac{1}{\lambda_s} \lim_T \lambda_T q_T$, is the speculative bubble. Hence, the price of money can be positive due to positive shadow prices of borrowing constraints or due to a bubble. If Euler equations (EE) hold at every date, as it is the case in both examples, shadow prices are zero and the positive price for money (given by $q_s = 1/p_s > 0$ in the examples), must be due to a speculative bubble.

It is well-known that examples of speculative bubbles with zero net supply are easier to construct (see, for instance, Magill-Quinzii [22]). However, example 2 is still important since some authors (Huang and Werner [19]) had claimed that Arrow-Debreu equilibrium with non-summable prices could not be implemented in a sequential economy. Example 2 shows that the sequential implementation becomes possible once we use transversality conditions that are appropriate for Mackey-discontinuous preferences.

In the case of assets in positive net supply, the traditional context of Mackey-continuous preferences was rather hostile to bubbles (see Santos-Woodford [25] and Magill-Quinzii [22], except when default was allowed (as in Araujo, Páscoa and Torres-Martinez [2]). Let us recall the reason and see what changes when we use new transversality conditions, suitable for Mackey-discontinuous preferences.

When $\beta = 0$, the transversality condition requires $\lim_s \lambda_s q_s \overline{z}_s^i = 0 \quad \forall i$. If $\sum_i \overline{z}_0^i = e > 0$, then, in a sequential equilibrium, $\sum_i \overline{z}_s^i = e \quad \forall s$. So,

$$0 = \sum_i \lim_s \lambda_s q_s \overline{z}_s^i = \lim_s \lambda_s q_s \sum_i \overline{z}_s^i = \lim_s \lambda_s q_s e$$

Hence $\lim_s \lambda_s q_s = 0$ and, therefore, bubbles can not occur.

Even when $\beta > 0$ a similar argument can be done if $\quad \overline{x}_s^i \downarrow \inf x_s^i > 0 \quad$ and $\quad \overline{\omega}_s^i \downarrow \inf \omega_s^i > 0$ and bubbles can still be ruled out. Indeed, at an optimal plan, for which all suitable directions are P-admissible, we get

$$\lim_s \lambda_s q_s \overline{z}_s^i = \beta \lim_s u'(\overline{x}_s) q_s (\overline{z}_{s-1}^i - \overline{z}_s^i) \quad \forall i$$

and, again, adding across agents, recalling that $u'(\overline{x}_s) = \lambda_s / \delta^{s-1}$ and that $\sum_i \overline{z}_s^i = e > 0$, we get $\sum_i \lim_s \lambda_s q_s = 0$ and bubbles are still avoided.

However, when $\limsup \omega_s > \liminf \omega_s$ the above argument fails since the transversality condition has now a weaker form:

$$\lim_s \lambda_s q_s \overline{z}_s^i \in [\beta \liminf_s u'(\overline{x}_s^i) q_s (\overline{z}_{s-1}^i - \overline{z}_s^i), \beta \limsup_s u'(\overline{x}_s^i) q_s (\overline{z}_{s-1}^i - \overline{z}_s^i)] \quad \forall i$$

that does not imply $\lim_s \lambda_s q_s = 0$. In fact, when we add cross agents we get $\lim_s \lambda_s q_s e \leq \sum_i \beta \limsup_s u'(\overline{x}_s^i) q_s (\overline{z}_{s-1}^i - \overline{z}_s^i)$, but the right-hand side is only know to be greater or equal to $\beta \limsup_s \lambda_s / \delta^{s-1} q_s \sum_i (\overline{z}_{s-1}^i - \overline{z}_s^i)$ which is zero. Therefore, bubbles with positive net supply are possible. Note that it is the case in Example 1 since ω^i does not converge. Notice that in this example the present value of aggregate wealth is finite, as it is the case for any sequential equilibrium that implements an Arrow-Debreu equilibrium according to part (ii) of the Theorem (since the deflator process in the summable component of the A-D equilibrium and endowments are uniformly bounded).

1.7 Concluding Remarks

In this chapter we consider a very simple economy, a deterministic economy with infinite-lived agents trading a single good and a single asset, fiat money. However, the results that we obtained in this simple context changed the way some important issues in general equilibrium theory have been addressed. First, we established that non-summable Arrow-Debreu equilibrium prices can be implemented in a sequential equilibrium with asset trades, contrary to previous claims (by Huang and Werner [19]). In fact, the argument against sequential implementation assumed traditional borrowing constraints that guarantee the standard transversality condition of Mackey continuous preferences. Having derived the appropriate transversality condition (and some related borrowing constraints) for Mackey discontinuous preferences, we were able to put in a one-to-one correspondence the Arrow-Debreu equilibria and the sequential equilibria.

Secondly, we characterized the class of non-summable Arrow-Debreu equilibrium prices, for an interesting type of Mackey-discontinuous preferences that represents Knightian impatience, and used this characterization to construct examples of non-summable Arrow-Debreu equilibria and of the respective sequential equilibria.

Third, we were able to relate two concepts of bubbles that, according to the previous literature (see Santos-Woodford [25] and Magill-Quinzii [22]), seemed to be unrelated. We showed that the Gilles-LeRoy [17] bubble, which was just the pure charge component of a non-summable Arrow-Debreu price functional, can be rewritten as the bubble in the price of the asset that completes the markets sequentially.

Fourth, we found robust examples of bubbles in the price of fiat money in positive net supply, even when the present value of aggregate wealth is finite. Previous examples of monetary equilibrium in deterministic economies with infinite-lived agents were generated by positive shadow prices of borrowing constraints and, therefore, due to a positive fundamental value rather than to a bubble (and actually in stochastic economies a bubble for some deflator could always be reinterpreted as a positive fundamental value for some other defaltor). Moreover, the previous literature claimed that infinite present values of aggregate wealth were necessary for the occurrence of asset price bubbles of positive net supply assets in deterministic (or for unambiguous bubbles in stochastic economies), under borrowing constraints consistent with the necessary transversality conditions (see Santos-Woodford [25]). However, when preferences are more general, allowing for Mackey discontinuity, and the transversality conditions become also more general, this claim no longer holds, as our first example shows.

2. KNIGHTIAN ASSET BUBBLES: EXTENSIONS TO STOCHASTIC CASE

In this Chapter we will extend the previous precautionary model to a stochastic setting. It is supposed that exists a probability measure over the possible states of nature. The results that were shown in Chapter 1 are generalized (so, speculative bubbles Knightian impatience are "robust" to uncertainty presence), but with one advantage: the hypotheses become more naturals because they need hold only at a specific path of all possibles.

2.1 Extension of The Utility Functional

There are countably many dates indexed by $t \in \mathbb{N}$. At each date $t \geq 2$ a state of nature is observed by agents. The set of possible states of nature is $S = \{1, ..., N\}$. At $t = 1$ the state realization is already known and it will be denoted by 1.

We say to be a node the path $s^t \in S^t$ of t state realizations. Given $s^t = (s_1, ..., s_{t-1}, s_t)$ we will call the node $s^t_- = (s_1, ..., s_{t-1})$ as the predecessor node of s^t and a node such that $s^t_+ \in \{(s^t, s) : s \in S\}$ as one of the successor nodes of s^t.

Let us denote by S^∞ the set $\Pi_{n \in \mathbb{N}} S$ of all infinite paths of state realizations and by $\overline{S}$ the set $\cup_{\tau=1}^\infty S^\tau$ of all nodes.

For every node s^t, the cylinder with base on s^t is the set

$$C(s^t) = \{s^\infty \in S^\infty : \exists \sigma^\infty \in S^\infty \quad \text{such} \quad \text{that} \quad s^\infty = (s^t, \sigma^\infty)\},$$

ie, the set of all infinite paths of state realizations whose t initial values are s^t. Moreover, for $n \geq t$, the $n - t$-dimensional cylinder with base on s^t is the set

$$C^n(s^t) = \{s^n \in S^n : \exists \sigma^{t-n} \in S^{t-n} \quad \text{such} \quad \text{that} \quad s^n = (s^t, \sigma^{t-n})\},$$

the one of all finite path with n realizations such its t initial states coincide with s^t. Let us denote $\overline{S}_{s_t}$ the union $\cup_{\tau \geq t} C^{\tau}(s^t)$ that is the set of all nodes that remains possible after the s^t realizations.

Let $\mathcal{F}_t$ be the σ-field that consists of all finite unions of cylinders with base on S^t. Then $(\mathcal{F}_t)_{t \in \mathbb{N}}$ defines the filtration

$$\mathcal{F}_1 \subset \mathcal{F}_2 \subset ... \subset \mathcal{F}_t \subset ... \subset \mathcal{F}$$

where $\mathcal{F}_1$ is the trivial σ-field and $\mathcal{F}$ is the σ-field generated by the field $\cup_t \mathcal{F}_t$.

In this setup, a $((\mathcal{F}_t)_{t \in \mathbb{N}}$-adapted) stochastic process $\psi : S^{\infty} \times \mathbb{N} \to \mathbb{R}^m$ can be seen as a function on $\overline{S}$ such that $\psi_{s^t} := \psi(s^{\infty}, t)$ for $s^{\infty} = (s^t, \sigma^{\infty})$ since this value is completely determined by s^t.

We assume that there is a positive probability measure p on the set S, (i.e., $p_s > 0$ $\forall s = 1, .., N$ and $\sum_{s=1}^{N} p_s = 1$) that, for simplicity, is the same at every node. Let $I\!P$ be the probability measure on $(S^{\infty}, \mathcal{F})$ induced by p.

A consumption plan x will be a real function on $\overline{S}$ that is bounded and non-negative (i.e., $x \in \ell_+^{\infty}(\overline{S})$).

Given a above-bounded real function f on S^{∞}, let E be the expectation operator with respect to $I\!P$, i.e.,

$$E[f] = \int_{S^{\infty}} f \, dI\!P,$$

a natural extension of utility function (1.1) to this framework is

$$U(x) = E\left[\sum_{t \geq 1} \delta^{t-1} u\left(x_{s^t}\right)\right] + \beta \inf_{s^t \in \overline{S}} u(x_{s^t}) \tag{2.1}$$

again with $\delta \in (0, 1)$, $\beta \in [0, \infty)$, where the real function u on $\mathbb{R}_+$ has the same properties of section (1.2).

As previously, can be shown that U defined by (2.1) is not Mackey-continuous. Let us rewrite this expression by noting that $E\left[\sum_{t \geq 1} \delta^{t-1} u\left(x_{s^t}\right)\right] = \sum_{t \geq 1} \delta^{t-1} E\left[u\left(x_{s^t}\right)\right]$ [1] and so

$$U(x) = \sum_{t \geq 1} \delta^{t-1} E\left[u\left(x_{s^t}\right)\right] + \beta \inf_{s^t \in \overline{S}} u(x_{s^t}) \tag{2.2}$$

[1] Note that $\lim_n E[\sum_{t \geq n} \delta^{t-1} u\left(x_{s^t}\right)] = 0$ since u is above-bounded. If $\inf u \circ \widehat{x} = -\infty$ then $\widehat{x}$ could be discarded since it is not the optimal.

By Proposition (15) of the appendix section (A) we know that (2.2) is the infimum of the Lebesgue Integrals of $u \circ x$ with respect to all measures ν defined on the subsets of $\overline{S}$ and such that $\nu(s^t) \geq \delta^{t-1} I\!P(s^t)\ \forall s^t \in \overline{S}$ and $\nu(\overline{S}) = \frac{1}{1-\delta} + \beta$. In a similar interpretation as made before, $\delta^{t-1} I\!P(s^t)$ represent the minimal weight [2] given by agent to consumption at node s^t whereas $\delta^{t-1} I\!P(s^t) + \beta$ is the maximal weight.

2.2 Stochastic Economy

The agents can make use of a financial structure to transfer wealth. But besides transfer income across different dates through, they can transfer across states too (at a same date as well). There are N numeraire assets (ie, there are as many assets as states). Your pay-offs are bounded functions on $\overline{S}$. At each node s^t the asset j pay-off $R_{s^t,j}$ is non-negative. Let $q_{s^t,j} \geq 0$ be the price of asset j at node s^t. The budget constraint faced by agent at node s^t is

$$x_{s^t} + \sum_{j=1}^{N} q_{s^t,j} z_{s^t,j} \leq \omega^i_{s^t} + \sum_{j=1}^{N} (q_{s^t,j} + R_{s^t,j}) z_{s^t_-,j} \tag{2.3}$$

where $\omega^i \in \ell^\infty_+(\overline{S})$ is the endowment of the agent i and the real function z_j on $\overline{S} \cup \{0\}$ is the asset j position in the portfolio plan $z = (z_1, ..., z_N)$. z_j must hold the initial condition $z_{0,j} = \overline{z}_{0,j} \geq 0$.

Suppose that we already know that the Property P over z avoid Ponzi Schemes. The problem of agent is:

$$\begin{aligned} \max \quad & U(x) \\ s.t. \quad & x \in B_P(q, \omega, \overline{z}_0) \end{aligned} \tag{2.4}$$

where

$$B_P(q, \omega, \overline{z}_0)) = \left\{ x \in \ell^\infty_+(\overline{S}) : \exists z : \overline{S} \cup \{0\} \to \mathbb{R}^N \text{ such that }, z_0 = \overline{z}_0,\ (2.3) \text{ and } P \text{ is valid for } x \text{ and z} \right\}.$$

Since u is increasing, for each portfolio plan z we see that there is only one consumption plan $x(z)$ that does not waste wealth:

$$x_{s^t}(z) := \omega^i_{s^t} + \sum_{j=1}^{J} \left[R_{s^t,j} z_{s^t_-,j} + q_{s^t,j} (z_{s^t_-,j} - z_{s^t,j}) \right] \quad \forall s^t \tag{2.5}$$

[2] Now this minimal weight now depends as on the factor δ^{t-1} as on probability $I\!P(s^t)$.

So, problem (2.4) can be restated as:

$$\begin{array}{ll} \max & \sum_{s\geq 1} \delta^{t-1} E\left[u \circ x_{s^t}(z)\right] + \beta \inf_{s^t \in \overline{S}} u \circ x_{s^t}(z) \\ s.t. & 0 \leq \omega^i_{s^t} + \sum_{j=1}^N \left[R_{s^t,j} z_{s^t_-,j} + q_{s^t,j}(z_{s^t_-,j} - z_{s^t,j})\right] \quad \forall s^t \\ & (P) \text{ is valid for } z \\ & x(z) \in \ell^\infty_+(\overline{S}) \\ & z_{0,j} = \overline{z}_{0,j} \geq 0 \quad \forall j \end{array} \tag{2.6}$$

2.3 Optimality in a Stochastic Economy

In order to calculate the Gâteux derivative at a solution of (2.6) we need extend the definitions made in section (1.4):

Definition: A function $v : \overline{S} \to \mathbb{R}^N$ is said to be a *P-admissible direction at $z \in B_P(q,\omega)$* when $\exists\, \varepsilon_0 > 0$ such that $x \circ (z + rv) \in B_P(q,\omega)\ \forall r \in (-\varepsilon_0, \varepsilon_0)$. □

Definition: A function $z : \overline{S} \cup \{0\} \to \mathbb{R}^N$ is said to be a *P-feasible portfolio* when $z_{0,j} = \overline{z}_{0,j} \geq 0$, $x(z) \in \ell^\infty_+(\overline{S})$ and P is valid for z. □

2.3.1 Necessary Conditions for Optimality

Let z^* be optimal on (2.6). For every v P-admissible direction at z^* it must hold $D(U \circ x \circ z)(z^*, v) = 0$ when this limit exists. So we can state an Euler Equation and a Transversality Condition for the the agent's problem.

Stochastic Euler Equation (SEE)

Proposition 7: *Let z^* be optimal for problem (2.6) and $x^* := x(z^*)$.*
Suppose that i) $x^* > 0$ and ii) $\inf x^*$ is not attained.
Let us denote by $v_{s^t,j}$ the function on $\overline{S}$ that is equal to zero except for s^t which is

$$(0,..,0,\underbrace{1}_{j^{th}},0,...,0).$$

If $v_{s^t,j}$ is a P-admissible direction, then it is true that

$$q_{s^t,j} u'(x^*_{s^t}) = \delta E^p[(q_{s^t_+,j} + R_{s^t_+,j}) u'(x^*_{s^t_+})] \tag{2.7}$$

where $E^p[.] = \int_S (.) dp$. □

PROOF: : Continuity of u implies that $\inf_{s^t} u(x^*_{s^t})$ is not attained either. For r with $|r|$ small enough it is true that

$$\inf_{s^t} u(x_{s^t}(z^* + rv_{s^t,j})) = \inf_{s^t} u(x_{s^t}(z^*))$$

thus, we have

$$\begin{aligned} 0 = D(U \circ x \circ z)(z^*, v_{s^t,j}) = \quad & \lim_{r\to 0} \tfrac{1}{r} \{\delta^{t-1} [u(x^*_{s^t} - rq_{s^t,j}) - u(x^*_{s^t})] I\!P_{s^t} + \\ & +\delta^t \textstyle\sum_{s=1}^N \left[u(x^*_{(s^t,s)} + rq_{(s^t,s),\,j} + rR_{(s^t,s),\,j})) - u(x^*_{(s^t,s)})\right] I\!P_{(s^t,s)}\Big\} \\ = \quad & -\delta^{t-1} q_{s^t,\,j} u'(x^*_{s^t}) I\!P_{s^t} + \delta^t \textstyle\sum_{s=1}^N (q_{(s^t,s),\,j} + R_{(s^t,s),\,j}) u'(x^*_{(s^t,s)}) I\!P_{s^t} p_s \\ = \quad & -\delta^{t-1} I\!P_{s^t} q_{s^t,\,j} u'(x^*_{s^t}) + \delta^t I\!P_{s^t} E^p \left[(q_{s^t_+,j} + R_{s^t_+,j}) u'(x^*_{s^t_+})\right] \end{aligned}$$

and so we are done. ■

Stochastic Transversality Condition (STC)

Proposition 8: *Let z^* be optimal for problem (2.6) such that (i) $x^* := x(z^*) \gg 0$, (ii) $\inf x^*$ is not attained and there is a path $\sigma^\infty = (\sigma_1, ..., \sigma_\tau, ...) \in S^\infty$ such that (iii) $\lim_\tau x^*_{\sigma^\tau} = \inf_{s^t\in\overline{S}} x^*_{s^t}$, (iv) $\liminf_t \min_{s^t\in\overline{S}}(x^*_{s^t} - \omega^i_{s^t}) = \underline{\lim}_\tau (x^*_{\sigma^\tau} - \omega^i_{\sigma^\tau})$ and (v) $\limsup_t \max_{s^t\in\overline{S}}(x^*_{s^t} - \omega^i_{s^t}) = \overline{\lim}_\tau (x^*_{\sigma^\tau} - \omega^i_{\sigma^\tau})$, where we denoted $\sigma^\tau = (\sigma_1, ..., \sigma_\tau)$.*

Suppose that (SEE) holds for every s^t at each asset j of a portfolio plan z^* which is also a P-admissible direction at z^*. Then (denoting $q_{s^t} z^*_{s^t} = \sum_j q_{s^t,j} z^*_{s^t,}$ and $R_{s^t} z^*_{s^t} = \sum_j R_{s^t,j} z^*_{s^t,}$)

$$\lim_t \delta^{t-1} E\left[u'(x^*_{s^t}) q_{s^t} z^*_{s^t}\right]$$

belongs to

$$\left[\beta \liminf_\tau u'(x^*_{\sigma^\tau}) \left((q_{\sigma^\tau} + R_{\sigma^\tau}) z^*_{\sigma^\tau_-} - q_{\sigma^\tau} z^*_{\sigma^\tau}\right), \beta \limsup_\tau u'(x^*_{\sigma^\tau}) \left((q_{\sigma^\tau} + R_{\sigma^\tau}) z^*_{\sigma^\tau_-} - q_{\sigma^\tau} z^*_{\sigma^\tau}\right)\right] \tag{2.8}$$

If $(\omega_{\sigma^\tau})_{\tau\in\mathbb{N}}$ converges then

$$\lim_t \delta^{t-1} E\left[u'(x^*_{s^t}) q_{s^t} z^*_{s^t}\right] = \beta \lim_\tau u'(x^*_{\sigma^\tau}) \left((q_{\sigma^\tau} + R_{\sigma^\tau}) z^*_{\sigma^\tau_-} - q_{\sigma^\tau} z^*_{\sigma^\tau}\right)$$

□

Remark: : The term $\delta^{t-1} u'(x^*_{s^t}) I\!P_{s^t}$ can be interpreted as a deflator of node s^t to node 1. It is clear that when $\beta = 0$, we get $\lim_t \delta^{t-1} E\left[u'(x^*_{s^t}) q_{s^t} z^*_{s^t}\right] = 0$, which is the traditional transversality condition. Again, in general the transversality condition that we derived does not require that at an optimal portfolio z^* the mean present value of portfolio positions must be asymptotically zero.

PROOF: We will estimate $\lim_{r\uparrow 0}$ and $\lim_{r\downarrow 0}$ of $\frac{1}{r}[U \circ x(z^* + rz^*) - U \circ x(z^*)]$. In section (B.1) of appendix is given a demonstration that [3]

$$\lim_{r\to 0} \frac{1}{r}\left\{E\left[u(x^*_{s^1} - rq_{s^1}z^*_{s^1}) - u(x^*_{s^1})\right] + \sum_{t\geq 2}\delta^{t-1}E\left[u(x^*_{s^t} + r(q_{s^t} + R_{s^t})z^*_{s^t_-} - q_{s^t}z^*_{s^t}) - u(x^*_{s^t})\right]\right\}$$
$$= -\lim_{t\to\infty}\delta^{t-1}E\left[u'(x^*_{s^t})q_{s^t}z^*_{s^t}\right] \tag{2.9}$$

and all that is lacking to estimate is $\lim_{r\downarrow 0}$ and $\lim_{r\uparrow 0}$ of

$$\frac{1}{r}[\inf_{s^t\in\overline{S}} u(x^*_{s^t} + r(q_{s^t} + R_{s^t})z^*_{s^t} - q_{s^t}z^*_{s^t})) - \inf_{s^t\in\overline{S}} u(x^*_{s^t})].$$

Affirmation:

(i) $\displaystyle\lim_{r\to 0^-}\frac{1}{r}[\inf_{s^t\in\overline{S}} u(x^*_{s^t}+r((q_{s^t}+R_{s^t})z^*_{s^t_-}-q_{s^t}z^*_{s^t}))-\inf_{s^t\in\overline{S}} u(x^*_{s^t})] = \limsup_{\tau} u'(x^*_{\sigma^\tau})\left((q_{\sigma^\tau} + R_{\sigma^\tau})z^*_{\sigma^\tau_-} - q_{\sigma^\tau}z^*_{\sigma^\tau}\right)$

(ii) $\displaystyle\lim_{r\to 0^+}\frac{1}{r}[\inf_{s^t\in\overline{S}} u(x^*_{s^t}+r((q_{s^t}+R_{s^t})z^*_{s^t_-}-q_{s^t}z^*_{s^t}))-\inf_{s^t\in\overline{S}} u(x^*_{s^t})] = \liminf_{\tau} u'(x^*_{\sigma^\tau})\left((q_{\sigma^\tau} + R_{\sigma^\tau})z^*_{\sigma^\tau_-} - q_{\sigma^\tau}z^*_{\sigma^\tau}\right)$

(the proof of this affirmation is given in section (B.3) of appendix)

Let $r < 0$. By optimality of z^* we get $\frac{1}{r}[U \circ x(z^* + rz^*) - U \circ x(z^*)] \geq 0$. Thus, the above affirmation together with equation (2.9) imply

$$\lim_{t\to\infty}\delta^{t-1}E\left[u'(x^*_{s^t})q_{s^t}z^*_{s^t}\right] \leq \beta\limsup_{t} u'(x^*_{\sigma^t})\left((q_{\sigma^t} + R_{\sigma^t})z^*_{\sigma^t_-} - q_{\sigma^t}z^*_{\sigma^t}\right)$$

The demonstration of the other inequality is analogous.

When $(\omega_{\sigma^\tau})_{\tau\in\mathbb{N}}$ converges (say to $\widetilde{\omega}$), we have that $(q_{\sigma^\tau} + R_{\sigma^\tau})z^*_{\sigma^\tau_-} - q_{\sigma^\tau}z^*_{\sigma^\tau} = x^*_{\sigma^\tau} - \omega_{\sigma^\tau} \to \inf_{s^t} x^*_{s^t} - \widetilde{\omega}$ and the interval in condition (2.8) collapses into $\beta\lim_\tau u'(x^*_{\sigma^\tau})\left((q_{\sigma^\tau} + R_{\sigma^\tau})z^*_{\sigma^\tau_-} - q_{\sigma^\tau}z^*_{\sigma^\tau}\right)$.■

2.3.2 *Sufficient Conditions for Optimality*

Until the end of this subsection we will suppose that $\widetilde{z}$ is a P-feasible portfolio process such that: (i) $\widetilde{x} := x(\widetilde{z}) \gg 0$, (ii) $\inf\widetilde{x}$ is not attained and there is a path (iii) $\sigma^\infty =$

[3] In fact, this is the well-know Stochastic TC for the case $\beta = 0$ since the left hand side term must be zero if z^* is optimal. See, for instance, Stokey and Lucas [29].

$(\sigma_1, ..., \sigma_\tau, ...) \in S^\infty$ such that $\lim_\tau \widetilde{x}_{\sigma^\tau} = \inf_{s^t \in \overline{S}} \widetilde{x}_{s^t}$, (iv) $\inf_{s^t \in \overline{S}}(\widetilde{x}_{s^t} - \omega^i_{s^t}) = \underline{\lim}_\tau(\widetilde{x}_{\sigma^\tau} - \omega^i_{\sigma^\tau})$ and (v) $\sup_{s^t \in \overline{S}}(\widetilde{x}_{s^t} - \omega^i_{s^t}) = \overline{\lim}_\tau(\widetilde{x}_{\sigma^\tau} - \omega^i_{\sigma^\tau})$, where we denoted $\sigma^\tau = (\sigma_1, ..., \sigma_\tau)$.

We already know that if $\widetilde{z}$ is the optimal solution then (SEE) is true for the asset j at every node s^t for which the direction $v_{s^t,j}$ is admissible. Now, supposing that (SEE) is true at every node for all assets, we investigate what is the other condition that should be added in order to ensure that $\widetilde{z}$ will indeed be a solution.

Lemma 9: *Suppose that for all z P-feasible portfolio we have*

$$\begin{aligned} &\beta \limsup_\tau u'(\widetilde{x}_{\sigma^\tau})[(q_{\sigma^\tau} + R_{\sigma^\tau})z_{\sigma^{\tau-1}} - q_{\sigma^\tau} z_{\sigma^\tau}] - \lim_t \delta^{t-1} E\left[u'(\widetilde{x}_{s^t}) q_{s^t} z_{s^t}\right] \leq \\ &\leq \beta \liminf_\tau u'(\widetilde{x}_{\sigma^\tau})[(q_{\sigma^\tau} + R_{\sigma^\tau})\widetilde{z}_{\sigma^{\tau-1}} - q_{\sigma^\tau} \widetilde{z}_{\sigma^\tau}] - \lim_t \delta^{t-1} E\left[u'(\widetilde{x}_{s^t}) q_{s^t} \widetilde{z}_{s^t}\right] \end{aligned} \tag{2.10}$$

If $(\omega_{\sigma^\tau})_\tau$ converges and $\widetilde{z}$ satisfies (SEE) at every node then it is a solution for the problem (2.6). □

Proof: If we denote $D_1 = \sum_t \delta^{t-1} E\left[u(x_{s^t}(\widetilde{z})) - u(x_{s^t}(z))\right]$, can be shown[4] that

$$D_1 \geq \lim_t \delta^{t-1} E\left[u'(x_{s^t}(\widetilde{z})) q_{s^t} z_{s^t}\right] - \lim_t \delta^{t-1} E\left[u'(x_{s^t}(\widetilde{z})) q_{s^t} \widetilde{z}_{s^t}\right] \tag{2.11}$$

Let $D_2 = \beta[\inf_{s^t \in \overline{S}} u(x_{s^t}(\widetilde{z})) - \inf_{s^t \in \overline{S}} u(x_{s^t}(z))]$, which is bounded from below, as shown by the next lemma:

Affirmation:

$$D_2 \geq \beta \underline{\lim}_\tau u'(\widetilde{x}_{\sigma^\tau})[(q_{\sigma^\tau} + R_{\sigma^\tau})\widetilde{z}_{\sigma^{\tau-1}} - q_{\sigma^\tau} \widetilde{z}_{\sigma^\tau}] + \beta \underline{\lim}_\tau u'(\widetilde{x}_{\sigma^\tau})[q_{\sigma^\tau} z_{\sigma^\tau} - (q_{\sigma^\tau} + R_{\sigma^\tau}) z_{\sigma^{\tau-1}}]$$

The proof is given in Appendix B.4.

So,

$$\begin{aligned} D_1 + D_2 \geq \; & \lim_t \delta^{t-1} E\left[u'(x_{s^t}(\widetilde{z})) q_{s^t} z_{s^t}\right] - \lim_t \delta^{t-1} E\left[u'(x_{s^t}(\widetilde{z})) q_{s^t} \widetilde{z}_{s^t}\right] + \\ & + \beta \underline{\lim}_\tau u'(\widetilde{x}_{\sigma^\tau})[(q_{\sigma^\tau} + R_{\sigma^\tau})\widetilde{z}_{\sigma^{\tau-1}} - q_{\sigma^\tau} \widetilde{z}_{\sigma^\tau}] - \beta \overline{\lim}_\tau u'(\widetilde{x}_{\sigma^\tau})[(q_{\sigma^\tau} + R_{\sigma^\tau}) z_{\sigma^{\tau-1}} - q_{\sigma^\tau} z_{\sigma^\tau}] \end{aligned}$$

and by condition (2.10) we get $D_1 + D_2 \geq 0$ ■

Proposition 10: *Suppose that ω converges. If P is such that*

$$\lim_t \delta^{t-1} E\left[u'(\widetilde{x}_{s^t}) q_{s^t} z_{s^t}\right] - \beta \limsup_\tau u'(\widetilde{x}_{\sigma^\tau})[(q_{\sigma^\tau} + R_{\sigma^\tau}) z_{\sigma^{\tau-1}} - q_{\sigma^\tau} z_{\sigma^\tau}] \geq 0 \tag{2.12}$$

then (SEE) at every date and (STC) become sufficient for optimality of $\widetilde{z}$. □

Proof: By the previous lemma and the second part of proposition (8), (STC) reduces to the claimed condition. ■

[4] See the Appendix (B.2).

2.4 Stochastic Sequential Equilibria

Let us define the conceit of stochastic sequential equilibrium under the requirement of a property (borrowing constraint) P that avoids Ponzi schemes.

Given $(\delta, \beta, (u_i, \omega^i, \overline{z}_0^i)_{i=1}^I, (R_j)_{j=1}^N) \in (0,1) \times \mathbb{R}_+ \times (\mathcal{U} \times \ell_+^\infty(\overline{S}) \times \mathbb{R}_+^N)^I \times (\ell_+^\infty(\overline{S}))^N$, a couple composed of a node-contingent asset price $q : \overline{S} \to \mathbb{R}_+^N$ and a portfolio plan $\overline{z}^i : \overline{S} \cup \{0\} \to \mathbb{R}^N$ for each agent is said to be a Sequential P-Equilibrium when

1. $\overline{z}^i$ is U^i-maximal in $B_P(q, \omega^i, \overline{z}_0^i) \quad \forall i$
2. $\sum_{i=1}^I \overline{z}_{s^t}^i = \sum_{i=1}^I \overline{z}_0^i \geq 0 \quad \forall s^t$
3. $\sum_{i=1}^I x_{s^t}(\overline{z}^i) = \sum_{i=1}^I (\omega_{s^t}^i + R_{s^t} \overline{z}_0^i) \quad \forall s^t$

Remark: Since it is possible that exists a asset in positive net supply with positive pay-off (of good units) at node s^t, we need include this quantity in supply side of the real market-clearing equation.

Remark: It is clear that (2) implies (3) and also that, when $q > 0$, (3) implies (2) in the above definition.

2.5 Borrowing Constraints

As we made in section (1.4.3), let us discuss some borrowing constraints that generalize the traditional ones (case $\beta = 0$) for stochastic economies. To rule out Ponzi schemes, we will propose new borrowing constraints that force all portfolios to mimic the new transversality condition that the optimal portfolios must satisfy.

In fact, let us now choose a constraint (P) on short-sales such that (2.12) be true, so by Proposition (10) we have that (SEE) and (STC) become sufficient to optimality. We suggest three kinds of constraints that impose (2.12):

$(P = \beta TC)$: Following the traditional transversality condition approach we will impose (2.8) (which we know that a suitable z^* must satisfy if it is optimal) to every portfolio that can be chosen by the agent, i.e., $\lim_t \delta^{t-1} E\left[u'(x_{s^t}^*) q_{s^t} z_{s^t}^*\right] = \beta \limsup_\tau u'(x_{\sigma^\tau}^*) \left((q_{\sigma^\tau} + R_{\sigma^\tau}) z_{\sigma_-^\tau}^* - q_{\sigma^\tau} z_{\sigma^\tau}^* \right)$, where $(\sigma^t)_t$ is a sequence of nodes that holds the conditions (iii), (iv) and (v) of Proposition (8). Thus, (2.12) is true for $P = \beta TC$ (and z^* is βTC-feasible if ω^i converges in the sequence $(\sigma^t)_t$).

Remark: When $P = \beta TC$, Proposition (10) implies that (EE) is sufficient condition to get z^* optimal.

$(P = \beta DC)$: The second is a explicit upper bound on borrowing (that depends on the $(x(z) - \omega)$ behavior at $(\sigma^t)_t$): one requests that

$$\delta^{s-1} u'(x^*_{s^t}) q_{s^t} z_{s^t} \geq \beta u'(x^*_{\sigma^t}) q_{\sigma^t}(x_{\sigma^t}(z) - \omega_{\sigma^t}) \ \forall s^t \tag{2.13}$$

So z βDC-feasible portfolio implies that
$\beta u'(x^*_{\sigma^t})[q_{\sigma^t}(z_{\sigma^t} - z_{\sigma^t_-}) - R_{\sigma^t} z_{\sigma^t_-}] + \delta^{t-1} E[u'(x^*_{s^t_-}) q_{s^t_-} z_{s^t_-}] \geq 0 \quad \forall s \geq \bar{s}$,
therefore, $\lim_t \delta^{t-1} E[u'(x^*_{s^t}) q_{s^t} z_{s^t}] + \beta \underline{\lim}_t u'(x^*_{\sigma^t})[q_{\sigma^t}(z_{\sigma^t} - z_{\sigma^t_-}) - R_{\sigma^t} z_{\sigma^t_-}] \geq 0$.

$(P = \beta W)$: The third is a variant of the credit dependent on future wealth (now in stochastic setting). Require portfolio values $q^t_s z^t_s$ to be bounded from below by the ability to repay out of your deflated future wealth $-\frac{1}{\lambda^t_s} \sum_{s^\tau \in \overline{S}_{s^t} \setminus \{s^t\}} \lambda_{s^\tau} \omega_{s^\tau}$ and a additional term that again depends on behavior of $(z_{s^t_-} - z_{s^t})$ at the sequence of nodes $(\sigma^t)_t$:

$$\lambda^t_s q_{s^t} z_{s^t} \geq - \sum_{s^\tau \in \overline{S}_{s^t} \setminus \{s^t\}} \lambda_{s^\tau} \omega_{s^\tau} + \beta I\!P_{s^t} \limsup_t \lambda_{\sigma^t} \delta^{-(t-1)} [R_{\sigma^t} z_{\sigma^t_-} + q_{\sigma^t}(z_{\sigma^t_-} - z_{\sigma^t}))], \tag{2.14}$$

where $\lambda_{s^t} = \delta^{t-1} u'(x^*_{s^t}) I\!P_{s^t}$

It is immediate to see that this constraint implies condition (2.12) since $\lambda\omega \in \ell^1(\overline{S})$.

2.6 Speculative Bubbles in Asset Prices

Let us review some definitions and results that will be useful later. We follow the same approach in Duffie [12], Magill-Quinzii [22], Santos-Woodford [25] and others.

Given asset prices $q : \overline{S} \to \mathbb{R}^N_+$ and asset pay-offs $R : \overline{S} \to \mathbb{R}^N_+$ associated to an P-sequential equilibrium, let $T_{s^t} : \mathbb{R}^N \to \mathbb{R}^N$ be the linear operator defined by the $N \times N$-matrix $(R_{s^t,s} + q_{s^t,s})_{s \in S}$ that represents the asset returns. We say that the financial markets are *complete* at node s^t when $T_{s^t}(\mathbb{R}^N) = \mathbb{R}^N$, what means that all desired wealth-position at the successors of s^t

can be reached by a suitable choice of portfolio. The financial markets are said to be *complete* if they are complete at every s^t. Let us suppose that it is true.

We say that λ is a process of *node-deflators* (or *node-prices*) if it is a function $\lambda : \overline{S} \to \mathbb{R}_+$ such that

$$\lambda_{s^t} q_{s^t} = \sum_{s=1}^{N} \lambda_{(s^t,s)} (R_{(s^t,s)} + q_{(s^t,s)}) \tag{2.15}$$

at each node s^t. Since q is a price process associated to a P-equilibrium (and so arbitrage free), exists such process λ. Once we supposed that financial markets are complete, it is unique for a fixed λ normalization [5] .

If we iterate $(n-1)$ times the equation (2.15), we get

$$\lambda_{s^t} q_{s^t} = \sum_{\tau=t+1}^{t+n} \sum_{s^\tau \in C^\tau(s^t)} \lambda_{s^\tau} R_{s^\tau} + \sum_{s^{t+n} \in C^{t+n}(s^t)} \lambda_{s^{t+n}} q_{s^{t+n}}$$

Making $n \to \infty$ results [6]

$$\lambda_{s^t} q_{s^t,j} = \sum_{s^\tau \in \overline{S}_{s^t} \setminus \{s^t\}} \lambda_{s^\tau} R_{s^\tau,j} + \lim_T \sum_{s^T \in C^T(s^t)} \lambda_{s^T} q_{s^T,j}$$

As markets are complete and q is of P-equilibrium, the node-deflator λ_{s^t} must be positive at each node. Thus we can define the *fundamental value* $V_{s^t,j}$ and the *speculative bubble* $B_{s^t,j}$ of asset j at node s^t respectively as

$$\frac{1}{\lambda_{s^t}} \sum_{s^\tau \in \overline{S}_{s^t} \setminus \{s^t\}} \lambda_{s^\tau,j} R_{s^\tau,j} \tag{2.16}$$

and as

$$\frac{1}{\lambda_{s^t}} \lim_T \sum_{s^T \in C^T(s^t)} \lambda_{s^T} q_{s^T,j}. \tag{2.17}$$

when the residual term is is positive, we say that there is a *bubble component* in the price of asset j at this node.

Let us denote $B_{s^t} \in \mathbb{R}_+^N$ the vector $(B_{s^t,j})_{j=1}^N$. It is clear that

$$\lambda_{s^t} B_{s^t} = \sum_{s \in S} \lambda_{(s^t,s)} B_{(s^t,s)},$$

[5] A discussion about all of this can be found in Duffie [12].

[6] Since $\lambda_{s^t} q_{s^t,j} \geq \sum_{\tau=t+1}^{t+n} \sum_{s^\tau \in C^\tau(s^t)} \lambda_{s^\tau} R_{s^\tau,j} \quad \forall n$, the series (that has only non-negative terms) converges and thus $\exists \lim_T \sum_{s^T \in C^T(s^t)} \lambda_{s^T} q_{s^T,j} (\geq 0)$.

so, each asset bubble process must hold a martingale property.

When (SEE) (equation (2.7)) holds at every node, we can find the unique node-deflator process making $\lambda_{s^t} = \delta^{t-1} u_i'(x^*_{s^t} I\!P_{s^t})$ and use this expression to evaluate $V_{s^t,j}$. Therefore under hypotheses that imply (SEE), if $N = 1$ we have that the definition of fundamental value (and so of speculative bubble) given here coincides with the section (1.6) one.

2.7 Sequential Implementation with Bubbles

In this section we will whether an Arrow-Debreu equilibrium (with consumption space $\ell^\infty_+(\overline{S})$) can be implemented as a sequential P-equilibrium.

Let $\overline{x}$ be a consumption that is optimum in the AD budget constraint defined by a price $\pi \in (\ell^\infty_+(\overline{S}))^*$ and endowments $\omega^i \in \ell^\infty_+(\overline{S})$. Again, π price can be decomposed into a countably additive component $\rho \in \ell^1_{++}(\overline{S})$ and a pure charge ζ.

Fixed the asset pay-offs $R_j : \overline{S} \to \mathbb{R}_+$ for $j = 1, ..., N$, the initial position $\overline{z}^i_0 \in \mathbb{R}^N_+$ and a process $\widetilde{B} \equiv (\widetilde{B}_1, ..., \widetilde{B}_N) : (\overline{S}) \to \mathbb{R}^N_+$ such that $\rho_{s^t} B_{s^t} = \sum_{s \in S} \rho_{(s^t,s)} B_{(s^t,s)}$ at every node, we can define a price process for assets making

$$\rho_{s^t} q_{s^t,j} = \sum_{s^\tau \in \overline{S}_{s^t} \setminus \{s^t\}} \rho_{s^\tau} R_{s^\tau,j} + \rho_{s^t} \widetilde{B}_{s^t,j}, \tag{2.18}$$

what implies $\rho_{s^t} q_{s^t,j} = \sum_{s \in S} \rho_{(s^t,s)} (R_{(s^t,s),j} + q_{(s^t,s),j}) \quad \forall s^t$, i.e., the $\ell^1_{++}(\overline{S})$ component of price is a node-deflator process.

Let us suppose that financial markets with this prices are complete and find a portfolio $z^{\overline{x}}$ such that joined with x^* holds the budget constraints

$$\overline{x}_{s^t} + \sum_{j=1}^N q_{s^t,j} z^{\overline{x}}_{s^t,j} = \widetilde{\omega}^i_{s^t} + \sum_{j=1}^N (q^{\overline{x}}_{s^t,j} + R_{s^t,j}) z^{\overline{x}}_{s^t_-,j} \tag{2.19}$$

(where $\widetilde{\omega}_{s^t}$ is the adjusted endowment defined by $\omega_{s^t} - R_{s^t} \overline{z}^i_0$) and the initial condition $z^{\overline{x}}_0 = \overline{z}^i_0$.

If $z^{\overline{x}}_{s^t_-}$ is already defined, let us find $z^{\overline{x}}_{s^t}$. At each nodes $s^{t+1} \in C^{t+1}(s^t)$ and $s^{t+2} \in C^{t+2}(s^t)$ must be true that

$$(q_{s^{t+1}} + R_{s^{t+1}})z^{\overline{x}}_{s^t} = \overline{x}_{s^{t+1}} - \widetilde{\omega}^i_{s^{t+1}} + q_{s^{t+1}} z^{\overline{x}}_{s^{t+1}} \tag{2.20}$$

and

$$(q_{s^{t+2}} + R_{s^{t+2}})z^{\overline{x}}_{s^{t+1}} = \overline{x}_{s^{t+2}} - \widetilde{\omega}^i_{s^{t+2}} + q_{s^{t+2}} z^{\overline{x}}_{s^{t+2}} \tag{2.21}$$

Multiplying (2.21) by $p_{s^{t+2}}$ and summing up we get

$$p_{s^{t+1}} q_{s^{t+1}} z^{\overline{x}}_{s^{t+1}} = \sum_{s^{t+2} \in C^{t+2}(s^{t+1})} p_{s^{t+2}} (\overline{x}_{s^{t+2}} - \widetilde{\omega}^i_{s^{t+2}}) + \sum_{s^{t+2} \in C^{t+2}(s^{t+1})} p_{s^{t+2}} q_{s^{t+2}} z^{\overline{x}}_{s^{t+2}}. \tag{2.22}$$

Since for each $s^T \in C^T(s^{t+1})$ the analogous of (2.22) must holds for $p_{s^T} q_{s^T} z^{\overline{x}}_{s^T}$, we get

$$p_{s^{t+1}} q_{s^{t+1}} z^{\overline{x}}_{s^{t+1}} = \sum_{\tau=t+2}^{T+1} \sum_{s^\tau \in C^\tau(s^{t+1})} p_{s^\tau} (\overline{x}_{s^\tau} - \omega^i_{s^\tau} + R_{s^\tau} \overline{z}^i_0) + \sum_{s^{T+1} \in C^{T+1}(s^{t+1})} p_{s^{T+1}} q_{s^{T+1}} z^{\overline{x}}_{s^{T+1}}. \tag{2.23}$$

Making $T \to \infty$, equation (2.23) implies

$$p_{s^{t+1}} q_{s^{t+1}} z^{\overline{x}}_{s^{t+1}} = \sum_{s^\tau \in \overline{S}_{s^{t+1}} \setminus \{s^{t+1}\}} p_{s^\tau} (\overline{x}_{s^\tau} - \omega^i_{s^\tau} + R_{s^\tau} \overline{z}^i_0) + \lim_T \sum_{s^T \in C^T(s^{t+1})} p_{s^T} q_{s^T} z^{\overline{x}}_{s^T}. \tag{2.24}$$

Let us multiply the equation (2.20) by $p_{s^{t+1}}$ and use (2.24) to get:

$$\begin{aligned} p_{s^{t+1}}(q_{s^{t+1}} + R_{s^{t+1}})z^{\overline{x}}_{s^t} = p_{s^{t+1}}(\overline{x}_{s^{t+1}} - \widetilde{\omega}^i_{s^{t+1}}) + \textstyle\sum_{s^\tau \in \overline{S}_{s^{t+1}} \setminus \{s_{t+1}\}} p_{s^\tau} (\overline{x}_{s^\tau} - \omega^i_{s^\tau} + R_{s^\tau} \overline{z}^i_0) + \\ + \lim_T \textstyle\sum_{s^T \in C^T(s^{t+1})} p_{s^T} q_{s^T} z^{\overline{x}}_{s^T} \end{aligned} \tag{2.25}$$

So, to find $z^{\overline{x}}_{s^t}$ it is necessary know the value of the limit above at each node $s^{t+1} \in C^{t+1}(s^t)$. We will find it using the budget constraint at 1. In fact, since

$$q_1 z^{\overline{x}}_1 = (q_1 + R_1)\overline{z}^i_0 - \overline{x}_1 + \omega^i_1 - R_1 \overline{z}^i_0,$$

from the analogous to (2.24) for $p_1 q_1 z^{\overline{x}}_1$ we get

$$\begin{aligned} l(1) := \lim_T \textstyle\sum_{s^T \in S^T} p_{s^T} q_{s^T} z^{\overline{x}}_{s^T} &= \textstyle\sum_{s^\tau \in \overline{S} \setminus \{1\}} p_{s^\tau} (\omega^i_{s^\tau} - \overline{x}_{s^\tau} - R_{s^\tau} \overline{z}^i_0) + \\ &+ p_1(q_1 + R_1)\overline{z}^i_0 - p_1(\overline{x}_1 + \omega^i_1 - R_1 \overline{z}^i_0) \\ &= (p_1 q_1 - \textstyle\sum_{s^\tau \in \overline{S} \setminus \{1\}} p_{s^\tau} R_{s^\tau})\overline{z}^i_0 \\ &\quad + \textstyle\sum_{s^\tau \in \overline{S}} p_{s^\tau} (\omega^i_{s^\tau} - \overline{x}_{s^\tau}) \\ &= p_1 \widetilde{B}_1 \overline{z}^i_0 + \rho(\omega^i - \overline{x}) \\ &= p_1 \widetilde{B}_1 \overline{z}^i_0 + \zeta(\overline{x} - \omega^i) \end{aligned}$$

Let us define $l(s^t) = l(1)I\!P_{s^t}$ (hence $l(s^t) = \sum_{s^{t+1}\in C^{t+1}(s^t)} l(s^{t+1})$). Now we can find $z^{\overline{x}}_{s^t}$ from equation (2.25):

$$z^{\overline{x}}_{s^t} = T^{-1}_{s^t}\left[\left(\overline{x}_{s^{t+1}} - \widetilde{\omega}^i_{s^{t+1}} + \frac{1}{p_{s^{t+1}}}\Big(\sum_{s^\tau\in\overline{S}_{s^{t+1}}\setminus\{s_{t+1}\}} p_{s^\tau}(\overline{x}_{s^\tau} - \omega^i_{s^\tau} + R_{s^\tau}\overline{z}^i_0) + l(s^{t+1})\Big)\right)_{s^{t+1}\in C^{t+1}(s^t)}\right], \tag{2.26}$$

where T_{s^t} is the linear operator defined in Section (2.6).

Remark: Interpretation of equation $\lim_T \sum_{s^T\in S^T} p_{s^T} q_{s^T} z^{\overline{x}}_{s^T} = p_1\widetilde{B}_1\overline{z}^i_0 + \zeta(\overline{x} - \omega^i)$: (i) With traditional (TC) (i.e., $\lim_T \sum_{s^T\in S^T} p_{s^T} q_{s^T} z^{\overline{x}}_{s^T} = 0$) and with AD-equilibrium price $\pi \in \ell^1_+(\overline{S})$ (and so $\zeta(\overline{x} - \omega^i)$), the allocation $\overline{x}$ only can be implemented as a sequential equilibrium if $\widetilde{B}_1\overline{z}^i_0 = 0$, i.e., only could exist bubble components in price of assets with zero supply; (ii) Again with traditional (TC), generally only allocations related to summable equilibria can be implemented without bubbles for positive net supply assets.

At the next, we need a bit of notation. Given a sequence of nodes $(\sigma^t)_t$ let A the set of this nodes and $\widetilde{b}$ be the functional defined on $\ell^\infty(A)$ which coincides with the Banach limit b on $\mathcal{B} = \{x \in \ell^\infty(A) : \exists \lim_{n\to\infty} \frac{\sum_{t=1}^n x_{\sigma^t}}{n}\}$ and is given by $\limsup(x_{\sigma^t})_t$ for $x \notin \mathcal{B}$.

The following proposition establishes the sequential implementation of A-D equilibria and, conversely, that sequential equilibria can be seen as A-D equilibria, under a condition on the asymptotic behavior of portfolios.

Theorem 11: *Given $\beta > 0, \delta \in (0,1)$, $(u_i, \omega^i, \overline{z}^i_0)^I_{i=1} \in (\mathcal{U} \times \ell^\infty_+(\overline{S}) \times \mathbb{R}^N_+)^I$ such $(\omega^i_{\sigma^t})_t \in \ell^\infty(A)$ $\forall i$ and given a financial structure characterized by $R_j : \overline{S} \to \mathbb{R}_+$ for $j = 1, ..., N$. Suppose that Property P implies that for a certain node-deflator process $\lambda > 0$ we have*

$$\lim_T \sum_{s^T\in S^T} \lambda_{s^T} q_{s^T} z_{s^T} \geq \lambda_1\widetilde{B}_1\overline{z}^i_0 + \beta\lim_t \frac{\lambda_{\sigma^t}}{\delta^{t-1}I\!P_{s^t}}\widetilde{b}((R_{\sigma^t} + q_{\sigma^t})z_{\sigma^t_-} - q_{\sigma^t}z_{\sigma^t}). \tag{2.27}$$

i) If $(q, (x(\overline{z}^i), \overline{z}^i)^I_{i=1})$ is a Sequential P-Equilibrium (of economy with adjusted endowments) such that $\inf_{s^t} x_{s^t}(\overline{z}^i) > 0$ is not attained, $\lim_t x_{\sigma^t}(\overline{z}^i) = \inf_{s^t} x_{s^t}(\overline{z}^i)$ $\forall i$ and, at $x(\overline{z}^i)$, v_s is a P-admissible direction, for every s, then $(\pi, (\overline{x}^i)^I_{i=1}))$ is an Arrow-Debreu Equilibrium with π

defined by

$$\pi x = \sum_{t\geq 1}\sum_{s^t\in S^t} \delta^{t-1}u_i'(\overline{x}^i_{s^t})x_{s^t} + \beta u_i'(\inf_t \overline{x}^i_{\sigma^t})b((x_{\sigma^t})_t) \tag{2.28}$$

ii) If $(\pi, (\overline{x}^i)_{i=1}^I)$ *is an Arrow-Debreu Equilibrium with* π *given by equation (*2.28*) and such that* $\inf_{s^t} x_{s^t}(\overline{z}^i) > 0$ *is not attained,* $\lim_t x_{\sigma^t}(\overline{z}^i) = \inf_{s^t} x_{s^t}(\overline{z}^i)$ *and if financial markets are complete at prices* q *given by (*2.18*), then* $(q, x((\overline{z}^i), (\overline{z}^i)_{i=1}^I)$ *is a Sequential* P*-Equilibrium (of economy with adjusted endowments) with* $\rho = (\delta^s u_i'(\overline{x}^i_{s^t})I\!P_{s^t})_{s\in\overline{S}}$ *and* $\overline{z}^i = z^{\overline{x}^i}$ *(given by (*2.26*)).* □

PROOF: i)First, notice that for any budget feasible portfolio plan z we have

$\lim_T \sum_{s^T\in S^T} \lambda_{s^T} q_{s^T} z_{s^T} = \lambda_1 \widetilde{B}_1 \overline{z}^i_0 + \sum_{s^\tau\in\overline{S}} \lambda_{s^\tau}(\omega^i_{s^\tau} - x_{s^\tau}(z))$

By individual optimality in the sequential economy, (EE) holds at every node s^t and, therefore, $\lambda_{s^t} = \delta^{t-1}u_i'(\overline{x}^i_s)I\!P_{s^t}$. Then,

$$\begin{aligned}\pi^i(\omega^i - x^i(z)) &= \textstyle\sum_t \sum_{s^t\in S^t} \delta^{t-1}u_i'(\overline{x}^i_{s^t})(\omega^i_{s^t} - x^i_{s^t}(z)) + \beta u_i'(\inf \overline{x}^i) b((\omega^i_{\sigma^t} - x^i_{\sigma^t}(z))_t)\\ &\geq \textstyle\sum_{t\in\overline{S}} \lambda_{s^t}(\omega^i_{s^t} - x^i_{s^t}(z)) + \beta \lim_t \frac{\lambda_{\sigma^t}}{\delta^{t-1}\boldsymbol{P}_{s^t}} \widetilde{b}(q_{\sigma^t} z_{\sigma^t} - (R_{\sigma^t} + q_{\sigma^t}) z_{\sigma^t_-})\\ &= \textstyle\lim_T \sum_{s^T\in S^T} \lambda_{s^T} q_{s^T} z_{s^T} - \lambda_1 \widetilde{B}_1 \overline{z}^i_0 + \beta \lim_t \frac{\lambda_{\sigma^t}}{\delta^{t-1}\boldsymbol{P}_{s^t}} \widetilde{b}(q_{\sigma^t} z_{\sigma^t} - (R_{\sigma^t} + q_{\sigma^t}) z_{\sigma^t_-}) \geq 0\end{aligned}$$

by the assumption of this proposition. That $x(\overline{z}^i)$ is optimal in AD-budget constraint follows from an analogous argument to the one made to demonstrate Proposition (1).

(ii) For asset prices and portfolio positions constructed from the A-D equilibrium, as explained in the beginning of this section, we have,

$$\lim_T \sum_{s^T\in S^T} p_{s^T} q_{s^T} z^{\overline{x}}_{s^T} = p_1 \widetilde{B}_1 \overline{z}^i_0 + \beta u_i'(\inf \overline{x}^i) b((\overline{x}^i_{\sigma^t} - \omega^i_{\sigma^t})_t)$$

and so the borrowing constraint holds.

Commodity markets clearing follows from A-D market clearing.

By the same argument of item (i), the sequential budget set $B_p(q, \widetilde{\omega}^i, z^i_0)$ is contained in the A-D budget set at (π, ω^i). ■

2.8 Concluding Remarks

The simple deterministic set-up, that turned out not to be so hostile to speculation in fiat money, can be replaced by a more elaborated stochastic set-up with contingent endowments and several assets. Again be the same argument used in Chapter 1, the non-linearity of lim sup operator does not allow to infer, from the transversality condition, that for an asset in posite net supply speculation can be ruled out.

To sum up, we showed that precautionary behavior induces quite different transversality conditions in consumer's sequential optimization problems, which can accommodate bubbles in the prices of assets in posite net supply, both in deterministic and stochastic settings, even under finite values of aggregate wealth

The next stage of our research is try to replicate the same conclusion to a setting in that agents own a collection of beliefs about the future (Knightian uncertainty).

3. FINAL COMMENTS AND REMARKS

Here we discuss some themes related with the previous ones worked and that will be the aim of our future research. Along the next sections we present some conjectures, intuitions of why they are true and prove some preliminary statements.

First, we deepen in the analysis of the relationship between non-summable AD-equilibrium prices and speculative bubbles. We are looking for conditions that allow us to conclude that other types of Mackey-discontinuities for agents' utilities can generate speculative bubbles in the asset prices of an associated sequential economy.

Later, we will mention the analogies of economies with Knightian impatience agents and economies with the overlapping generations of agents. Besides we present some results of this last class of models that are not valid in traditional setup of general equilibrium that will be replicated in a framework with Knightian impatience.

3.1 Speculative Bubbles For General Preferences

In this section we keep the same framework of sections (2.1) and (2.2) except that other forms for utility functions are allowed. We will find sufficient conditions to interpret an AD-equilibrium (as defined in section (1.3)) with prices $\pi \notin \ell^1$ as a P-sequential equilibrium (as defined in section (2.2)) of a deterministic economy that has a speculative bubble on asset prices for a suitable choice of P in spite of a positive net supply. We conjecture that these conditions can be extended to the case in which uncertainty is modeled.

Let $(U^i, \omega^i)_{i=1}^I$ with U^i strongly increasing. If $((\overline{x}^i)_{i=1}^I, \pi)$ is an AD-equilibrium such that $\pi \in ba_+ \equiv (\ell^\infty)^*_+$ then, by the Yosida-Hewitt Theorem, we can decompose this equilibrium price as $\pi \equiv \rho + \zeta$ where $\rho \in ca_+$ and ζ is a non-negative pure charge. Since $\zeta(\{n\}) = 0 \quad \forall n \in \mathbb{N}$ we must have $\rho_n := \rho(\{n\}) > 0$.

Given an asset whose pay-offs are characterized by the deterministic path $R = (R_t)_t \in \ell_+^\infty$ and given $(z_0^i)^i \in \mathbb{R}_+^I$ the asset initial position of agents let us denote $\widetilde{\omega}_t^i = \omega_t^i - R_t z_0^i$ the adjusted endowment.

Now we will discuss the implementation of this AD-equilibrium as a P-sequential equilibrium (with P being an arbitrary borrowing constraint) of the deterministic sequential economy $((U^i, \widetilde{\omega}^i)_{i=1}^I, R)$[1], but we believe that the same results remain being true (and with similar proofs) for stochastic economies.

The next proposition says: whenever an AD-equilibrium with $\pi = \rho + \zeta \notin \ell^1$ can have its allocations $(\overline{x}^i)_{i=1}^I$ implemented as a P-equilibrium of the deterministic economy obttained from the initial by adding the asset, adjusting the endowments and imposing the borrowing constraint P, then for the existence of a speculative bubble it is enough to have the pure charge being positive at $(\overline{x}_t^i - \omega_t^i)$ for one agent i and that the portfolio otpimal position of this agent, at least at a subsequence of dates, does not explode.

Proposition 12: *Under the previous notations and hypothesis, let P be a borrowing constraint that implements $(\overline{x}^i)_{i=1}^I$ as a P-sequential equilibrium for the deterministic economy*

$$((U^i, \widetilde{\omega}^i, z_0^i)_{i=1}^I, R) \quad \textit{with} \quad \sum_i^I z_0^i = z_0 \geq 0$$

$$\textit{and} \qquad \rho_t q_t = \rho_{t+1}(q_{t+1} + R_{t+1}) \quad \forall t \geq 1 \tag{3.1}$$

If $\exists i_0$ such that $\zeta(\overline{x}^{i_0} - \omega^{i_0}) > 0$ and $(|\overline{z}_t^{i_0}|)_t$ does not go to ∞ then the asset prices have a speculative bubble. □

Proof: One can show by induction that $\rho_1 q_1 = \sum_{t\geq 2}^T \rho_t R_t + \rho_T q_T$ and so, making $T \to +\infty$, we get

$$\rho_1 q_1 = \sum_{t\geq 2} \rho_t R_t + \lim_{T\to+\infty} \rho_T q_T \tag{3.2}$$

Since $\rho_1 > 0$, we have that q_1 is the sum of present value $\frac{1}{\rho_1}\sum_{t\geq 2}\rho_t R_t$ (with respect to deflators $(\rho_t)_t$) [2] and the residual value $\frac{1}{\rho_1}\lim_T \rho_T q_T$. We will show that the last term is positive and so there is a bubble component in the asset price.

[1] It is clear that for do this we need $\widetilde{\omega}_t^i \geq 0 \quad \forall i, \forall t$.

[2] As seen in section (2.6)

The equilibrium consumption plan $\overline{x}^{i_0}$ holds $\forall t \geq 1$ that

$$\begin{aligned}\overline{x}_t^{i_0} &= \widetilde{\omega}_t^{i_0} + (q_t + R_t)\overline{z}_{t-1}^{i_0} - q_t\overline{z}_t^{i_0} \\ &= \omega_t^{i_0} - R_t z_0^{i_0} + (q_t + R_t)\overline{z}_{t-1}^{i_0} - q_t\overline{z}_t^{i_0}\end{aligned}$$

Multiplying by ρ_t at every date $t \geq 2$ results

$$\begin{aligned}\rho_t(\overline{x}_t^{i_0} - \omega_t^{i_0}) &= \rho_t(q_t + R_t)\overline{z}_{t-1}^{i_0} - \rho_t R_t z_0^{i_0} - \rho_t q_t \overline{z}_t^{i_0} \\ &= \rho_{t-1}q_{t-1}\overline{z}_{t-1}^{i_0} - \rho_t R_t z_0^{i_0} - \rho_t q_t \overline{z}_t^{i_0}\end{aligned}$$

Summing up all the dates:

$\rho(\overline{x} - \omega) = \sum_{t\geq 1} \rho_t(\overline{x}_t^{i_0} - \omega_t^{i_0})$
$= \rho_1(q_1 + R_1)z_0^{i_0} - \rho_1 R_1 z_0^{i_0} - \rho_1 q_1 \overline{z}_1^{i_0} + \sum_{t\geq 2}\left[\rho_{t-1}q_{t-1}\overline{z}_{t-1}^{i_0} - \rho_t R_t z_0^{i_0} - \rho_t q_t \overline{z}_t^{i_0}\right]$
$= (\rho_1 q_1 - \sum_{t\geq 2}\rho_t R_t)z_0^{i_0} - \lim_\tau \rho_\tau q_\tau \overline{z}_\tau^{i_0}$
where the limit of $\rho_\tau q_\tau \overline{z}_\tau^{i_0}$ exists because $\sum_{t\geq 2}\rho_t R_t$ converges (since $\rho \in \ell^1$ and $(R_t)_t \in \ell^\infty$). Once $((\overline{x}^i)_{i=1}^I, \pi)$ is an AD-Equilibrium for the economy with endowments $(\omega^i)_i$, $\pi(\overline{x}^i - \omega^i)$ and so $\rho(\overline{x}^i - \omega^i) = \zeta(\omega^i - \overline{x}^i)$ that give us

$$\lim_\tau \rho_\tau q_\tau \overline{z}_\tau^{i_0} = (\rho_1 q_1 - \sum_{t\geq 2}\rho_t R_t)z_0^{i_0} + \zeta(\overline{x}^{i_0} - \omega^{i_0})$$

By equation (3.2) we have $\rho_1 q_1 - \sum_{t\geq 2}\rho_t R_t \geq 0$. Since $\zeta(\overline{x}^{i_0} - \omega^{i_0}) > 0$ we get $\lim_\tau \rho_\tau q_\tau \overline{z}_\tau > 0$. Besides $|\overline{z}_t^i|$ does not go to infinity and thus $\lim_\tau \rho_\tau q_\tau > 0$. ■

Thus any choice of a borrowing constraint P that allows implement the allocations of an AD-equilibrium with non-summable prices and that holds the above conditions will ever leads to equilibrium asset prices with speculative bubbles. So, a natural development of this work is, given a particular (and relevant to the economic theory) non-Mackey-continuous utility, try to find some P of those with meaningful economy intuition inside the framework. If this search will has succeeds, we can get others good examples of bubbles with positive net supply.

Although this journey may seem difficult at first, we have already found an encouraging preliminary answer: we know that given an AD-equilibrium with $\pi \in ba_+$ there is P, although artificial for the moment, such that the allocations of this equilibrium can be implemented as a P-sequential equilibrium with a bubble component in the asset prices (the asset pay-offs being arbitrary). This will be shown by the next proposition:

Proposition 13: *An AD-equilibrium* $((\overline{x}^i)_{i=1}^I, \pi)$ *with* $\pi \in ba_+$ *can be implemented as a* P*-sequential equilibrium of a deterministic economy* $((U^i, \widetilde{\omega}^i, z_0^i)_{i=1}^I, R)$ *with an arbitrary financial structure* $R = (R_t)_t \in \ell_+^\infty$ *such that* $\widetilde{\omega}_t^i > 0 \quad \forall i, \forall t$ *and*

$$\rho_t q_t = \sum_{\tau > t} \rho_\tau R_t + k_0$$

where $k_0 > 0$ *and* P *is the borrowing constraint defined by the requirement:*

$$\lim_\tau \rho_\tau q_\tau z_\tau \geq k_0 z_0 + \zeta(x(z) - \omega^i) \tag{3.3}$$ □

PROOF: Since it is clear that $\rho_t q_t = \rho_{t+1} q_t + \rho_{t+1} R_t$ then by the same argument made in proof of proposition (12) we know that given z a P-feasible portfolio must be true that $\rho(x(z) - \omega^i) = (\rho_1 q_1 - \sum_{t \geq 2} \rho_t R_t) z_0 - \lim_\tau \rho_\tau q_\tau z_\tau$. If the condition (3.3) holds, we get

$$\pi(x(z) - \omega^i) = \rho(x(z) - \omega^i) + \zeta(x(z) - \omega^i) \leq 0$$

Let us define the portfolio plan $\overline{z}^i$ of the following way: $\overline{z}_0^i = z_0^i$ and given $\overline{z}_{t-1}^i$ we get $\overline{z}_t^i$ by solving $\overline{x}_t^i = \widetilde{\omega}_t^i + (q_t + R_t)\overline{z}_{t-1}^i - q_t \overline{z}_t^i$. By construction, $\overline{z}$ is P-feasible.

As $\overline{x}^i$ is maximal at $B(\pi, \omega^i)$ and every z P-feasible gives $x(z) \in B(\pi, \omega^i)$ then $\overline{x}^i$ is maximal for the sequential problem. ■

Note that even allocations of an AD-equilibrium with $\pi \in \ell^1$ (what results, for instance, from the utility functions of Chapter 2 in the case $\beta = 0$), could be implemented as P-sequential equilibrium with positive asset supply. In this case we would have $\lim_\tau \rho_\tau q_\tau \overline{z}_\tau \geq k_0 z_0 > 0$.

However, this does not contradict with the traditional TC condition $\lim_\tau \rho_\tau q_\tau \overline{z}_\tau = 0$, that is necessary to optimality for suitable solutions, since the directional derivative that must be evaluated in order to obtain TC can not be calculated at $\overline{z}$ once the direction $\overline{z}$ is not P-admissible.

In fact, as we saw in the proof, $\lim_\tau \rho_\tau q_\tau \overline{z}_\tau = k_0 z_0 + \zeta(\overline{x} - \omega^i)$. If $\pi \in \ell^1$, we get $\lim_\tau \rho_\tau q_\tau \overline{z}_\tau = k_0 z_0$. For the portfolio defined by $\overline{z} + r\overline{z}$ be P-feasible it is necessary that $\lim_\tau \rho_\tau q_\tau (1+r)\overline{z}_\tau \geq k_0 z_0$ but $\lim_\tau \rho_\tau q_\tau (1+r)\overline{z}_\tau = (1+r) \lim_\tau \rho_\tau q_\tau \overline{z}_\tau = (1+r) k_0 z_0$ so if $k_0 z_0 > 0$, the constraint P holds only if $r > 0$. Thus the optimality of portfolio $\overline{z}$ only that $\lim_\tau \rho_\tau q_\tau \overline{z}_\tau > 0$.

Of course, the choice of this requirement P as borrowing constraint for the case $\pi \in \ell^1$ is not natural. But when there are non-Mackey-continuous utilities and so the TC condition changes, choose P or some variation could become suitable.

3.2 Knightian Impatience vs Overlapping Generations

As we saw in Chapter 1, Knightian impatience allows that agents discount the future of a special way: consumption at very distant dates do not need have marginal present value asymptoticly vanished. This could connect the analysis of General Equilibrium (GE) with the Overlapping Generations (OG) literature since in this class of models the welfare of future generations of agents has the same weight of the present. So we conjecture that is possible replicate in GE models some OG propositions that are not true for the case that agents have traditional Mackey-continuous utilities. This theme will be future object of our research. One example of such statement is the existence of speculative bubbles for assets with positive net supply that is well-known for OG models (See Tirole [30]) and we dealt in GE along Chapter 1 and 2. Other possibilities can be found at Geanakoplos and Polemarchakis [14].

We will note another interesting fact due to Knightian impatience. We will change the model of Chapter 1 in order to include the presence of a financial authority with power to make wealth transferences among the agents through fiscal policies. As it sometimes happens in Overlapping Generation models, even spendthrift governments could obtain besides nominal effects, also real effects with their fiscal policies (See Balasko and Shell [3]).

3.2.1 Bonafide Equilibria

Under the framework of Chapter 1, we will make an adaptation: the existence of a central monetary authority - the government.

Only the government can issue fiat money. At every date t he can makes wealth transferences through a fiscal policy, i.e., a vector of fiat money amounts $(m_t^i)_{i=1}^I$. If $m_t^i < 0$, the government is tax the agent i wealth by the requirement of this money amount. If $m_t^i > 0$ the agent receives this amount. Let m_t be the lump sum $\sum_i m_t^i$ of the money issued at date t. So $M_t = \sum_{\tau=1}^t m_\tau$ is the total supply of fiat money at date t.

Now, the sequential budget constraints for agent i are

$$x_t - \omega_t^i \leq q_t(z_{t-1}^i - z_t^i) + q_t m_t^i \quad \forall t > 0 \tag{3.4}$$

with the condition that $z_0^i = 0$ and $z_t^i \geq 0 \; \forall t \geq 1$. This last condition says that inside-money is not created.

Definition: A vector $(\overline{q}, (\overline{x}^i, \overline{z}^i, \overline{m}^i)_i)$ where $\overline{q} = (\overline{q}_t)$ is the money price, $(\overline{x}^i, \overline{z}^i)$ is a pair consumption-portfolio and $(\overline{m}_i)^i = (\overline{m}^i_t)_t$ is a path of fiscal policies is said to be a *Bonafide-equilibrium* for the deterministic economy $(U^i, \omega^i)_i$ when (i)$\forall i$ $\overline{x}^i$ maximizes U^i s.t. (3.4) with prices $\overline{q}$ and monetary transference $\overline{m}^i$, (ii) $\sum_i(\overline{x}^i - \omega^i) = 0$, (iii) $\sum_i \overline{z}^i_t = \overline{M}_t$ $\forall t$, (iv) $(\overline{q}_t)_t > 0$ and (v) $\overline{q}_t(\overline{z}^i_{t-1} - \overline{z}^i_t) + \overline{q}_t\overline{m}^i_t + \omega^i_t \geq 0 \quad \forall t, \forall i$. □

So, in a Bonafide-equilibrium fiat money has a positive price. Condition (v) says that at equilibrium the government can not tax agents by more than agents' wealth. Since in our framework every U^i is strongly increasing, conditions (i), (ii) and (iv) implies the market-clearing in monetary market at each date (condition (iii)).

Now we can state that if $\beta = 0$, the agents only assign value to fiat money that government will issue if the fiscal police plan is balanced, i.e., such that at least asymptotically the public expenditure has been financed by revenue receivables from taxes:

Proposition 14: ***Asymptotic Ricardian Equivalence***
Supposes $\beta = 0$ and let $(\overline{q}, (\overline{x}^i, \overline{z}^i, \overline{m}^i)_i)$ be a Bonafide-equilibrium such that $\overline{z}^i_t > 0$ $\forall i, \forall t$. Then it is true that $\lim_t \overline{M}_t = 0$. □

Proof: Since each $\overline{x}^i$ is an interior optimal, it can be show in a analogous way to section (1.4.1) that $\overline{q}_t(u^i)'(\overline{x}^i_t) = \delta\overline{q}_{s+1}(u^i)(\overline{x}^i_{t+1})\,\forall t$ and $\lim_t \delta^t(u^i)(\overline{x}^i_t)\overline{q}_t\overline{z}^i_t = 0$ must be true for every agent. So, by defining $\lambda_t = \delta^t(u^i)(\overline{x}^i_t)$ as the normalized deflator of date t (that is independent of i), we get $\lim_t \lambda_t\overline{q}_t\overline{z}^i_t = 0$ for each agent. Thus

$$0 = \sum_i \lim_t \lambda_t\overline{q}_t\overline{z}^i_t = \lim_t \lambda_t\overline{q}_t\overline{M}_t \tag{3.5}$$

On other side $0 < \lambda_1\overline{q}_1 = \lambda_t\overline{q}_t$ $\forall t$ soon

$$\lambda_1\overline{q}_1 = \lim_t \lambda_t\overline{q}_t > 0 \tag{3.6}$$

finally, equations (3.5) and (3.6) imply $\overline{M}_t \to 0$. ■

However this proposition is not longer true if $\beta > 0$. An example of non-balanced fiscal policy compatible with Bonafide-equilibrium can be constructed from the example 1 with two agents of Chapter 1. It is enough define m^i_1 as the initial fiat money endowment of agent i and $m^i_t = 0$ for $t \geq 2$. As it was commented in section (1.1) we can choose this asset initial position such that $z^i_t > 0$. Then exists a money issuing at date $t = 1$ that is not taken back any date.

Remark: In fact, a Bonafide-equilibrium can be seen as deterministic P-Sequential equilibrium with a speculative bubble for fiat money but with a total supply that can change along the dates. When $\beta = 0$, if the money total supply is positive it must tend to zero when t goes to infinite. When $\beta > 0$, as we already seen, this is not the case and thus is natural that Ricardian equivalence is no longer true.

APPENDIX

A. EPSILON-CONTAMINATION CAPACITY

Let S be an arbitrary set and $\mathcal{S}$ a σ-field of S subsets. First, we will define the *core* of a capacity ν at $(S, \mathcal{S})$:

$$\text{core}(\nu) = \{\eta \in ba(S, \mathcal{S}) : \eta \geq \nu, \; \eta(S) = \nu(S)\}$$

Let us define for $r \geq 0$ the set $M^r = \{\eta \in ca_+(S, \mathcal{S}) : \eta(S) = r\}$. Now we can state:

Proposition 15: *Let $\mu \in ca_+(S, \mathcal{S})$, $\beta \in [0, +\infty)$ and ν the capacity defined by $\nu(A) = \mu(A)$ for $A \in \mathcal{S}$, $A \neq S$ and $\nu(S) = \mu(S) + \beta$. It is true that* [1]

$$\min_{\eta \in \text{core}(\nu)} \int_S u \circ x \, d\eta = \inf_{\substack{\eta \in M^{\mu(S)+\beta} \\ \eta \geq \nu}} \int_S u \circ x \, d\eta = \int_S u \circ x \, d\mu + \beta \inf_S u \circ x$$

$\square$

Proof: Given $x \in B(S, \mathcal{S})$ such that $u \circ x \in B(S, \mathcal{S})$, let $I(\nu) = \min_{\eta \in \text{core}(\nu)} \int_S u \circ x \, d\eta$ and $F(\nu) = \inf_{\substack{\eta \in M^{\mu(S)+\beta} \\ \eta \geq \nu}} \int_S u \circ x \, d\eta$. It is clear that $F(\nu) \geq I(\nu)$.

Let $\eta \in ba(S, \mathcal{S})$ such that $\eta \geq \nu$ and $\eta(S) = \nu(S)$. Thus $(\eta - \mu) \in ba_+$. So we get

$$\begin{aligned} \int u \circ x d\eta \;\; &= \int u \circ x d(\eta - \mu) + \int u \circ x d\mu \\ &\geq (\eta(S) - \mu(S)) \inf u \circ x + \int u \circ x d\mu \\ &= \beta \inf u \circ x + \int u \circ x d\mu \end{aligned}$$

hence $I(\nu) \geq \int_S u \circ x \, d\mu + \beta \inf_S u \circ x$. On other hand, let (x_n) a sequence in $x(S)$ such that $x_n \longrightarrow \inf x$.

Let us define $\varsigma_n = \mu + \beta \delta_{x_n}$, where δ_{x_n} is the Dirac delta probability measure with mass at x_n. So $\varsigma_n \in M^{\mu(A)+\beta}$, $\varsigma_n \geq \nu$ and

$$I_n := \int u \circ x d\varsigma_n = \int_S u \circ x \, d\mu + \beta u(x_n) \geq F(\nu)$$

since $I_n \longrightarrow \int u \circ x d\mu + \beta \inf u \circ x$ we are done because we get

$$\int u \circ x d\mu + \beta \inf u \circ x \geq F(\nu) \geq I(\nu) \geq \int u \circ x d\mu + \beta \inf u \circ x$$

$\blacksquare$

[1] If $\phi \in ca_+^1$ and $\epsilon \in (0, 1)$, making $\mu = \epsilon\phi$ and $\beta = 1 - \epsilon$, we get the ϵ-contamination formula for the Choquet integral.

B. EULER EQUATION AND TRANSVERSALITY CONDITION

B.1 TC is necessary for $\beta = 0$

Proof of equation (2.9):

$$\lim_{r\to 0} \tfrac{1}{r} \left\{ E\left[u(x^*_{s^1} - rq_{s^1}z_{s^1}) - u(x^*_{s^1})\right] + \textstyle\sum_{t\geq 2} \delta^{t-1} E\left[u(x^*_{s^t} + r(q_{s^t} + R_{s^t})z_{s^t_-} - q_{s^t}z_{s^t}) - u(x^*_{s^t})\right]\right\}$$
$$= E\left[\lim_{r\to 0} \tfrac{1}{r}\left(u(x^*_{s^1} - rq_{s^1}z_{s^1}) - u(x^*_{s^1})\right)\right] +$$
$$+ \textstyle\sum_{t\geq 2} \delta^{t-1} E\left[\lim_{r\to 0} \tfrac{1}{r}\left(u(x^*_{s^t} + r(q_{s^t} + R_{s^t})z_{s^t_-} - q_{s^t}z_{s^t}) - u(x^*_{s^t})\right)\right]$$
$$= E\left[-u'(x^*_{s^1})q_{s^1}z_{s^1})\right] + \textstyle\sum_{t\geq 2} \delta^{t-1} E\left[u'(x^*_{s^t})\left((q_{s^t} + R_{s^t})z_{s^t_-} - q_{s^t}z_{s^t}\right)\right]$$
$$= \underbrace{E\left[-u'(x^*_{s^1})q_{s^1}z_{s^1})\right] + \sum_{t\geq 2}\left\{\delta^{t-1}E\left[u'(x^*_{s^t})(q_{s^t} + R_{s^t})z_{s^t_-}\right] - \delta^{t-1}E\left[u'(x^*_{s^t})q_{s^t}z_{s^t}\right]\right\}}_{:=a}$$

but

$$\delta E\left[u'(x^*_{s^t})(q_{s^t} + R_{s^t})z_{s^t_-}\right]$$
$$= \delta \textstyle\sum_{s^t\in S^t} I\!P_{s^t} u'(x^*_{s^t})(q_{s^t} + R_{s^t})z_{s^t_-}$$
$$= \delta \textstyle\sum_{s^t_-\in S^{t-1}} \sum_{s=1}^{N} I\!P_{s^t_-} p_s u'(x^*_{(s^t_-,s)})(q_{(s^t_-,s)} + R_{(s^t_-,s)})z_{s^t_-}$$
$$= \delta \textstyle\sum_{s^{t-1}\in S^{t-1}} I\!P_{s^{t-1}} z_{s^{t-1}} E^p\left[u'(x^*_{s^{t-1}_+})(q_{s^{t-1}_+} + R_{s^{t-1}_+})\right]$$
$$= \textstyle\sum_{s^{t-1}\in S^{t-1}} I\!P_{s^{t-1}} u'(x^*_{s^{t-1}})q_{s^{t-1}}z_{s^{t-1}}$$
$$= E\left[u'(x^*_{s^{t-1}})q_{s^{t-1}}z_{s^{t-1}}\right]$$

So

$$a = E\left[-u'(x^*_{s^1})q_{s^1}z_{s^1})\right] + \textstyle\sum_{t\geq 2}\left\{\delta^{t-2}E\left[u'(x^*_{s^{t-1}})q_{s^{t-1}}z_{s^{t-1}}\right] - \delta^{t-1}E\left[u'(x^*_{s^t})q_{s^t}z_{s^t}\right]\right\}$$
$$= -\lim_{t\to\infty} \delta^{t-1}E\left[u'(x^*_{s^t})q_{s^t}z_{s^t}\right]$$

B.2 TC is sufficient for β=0

$\sum_{t=1}^{\infty} \delta^{t-1} E\left[u(x_{s^t}(\widetilde{z})) - u(x_{s^t}(z))\right] \geq \sum_{t=1}^{\infty} \delta^{t-1} E\left[u'(x_{s^t}(\widetilde{z}))(x_{s^t}(\widetilde{z}) - x_{s^t}(z))\right]$

$= \sum_{t=1}^{\infty} \delta^{t-1} E\left[u'(x_{s^t}(\widetilde{z}))[(q_{s^t} + R_{s^t})(\widetilde{z}_{s^t_-} - z_{s^t_-}) - q_{s^t}(\widetilde{z}_{s^t} - z_{s^t})]\right]$

$= \sum_{t=1}^{\infty} \left\{\delta^{t-1} E\left[u'(x_{s^t}(\widetilde{z}))(q_{s^t} + R_{s^t})(\widetilde{z}_{s^t_-} - z_{s^t_-})\right] - \delta^{t-1} E\left[u'(x_{s^t}(\widetilde{z}))q_{s^t}(\widetilde{z}_{s^t} - z_{s^t})\right]\right\}$

$= u'(x_1(\widetilde{z}))(q_1 + R_1)(\widetilde{z}_0 - z_0) - u'(x_1(\widetilde{z}))q_1(\widetilde{z}_1 - z_1) +$
$+ \sum_{t=2}^{\infty} \left\{\delta^{t-2} E\left[u'(x_{s^{t-1}}(\widetilde{z}))q_{s^{t-1}}(\widetilde{z}_{s^t_-} - z_{s^t_-})\right] - \delta^{t-1} E\left[u'(x_{s^t}(\widetilde{z}))q_{s^t}(\widetilde{z}_{s^t} - z_{s^t})\right]\right\}$

$= -\lim_t \delta^{t-1} E\left[u'(x_{s^t}(\widetilde{z}))q_{s^t}(\widetilde{z}_{s^t} - z_{s^t})\right]$

where we are using the equality $\delta E\left[u'(x_{s^t}(\widetilde{z}))(q_{s^t} + R_{s^t})(\widetilde{z}_{s^t_-} - z_{s^t_-})\right] = E\left[u'(x_{s^{t-1}}(\widetilde{z}))q_{s^{t-1}}(\widetilde{z}_{s^t_-} - z_{s^t_-})\right]$ that is due to (EE) (see the section (B.1)).

B.3 Proof of Affirmation In Proposition (8)

Affirmation:

$(i) \lim_{r\to 0^-} \frac{1}{r}[\inf_{s^t\in\overline{S}} u(x^*_{s^t}+r(q_{s^t}+R_{s^t})z^*_{s^t}-q_{s^t}z^*_{s^t_-}))-\inf_{s^t\in\overline{S}} u(x^*_{s^t})] = \lim\sup_{\tau} u'(x^*_{\sigma^\tau})\left((q_{\sigma^\tau} + R_{\sigma^\tau})z^*_{\sigma^{\tau-1}} - q_{\sigma^t}z^*_{\sigma^\tau}\right)$

$(ii) \lim_{r\to 0^+} \frac{1}{r}[\inf_{s^t\in\overline{S}} u(x^*_{s^t}+r(q_{s^t}+R_{s^t})z^*_{s^t}-q_{s^t}z^*_{s^t_-}))-\inf_{s^t\in\overline{S}} u(x^*_{s^t})] = \lim\inf_{t} u'(x^*_{\sigma^\tau})\left((q_{\sigma^\tau} + R_{\sigma^\tau})z^*_{\sigma^{\tau-1}} - q_{\sigma^\tau}z^*_{\sigma^\tau}\right)$

Proof: Let us denote:

$$\underline{x}^* = \inf_{s^t} x^*_{x_{s^t}}$$
$$\Delta_{s^t} = (q_{s^t} + R_{s^t})z^*_{s^t_-} - q_{s^t}z^*_{s^t}$$

Let ε_0 be such that $x(z^* + rz^*) \in B_P(q,\omega) \quad \forall r \in (-\varepsilon_0, \varepsilon_0)$

We will prove the item (i) of affirmation.

There exists [1]

$$\lim_{r\uparrow 0} \frac{1}{r}[\inf_{s^t} u(x^*_{s^t} + r\Delta_{s^t}) - u(\underline{x}^*)]$$

since $\inf(.) : \ell^\infty(\overline{S}) \to \mathbb{R}$ is a concave function.

[1] This is a direct consequence of lemma 6.14, page 237, Simonsen [28].

Fixed $r \in (-\varepsilon_0, 0)$ and given $\epsilon > 0$ is valid for all τ large enough that

$$\begin{array}{l}\frac{1}{r}[\inf_s u(x^*_{s^t} + r\Delta_{s^t}) - u(\underline{x}^*)] + \epsilon = \\ \left(-\frac{1}{r}\right)[u(\underline{x}^*) - \epsilon r - \inf_{s^t} u(x^*_{s^t} + r\Delta_{s^t})] \geq \\ \left(-\frac{1}{r}\right)[u(x^*_{\sigma^\tau}) - u(x^*_{\sigma^\tau} + r\Delta_{\sigma^\tau})] \geq \\ u'(x^*_{\sigma^\tau})\Delta_{\sigma^\tau}\end{array}$$

Making $\tau \to \infty$ we get

$$\frac{1}{r}[\inf_{s^t} u(x^*_{s^t} + r\Delta_{s^t}) - u(\underline{x}^*)] + \epsilon \geq \limsup_{\tau} u'(x^*_{\sigma^\tau})\Delta_{\sigma^\tau}$$

So, we get

$$\lim_{r\uparrow 0} \frac{1}{r}[\inf_{s^t} u(x^*_{s^t} + r\Delta_{s^t}) - u(\underline{x}^*)] + \epsilon \geq \limsup_{t} u'(x^*_{\sigma^t})\Delta_{\sigma^\tau} \tag{B.1}$$

that were shown for an arbitrary $\epsilon > 0$.

Now we will prove the reverse inequality.

Claim 1: For $r \in (-\varepsilon_0, \varepsilon_0)$, let $\underline{x}^r := \inf_{s^t}(x^*_{s^t} + r\Delta_{s^t})$. It is true that $\lim_{r\to 0} \underline{x}^r = \underline{x}^*$.

Proof: We have

$$\underline{x}^r \geq \underline{x}^* + r \inf_{s^t} \Delta_{s^t}$$

Since $(\Delta_{s^t})_{s^t} \in \ell^\infty(\overline{S})$ then $\underline{\lim}_{r\to 0}\underline{x}^r \geq \underline{x}^*$.
On other hand, $\underline{x}^* = \inf_{s^t}(x^*_{s^t} + r\Delta_{s^t} - r\Delta_{s^t}) \geq \underline{x}^r + \inf_{s^t}(-r\Delta_{s^t})$ and again by boundness of $(\Delta_{s^t})_{s^t}$ we get $\underline{x}^* \geq \overline{\lim}_{r\to 0}\underline{x}^r$. ∎

So, $\lim_{r\uparrow 0} u(\underline{x}^r) = u(\underline{x}^*)$. Once that $\underline{x}^* > 0$, for r near enough to zero is valid $\underline{x}^r > 0$ and thus we can write

$$(-\frac{1}{r})[u(\underline{x}^*) - u(\underline{x}^r)] \leq (-\frac{1}{r})u'(\underline{x}^r)[\underline{x}^* - \underline{x}^r]$$

Let $(r_n)_{n\in\mathbb{N}}$ be a real sequence such that $r_n \uparrow 0$, $x(z^* + r_n z^*) \in B_P(q,\omega)$ $\forall n$ and $\underline{x}^{r_n} > 0$ $\forall n$. It is true that

$$\begin{array}{l}\lim_n \frac{1}{r_n}[\inf_{s^t} u(x^*_{s^t} + r_n\Delta_{s^t}) - \inf_{s^t} u(x^*_{s^t})] = \\ \lim_{r\uparrow 0} \frac{1}{r}[\inf_{s^t} u(x^*_{s^t} + r\Delta_{s^t}) - \inf_{s^t} u(x^*_{s^t})]\end{array} \tag{B.2}$$

$$\lim_n \underline{x}^{r_n} = \underline{x}^* \quad \text{and so} \quad \lim_n u(\underline{x}^{r_n}) = u(\underline{x}^*), \quad \lim_n u'(\underline{x}^{r_n}) = u'(\underline{x}^*), \tag{B.3}$$

$$\text{and} \quad \left(-\frac{1}{r_n}\right)[u(\underline{x}^*) - u(\underline{x}^{r_n})] \leq \left(-\frac{1}{r_n}\right) u'(\underline{x}^{r_n})\,[\underline{x}^* - \underline{x}^{r_n}]\,. \tag{B.4}$$

Given $n \in \mathbb{N}$ and $\varepsilon > 0$ there is $\sigma(n,\epsilon) \in \overline{S}$ such that

$$\underline{x}^{r_n} - r_n\epsilon > x^*_{\sigma(n,\epsilon)} + r_n\Delta_{\sigma(n,\epsilon)}$$

and thus we get

$$(-\frac{1}{r_n})(\underline{x}^* - \underline{x}^{r_n}) - \varepsilon \leq (-\frac{1}{r_n})[x^*_{\sigma(n,\epsilon)} - (x^*_{\sigma(n,\epsilon)} + r_n\Delta_{\sigma(n,\epsilon)})] \tag{B.5}$$

Claim 2: $\lim_n x_{\sigma(n,\varepsilon)} = \underline{x}^*$

Proof: Given $\epsilon > 0$

$$\begin{aligned} |x_{\sigma(n,\varepsilon)} - \underline{x}^*| \leq & \ |r_n\Delta_{\sigma(n,\varepsilon)}| + |x_{\sigma(n,\varepsilon)} + r_n\Delta_{\sigma(n,\varepsilon)} - \underline{x}^{r_n}| + |\underline{x}^{r_n} - \underline{x}^*| \\ \leq & \ C|r_n| + \varepsilon|r_n| + |\underline{x}^{r_n} - \underline{x}^*| \end{aligned}$$

where $C \geq \sup_n |\Delta_{\sigma(n,\varepsilon)}|$.
Therefore $\exists \hat{n} \in \mathbb{N}$ such that $n \geq \hat{n} \Rightarrow |x_{\sigma(n,\varepsilon)} - \underline{x}^*| < \epsilon$. ■

Since $\underline{x}^*$ is not attained, the last claim implies that for each $t \in \mathbb{N}$ there are at most finite indexes n such that $\sigma(n,\varepsilon) \in S^t$ and so

$$\liminf_t (\min_{s^t \in S^t} \Delta_{s^t}) \leq \liminf_n \Delta_{\sigma(n,\varepsilon)} \leq \limsup_n \Delta_{\sigma(n,\varepsilon)} \leq \limsup_t (\max_{s^t \in S^t} \Delta_{s^t})$$

Then, from (B.3), (B.4) and (B.5) we get

$$\begin{aligned} \lim_n(-\tfrac{1}{r_n})[u(\underline{x}^*) - u(\underline{x}^{r_n})] \leq & \ \lim_n(-\tfrac{1}{r_n})u'(\underline{x}^{r_n})[\underline{x}^* - \underline{x}^{r_n}] \\ = & \ \lim_n u'(\underline{x}^{r_n}) \ \limsup_n(-\tfrac{1}{r_n})[\underline{x}^* - \underline{x}^{r_n}] \\ = & \ u'(\underline{x}^*) \ \limsup_n(-\tfrac{1}{r_n})[\underline{x}^* - \underline{x}^{r_n}] \\ \leq & \ u'(\underline{x}^*) \limsup_n(\Delta_{\sigma(n,\epsilon)} + \varepsilon) \\ \leq & \ u'(\underline{x}^*) \limsup_\tau \Delta_{\sigma^\tau} + u'(\underline{x}^*)\varepsilon \end{aligned} \tag{B.6}$$

with $\varepsilon > 0$ arbitrary.

The proof of item (ii) is analogous. ■

B.4 Proof of Affirmation In Lemma (2.9)

Let us denote $\widetilde{x}_{s^t} = x_{s^t}(\widetilde{z})$
By hypothesis $\lim_t \widetilde{x}_{\sigma^t} = \inf_{s^t} \widetilde{x}_{s^t}$
Let z be a P-feasible portfolio. Then

$$\begin{aligned} D_2 = &\quad \beta[\inf_{s^t} u(\widetilde{x}_{s^t}) - \inf_s u(x_{s^t}(z))] \\ \geq &\quad \beta[\inf_s u(\widetilde{x}_{s^t}) - u(x_{s^t}(z))] \qquad \forall s^t \end{aligned}$$

Given $\epsilon > 0$ we get for all t large enough

$$\begin{aligned} D_2 + \epsilon \geq &\quad \beta[u(\widetilde{x}_{\sigma^t}) - u(x_{\sigma^t}(z))] \\ \geq &\quad \beta u'(\widetilde{x}_{\sigma^t}) \{(q_{\sigma^t} + R_{\sigma^t})\widetilde{z}_{\sigma^{s-1}} - q_{\sigma^t}\widetilde{z}_{\sigma^t} - [(q_{\sigma^t} + R_{\sigma^t})z_{\sigma^{t-1}} - q_{\sigma^t}z_{\sigma^t}]\} \end{aligned}$$

and thus

$$\begin{aligned} D_2 + \epsilon \geq &\quad \beta\underline{\lim}_t u'(\widetilde{x}_{\sigma^t}) \{(q_{\sigma^t} + R_{\sigma^t})\widetilde{z}_{\sigma^{s-1}} - q_{\sigma^t}\widetilde{z}_{\sigma^t} - [(q_{\sigma^t} + R_{\sigma^t})z_{\sigma^{t-1}} - q_{\sigma^t}z_{\sigma^t}]\} \\ \geq &\quad \beta\underline{\lim}_t u'(\widetilde{x}_{\sigma^t}) \{(q_{\sigma^t} + R_{\sigma^t})\widetilde{z}_{\sigma^{s-1}} - q_{\sigma^t}\widetilde{z}_{\sigma^t}\} + \beta\underline{\lim}_t u'(\widetilde{x}_{\sigma^t}) \{q_{\sigma^t}z_{\sigma^t} - (q_{\sigma^t} + R_{\sigma^t})z_{\sigma^{t-1}}\} \end{aligned}$$

What were shown for an arbitrary $\epsilon > 0$.

C. SEQUENTIAL IMPLEMENTATION

C.1 Borrowing Constraints

Now, we show that with last explicit short-sale constraint x^* is optimal. Can be proved that $Ux^* - Ux \geq \beta[\lim u'(x^*_s)x^*_s - \underline{\lim} u'(x^*_s)x_s] + \lim_s \lambda_s q_s(z_s - z^*_s)$

Let $c = \beta \lim u'(x^*_s)x^*_s - \lim_s \lambda_s q_s z^*_s$

For $Ux^* - Ux \geq 0$, it is sufficient that

$c \geq \beta \underline{\lim} u'(x^*_s)x_s - \lim_s \lambda_s q_s z_s$.

Taking a constant M and a subsequence $x_{s_k} \to \underline{\lim} x_s$ it is sufficient too that

$c + \frac{M}{s_k} \geq \beta u'(x^*_{s_k})x_{s_k} - \lambda_{s_k} q_{s_k} z_{s_k} =$

$\beta u'(x^*_{s_k})(x_{s_k} - \omega_{s_k}) + \beta u'(x^*_{s_k})\omega_{s_k} - \lambda_{s_k} q_{s_k} z_{s_k} =$

$\beta u'(x^*_{s_k}) q_{s_k}(z_{s_k - 1} - z_{s_k}) + \beta u'(x^*_{s_k})\omega_{s_k} - \lambda_{s_k} q_{s_k} z_{s_k} =$

$\beta \frac{\lambda_{s_k}}{\delta^{s_k}} q_{s_k}(z_{s_k - 1} - z_{s_k}) + \beta \frac{\lambda_{s_k}}{\delta^{s_k}} \omega_{s_k} - \lambda_{s_k} q_{s_k} z_{s_k}$

$c + \frac{M}{s_k} \;\; \geq \;\; \beta \frac{\lambda_{s_k}}{\delta^{s_k}} q_{s_k}(z_{s_k - 1} - z_{s_k}) + \beta \frac{\lambda_{s_k}}{\delta^{s_k}} \omega_{s_k} - \lambda_{s_k} q_{s_k} z_{s_k}$

if and only if

$\lambda_{s_k} q_{s_k} \{\frac{\beta}{\delta^{s_k}} + 1\} z_{s_k} \geq \beta \frac{\lambda_{s_k}}{\delta^{s_k}} q_{s_k} z_{s_k - 1} + \beta \frac{\lambda_{s_k}}{\delta^{s_k}} \omega_{s_k} - c - \frac{M}{s_k}$

$\lambda_{s_k} q_{s_k} z_{s_k} \geq \frac{\delta^{s_k}}{\delta^{s_k} + \beta} \{\beta \frac{\lambda_{s_k}}{\delta^{s_k}} q_{s_k} z_{s_k - 1} + \beta \frac{\lambda_{s_k}}{\delta^{s_k}} \omega_{s_k} - c - \frac{M}{s_k}\}$

$\lambda_{s_k} q_{s_k} z_{s_k} \geq \frac{\beta}{\delta^{s_k}+\beta} \lambda_{s_k-1} q_{s_k-1} z_{s_k-1} + \frac{\beta}{\delta^{s_k}+\beta} \lambda_{s_k} \omega_{s_k} - \frac{\delta^{s_k}}{\delta^{s_k}+\beta} \{c + \frac{M}{s_k}\}$

So the explicit short-sale constraint

$\lambda_{s_k} q_{s_k} z_{s_k} \geq \upsilon_{s_k} \lambda_{s_k-1} q_{s_k-1} z_{s_k-1} + \upsilon_{s_k} \lambda_{s_k} \omega_{s_k} - \chi_{s_k}$

is sufficient condition to optimality of x^*.

BIBLIOGRAPHY

[1] Araujo, A., M. Pascoa and J.P. Torres-Martinez (2002), "Collateral Avoids Ponzi Schemes in Incomplete Markets". *Econometrica* 70: 1613-38.

[2] Araujo, A., M. Pascoa and J.P. Torres-Martinez (2006), "Bubbles, Collateral and Monetary Equilibrium". *PUC Working-Paper*, Rio de Janeiro.

[3] Balasko, Y. e K. Shell (1981), "The Overlapping-Generation Model II: The Case of Pure Exchange with Money". *Journal of Economic Theory*, 24, 112-42.

[4] Barrios, R. (1997), "Equilíbrio, Impaciência e Neutralidade à Incerteza em uma Economia Tipo Bewley". *Informes de Matemática - IMPA*, Série F092, Rio de Janeiro.

[5] Berger, J. and M. Berliner (1986), "Robust Bayes and Empirical Analysis with Epsilon Contaminated Priors". *The Annals of Statistics*, 14: 461-86.

[6] Bewley, T. (1972), "Existence of Equilibrium in Economies with Infinitely Many Commodities". *Journal of Economic Theory* 4: 514-40.

[7] Bewley, T. (1980), "The Optimal Quantity of Money" in *Models of Monetary Economics*, ed. by J. Kareken and N.Wallace, Minneapolis.

[8] Bhaskara Rao, K. and M. Bhaskara Rao (1983), *Theory of charges*, Academic Press, New York.

[9] Boratynska, A. (1996), "On Bayesian Robustness with the Epsilon-Contamination Class of Priors". *Statistics and PProbability Letters*, 26: 323-28.

[10] Brézis, H. (1983), *Análisis Funcional: Teoría y Aplicaciones*, Alianza Editorial, Madrid.

[11] Dow,J. and S.Werlang (1992), "Uncertainty Aversion, Risk Aversion and the Optimal Choice of Portfolio". *Econometrica* 60: 197-204.

[12] Duffie, D. (1991), “The Theory of Value in Security Markets” in *The Handbook of Mathematical Economics*, volume IV, Chapter 31, edi, by Hildenbrand, W. and H. Sonnenschein, North-Holland, 1615-1682.

[13] Dunford, N. and J. Schwartz (1958), *Linear Operators, Part I: General Theory*, Interscience, New York.

[14] Geanakoplos, J. e H. Polemarchakis (1991): “Overlapping Generations" in: W. Hildenbrand e H. Sonnenschein (ed.), *Handbook of Mathematical Economics*, vol 4, cap 35, 1899-1960.

[15] Gilboa, I. (1989), “Expectation and Variation in Multi-period Decisions”. *Econometrica*, 57: 1153-69.

[16] Gilboa, I and D. Schmeidler (1989): “Maxmin Expected Utility with a Non-Unique Prior". *Journal of Mathematical Economics*, 110: 605-39.

[17] Gilles, C. and S. LeRoy (1992), “Bubbles and Charges”. *International Economic Review*, 33: 323-39.

[18] Hodges, J. and E.Lehmann (1952), “The Use of Previous Experience in Reaching Statitical Decisons”. *Annals of Mathematical Statistics*, 23: 396-407.

[19] Huang, K. and J. Werner (2004), “Implementing Arrow-Debreu Equilibria by Trading Infinitely-Lived Securities”. *Economic Theory*, 24: 603-23.

[20] Kocherlakota, N. (2007), “Injecting Rational Bubbles". *Journal of Economic Theory*, forthcoming.

[21] Magill, M. and M. Quinzii (1994), “Infinite Horizon Incomplete Markets", *Econometrica*, 62: 853-80.

[22] Magill, M. and M. Quinzii (1996),“Incomplete Markets Over an Infinite Horizon: Long Lived Securities and Speculative Bubbles”. *Journal of Mathematical Economics*, 26: 133-70.

[23] Knight, F. (1921), *Risks, Uncertainty and Profit*, Houghton-Mifflin, Boston.

[24] Santos, M. (2006), “ The Value of Money in a Dynamic Equilibrium Model”. *Economic Theory*, 27: 39-58.

[25] Santos, M. and M. Woodford (1997), "Rational Asset Pricing Bubbles". *Econometrica*, 65: 19-57.

[26] Schmeidler, D. (1986), "Integral Representation Without Additivity". *Proceeding of American Mathematical Society*, 97: 255-61.

[27] Schmeidler, D. (1989), "Subjective Probability and Expected Utility Without Additivity". *Econometrica*, 57: 571-587.

[28] Simonsen, M. H. (1983), *Dinâmica Macroeconômica*, McGraw-Hill, São Paulo.

[29] Stokey, N. and R. Lucas (1996), *Recursive Methods in Economic Dynamics*, Harvard University Press, Cambridge.

[30] Tirole, J. (1985), "Asset Bubbles and Overlapping Generations". *Econometrica*, 53: 1499-528.