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Myrian Beatriz Silva Petrassi

Three Essays in General Equilibrium

Tese de Doutorado

Tese apresentada ao Programa de Pós–graduação em Economia do Departamento de Economia da PUC–Rio como requisito parcial para obtenção do título de Doutor em Economia

Orientador: Prof. Juan Pablo Torres-Martínez

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Resumo

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O objetivo desta tese é entender o papel de fricções de crédito em modelos de Equilíbrio Geral com mercados financeiros incompletos. Primeiro, o trabalho estuda a importância das restrições ao endividamento dos agentes para a existência de equilíbrio monetário em uma economia com informação simétrica onde a moeda (aqui caracterizada como um ativo que não paga dividendos, pode ser vendido a descoberto e está em oferta líquida positiva) tem apenas o papel de reserva de valor e é o único ativo da economia. Além disso, mostra que, apesar da moeda possibilitar as transferências de riqueza intertemporais e entre os estados da natureza, o equilíbrio monetário ainda é Pareto ineficiente. A tese também caracteriza a hipótese de impaciência uniforme, um importante requerimento para a existência de equilibrio, em termos das propriedades assintóticas dos fatores de desconto intertemporal. Como conseqüência, mostra que desconto hiperbólico é incompatível com a impaciência uniforme de funções de utilidade separáveis. Finalmente, mostra que, se os ativos financeiros são colateralizados para a proteção dos emprestadores em caso de *default*, sempre existe equilíbrio em um modelo de dois períodos com mercados incompletos e informação assimétrica onde os ativos são nominais, independente da estrutura informacional.

Palavras–chave

Restrições ativas à dívida. Valor fundamental da moeda. Bolhas em preços de ativos. Informação assimétrica. Default. Colateral.

Abstract

Petrassi, Myrian Beatriz Silva; Torres-Martínez, Juan Pablo. **Three Essays in General Equilibrium**. Rio de Janeiro, 2008. 40p. PhD Thesis — Department of Economia, Pontifícia Universidade Católica do Rio de Janeiro.

This thesis aims to understand the role of credit frictions in general equilibrium models with incomplete financial markets. First, this work studies the importance of debt constraints for the existence of monetary equilibrium in an economy with symmetric information and where money has only the role of store of value. It also characterizes the uniform impatience assumption, an important requirement to equilibrium existence, in terms of asymptotic properties on intertemporal discount factors. Finally, it shows that, if assets are collateralized in order to protect lenders in the case of default, equilibrium always exists in a two-period incomplete markets model of asymmetric information with nominal assets, independently of the financial-informational structure.

Keywords

Binding debt constraints. Fundamental value of money. Asset pricing bubbles. Asymmetric information. Default. Collateral.

Sumário

1
Introduction

Restrictions that actually generate frictions in an economy have not been included in most General Equilibrium models with incomplete markets. However, in infinite horizon models, welfare gains can be generated if frictions are allowed. Binding constraints can be important in several aspects. Among others, it may allow for a positive fundamental value of money that creates room for wealth transfers across time and states of nature. Also, collateral requirements in an asymmetric information environment restrict the possibility of unbounded arbitrage opportunities.

Seminal papers by Santos & Woodford (1997) and Magill & Quinzii (1996) do include limits to economic agents indebtness; however, such constraints never bind in equilibrium and never generate frictions in the economy. Some recent work by Santos (2006) and Gimenez (2006) include liquidity restrictions. Those take the form of a cash-in-advance constraint in the first paper, and a no short-sales constraint in the second. They show that, under these restrictions and some hypothesis over preferences, money has a positive value that is a true fundamental value if they are binding infinitely often.

This thesis aims to understand the importance of some type of frictions in general equilibrium models, and is divided in three chapters.

In the first chapter, we consider the widely studied case of an asset paying no dividends (that we will, with all the necessary disclaimers, call money) and in which the size of loans are bounded by an explicit debt constraint. We show that, under uniform impatience of preferences, assets in positive net supply are free of price bubbles for deflators that yield finite present values of wealth. This, however, does not imply that equilibrium prices must coincide with present values of dividends. In fact, if borrowing constraints are binding, asset prices must take into account the shadow prices associated with such constraints.

In such a context, we prove that a positive asset price occurs at some node if and only if debt constraints are either binding at this node or at some future state of nature. Thus, binding debt constraints always induce frictions which create room for improving welfare by allowing money to have a role in transferring wealth across the event tree. Also, we develop a duality theory of individual optimization under infinite-horizon, necessary for the results here obtained.

In the second chapter, we discuss how restrictive is the uniform impatience hypothesis, a joint requirement on agent's preferences and endowments imposed in order to prove equilibrium existence, to the results. As already mentioned, this hypothesis, under Kuhn-Tucker multipliers, precludes the possibility of occurrence of speculative bubbles on assets in positive net supply. But it also can be too restrictive and precludes the possibility of time varying intertemporal discounting (e.g., hyperbolic discount functions). In this chapter, we characterize uniform impatience assumption in terms of asymptotic properties on intertemporal discount factors. As a consequence, we show that hyperbolic discounting is incompatible with uniform impatience of separable utility functions.

Chapter 2 suggests that models with default and collateral requirements – where uniform impatience assumptions are not necessary to prove the existence of equilibrium (as Araujo, Páscoa & Torres-Martínez (2002, 2007)) – may be more appropriate if hyperbolic discounting is expected.

Under this environment, we introduce, in the third chapter [1], assets backed by physical collateral that extends the Cornet & de Boisdeffre (2002) model of asymmetric information. Originally, the authors propose a decentralized mechanism where agents anticipate asset prices and refine their signals by precluding arbitrage opportunities. This procedure extends the classical non-arbitrage asset pricing. However, there are financial structures for which only asset prices that fully reveal information are equilibria. Differently, we show that, if we allow for default and collateralized assets, equilibrium always exists, independently of the financial-informational structure.

[1]This chapter is based on Petrassi & Torres-Martínez (2008).

2
Welfare-Improving Debt Constraints

2.1
Introduction

Long-lived assets in positive net supply, such as equity and fiat money, have been widely studied in general equilibrium models. In such models, the relation between dividends and equilibrium prices for those assets depends crucially on whether agents are finitely or infinitely lived. In overlapping generations models, for instance, prices may exceed fundamental values. In models with infinite-lived agents, however, prices tend to coincide with fundamental values – i.e, price bubbles cannot occur (see Magill & Quinzii (1996) and Santos & Woodford (1997)).

The source of the difference is the following. In overlapping generations models, the present value of wealth may be infinite, whereas, in models where agents are infinitely lived, the present value of wealth must be finite whenever the relevant deflators are Kuhn-Tucker multipliers. Indeed, it is for such deflators, and under a condition known as uniform impatience, that assets in positive net supply are not subject to bubbles.

Uniform impatience is a usual requirement for existence of equilibrium in economies with infinite lived debt-constrained agents (see Hernandez & Santos (1996) or Magill & Quinzii (1996)). This condition is satisfied whenever preferences are separable over time and across states so long as (i) the intertemporal discounted factor is constant, (ii) individual endowments are uniformly bounded away from zero, and (iii) aggregate endowments are uniformly bounded from above.

In this paper, we impose a constraint on the amount of debt an agent can have at any given point in time in a model with infinitely lived agents. It is then shown that prices of assets in positive net supply may differ from their expected discounted sum of dividends *even* under uniform impatience. The necessary and sufficient condition needed for this pricing deviation to occur is that each agent must have binding debt constraints now or at some point in the future.

Throughout the paper, we consider the case of an asset paying no dividends and with positive endowments at the initial node. Samuelson (1958) (see also Tirole (1985)) analyzed such an asset within an overlapping generations context, whereas Bewley (1980), among others (see also Santos & Woodford (1997)), consider this asset in the context of infinite-lived households. Araujo, Páscoa & Novinski (2007) studied this infinite-horizon environment under uncertainty about discounting rates and where, as a consequence, agents become precautious.

The benchmark in which the asset pays no dividends is both simpler and more intriguing. As usual, we call such an asset fiat money.[1] In our model, money may have positive price in equilibrium due to binding debt constraints, which are imposed to avoid Ponzi schemes. This is somehow related to some papers that consider the role of money as a medium of exchange. In Clower (1967), for example, money has positive price in equilibrium because of binding *liquidity constraints*. In a recent work along those lines, Santos (2006) showed that monetary equilibrium can only arise when *cash-in-advance constraints* are binding infinitely often for all agents. In our paper, in contrast, we consider a pure credit economy where money can still be positively valued as a result of agents' desire to take loans when they cannot (either because monetary loans are not allowed or because a debt ceiling has been hit) [2].

As shadow prices of debt constraints play a crucial role in our setting, we develop a duality theory for the households' dynamic programming problem. We identify the Euler and transversality conditions that characterize individual optimality and show that, under the Kuhn-Tucker multipliers, the present value of endowments must be always finite. Once we combine this property with the uniform impatience property, we rule out bubbles in the price of money. Money then can only have positive price if debt constraints are binding. This leads to a somewhat surprising result: credit frictions make room for welfare improvements through intertemporal and inter-states transfers of wealth that are only available when money has positive price.

Our monetary equilibrium is always Pareto inefficient. Indeed, if it were efficient, the agents' rates of intertemporal substitution would necessarily coincide. As money is in positive net supply, at least one agent must carry money, so that his debt constraint is non-binding (and the shadow price is

[1] It is widely recognized that money plays three roles: (i) it is a mean to transfer wealth across time and states (i.e., it stores value), (ii) it is a medium of exchange, and (iii) it is also an unit of account. What we call money in our model plays the role of transferring wealth across time and states. We, therefore, abstract throughout from roles (ii) and (iii) of money.

[2] In a similar context, Gimenez (2005) provided examples of monetary bubbles that can be reinterpreted as positive fundamental values in cashless economies with *no short-sales* restrictions.

zero). When the agents' rates of intertemporal substitution coincide and the shadow price for one agent is zero, the shadow prices for all agents must be zero, and, as a consequence, the price of money cannot be positive.

When money has a positive value, there exists a deflator, but not one of the Kuhn-Tucker deflators, under which the discounted value of aggregated wealth is infinite and a pure bubble appears. Also, independently of the non-arbitrage deflator, when aggregated endowments can be replicated by a portfolio trading plan, the discounted value of future wealth must be finite (see Santos & Woodford (1997)). Therefore, if we allow for an increasing number of non-redundant securities in order to assure that aggregated wealth can be replicated by the deliveries of a portfolio trading plan, money will have zero price. However, the issue of new assets, in order to achieve that efficacy of the financial markets, can be too costly.

Uniform impatience is key for the results we derive. In fact, we provide an example in which utility functions do not satisfy uniform impatience and speculation in an asset in positive net supply occurs, even for deflators that yield finite present values of wealth.

The rest of the chapter is organized as follows. Section 2 presents the basic model. In Section 3, our results are proved. In the Appendix A we develop the necessary mathematical tools: a duality theory of individual optimization. Other important results are proved in Appendices B and C.

2.2 Model

We consider an infinite horizon discrete time economy. The set of dates is $\{0,1,\ldots\}$ and there is no uncertainty at $t=0$. However, given a history of realizations of the states of nature for the first $t-1$ dates, with $t\geq 1$, $\overline{s}_t=(s_0,\ldots,s_{t-1})$, there is a finite set $S(\overline{s}_t)$ of states of nature that may occur at date t. A vector $\xi=(t,\overline{s}_t,s)$, where $t\geq 1$ and $s\in S(\overline{s}_t)$, is called a *node* of the economy. The only node at $t=0$ is denoted by ξ_0. Let D be the *event-tree*, i.e., the set of all nodes.

Given $\xi=(t,\overline{s}_t,s)$ and $\mu=(t',\overline{s}_{t'},s')$, we say that μ is a *successor* of ξ, and we write $\mu\geq\xi$, if $t'\geq t$ and $\overline{s}_{t'}=(\overline{s}_t,s,\ldots)$. We write $\mu>\xi$ to say that $\mu\geq\xi$ but $\mu\neq\xi$ and we denote by $t(\xi)$ the date associated with a node ξ. Let $\xi^+=\{\mu\in D:(\mu\geq\xi)\wedge(t(\mu)=t(\xi)+1)\}$ be the set of immediate successors of ξ. The (unique) predecessor of ξ is denoted by ξ^- and $D(\xi):=\{\mu\in D:\mu\geq\xi\}$ is the sub-tree with root ξ.

At each node, a finite set of perishable commodities is available for trade, L. Let $p=(p(\xi);\xi\in D)$, where $p(\xi):=(p(\xi,l);l\in L)$ denotes the commodity price at $\xi \in D$. We assume that there is only one asset, *money*, that can be traded at any node along the event-tree. Although this security does not deliver any payment, it can be used to make intertemporal transfers. Let $q = (q(\xi); \xi \in D)$ be the plan of state-dependent monetary prices. We assume that money is in positive net supply that does not disappear from the economy neither depreciates.

A finite number of agents, $h \in H$, can trade money and buy commodities along the event-tree. Agent h is characterized by his physical and financial endowments, $(w^h(\xi), e^h(\xi)) \in \mathbb{R}^L_{++} \times \mathbb{R}_+$, at each $\xi \in D$, and by his preferences over consumption, which are represented by an utility function $U^h : \mathbb{R}_+^{D\times L} \to \mathbb{R}_+ \cup \{+\infty\}$. For any $\xi \in D$, let $W_\xi = \sum_{h\in H} w^h(\xi)$ be the aggregated physical endowment at node ξ.

The consumption allocation of agent h at $\xi \in D$ is denoted by $x^h(\xi) := (x^h(\xi, l); l \in L)$. Analogously, the number $z^h(\xi)$ denotes the quantity of money that h negotiates at ξ. Thus, if $z^h(\xi) > 0$, he buys the asset, otherwise, he short sales money making future promises.

Given prices (p, q), let $B^h(p, q)$ be the choice set of agent $h \in H$, that is, the set of plans $(x, z) := ((x(\xi), z(\xi)); \xi \in D) \in \mathbb{R}_+^{D\times L} \times \mathbb{R}^D$, such that, at each $\xi \in D$, the following budget and debt constraints hold,

$$g^h_\xi(y^h(\xi), y^h(\xi^-); p, q) :=$$

$$\begin{aligned} p(\xi)\left(x^h(\xi) - w^h(\xi)\right) + q(\xi)\left(z^h(\xi) - e^h(\xi) - z^h(\xi^-)\right) &\leq 0, \\ q(\xi)z^h(\xi) + p(\xi)M &\geq 0, \end{aligned}$$

where $y^h(\xi) = (x^h(\xi), z^h(\xi))$, $z^h(\xi_0^-) = 0$ and $M \in \mathbb{R}^L_+$. Note that short sales of money are bounded by the exogenous *debt constraints* above in order to avoid Ponzi schemes. Agent's h individual problem is to choose a plan $y^h = (x^h, z^h)$ in $B^h(p, q)$ in order to maximize his utility functions U^h.

Definition 1. *An equilibrium for our economy is given by a vector of prices* (p, q) *jointly with individual allocations* $\left((x^h, z^h); h \in H\right)$, *such that,*

(a) *For each* $h \in H$, *the plan* $(x^h, z^h) \in B^h(p, q)$ *is optimal, at prices* (p, q),

(b) *Physical and asset markets clear,*

$$\sum_{h\in H}\left(x^h(\xi); z^h(\xi)\right) = \left(W_\xi, \sum_{h\in H}\left(e^h(\xi) + z^h(\xi^-)\right)\right).$$

2.3 Characterizing Monetary Equilibria

In our economy, a *pure spot market equilibrium*, i.e., an equilibrium with zero monetary price, always exists provided that preferences satisfy the first part of the following hypothesis. However, our objective is to determine conditions that characterize the existence of equilibria with positive price of money, called *monetary equilibria*. For this reason, we also assume that agents are *uniformly impatience*.

Assumption A.
A1. Preferences. *Let* $U^h(x) := \sum_{\xi\in D} u^h(\xi, x(\xi))$*, where for any* $\xi \in D$*,* $u^h(\xi,\cdot) : \mathbb{R}^L_+ \to \mathbb{R}_+$ *is a continuous, concave and strictly increasing function. Also,* $\sum_{\xi\in D} u^h(\xi, W_\xi)$ *is finite.*

A2. Uniform Impatience Assumption. *There are* $\pi \in [0, 1)$ *and* $(v(\mu); \mu \in D) \in \mathbb{R}^{D\times L}_+$ *such that, given a consumption plan* $(x(\mu); \mu \in D)$*, with* $0 \leq x(\mu) \leq W_\mu$*, for any* $h \in H$*, we have*

$$u^h(\xi, x(\xi) + v(\xi)) + \sum_{\mu>\xi} u^h(\mu,\, \pi'\, x(\mu)) > \sum_{\mu\geq\xi} u^h(\mu, x(\mu)), \qquad \forall \xi \in D,\ \forall \pi' \geq \pi.$$

Moreover, there is $\delta > 0$ *such that,* $w^h(\xi) \geq \delta v(\xi)$*,* $\forall \xi \in D$*.*

The requirements of impatience above depend on both preferences and physical endowments. As particular cases we obtain the assumptions imposed by Hernandez & Santos (1996) and Magill & Quinzii (1996). Indeed, in Hernandez & Santos (1996), for any $\mu \in D$, $v(\mu) = W_\mu$. Also, since in Magill & Quinzii (1996) initial endowments are uniformly bounded away from zero by an interior bundle $\underline{w} \in \mathbb{R}^L_+$, they suppose that $v(\mu) = (1, 0, \ldots, 0)$, $\forall \mu \in D$.

Under Assumption A1, Propositions A1 and B1 in the Appendices assure that, given an equilibrium $\left[(p, q); ((x^h, z^h); h \in H)\right]$, there are, for each $h \in H$, Kuhn-Tucker multipliers $(\gamma^h(\xi); \xi \in D)$, such that,

$$q(\xi) = F(\xi, q, \gamma^h) + \lim_{T\to+\infty} \sum_{\{\mu\geq\xi: t(\mu)=T\}} \frac{\gamma^h(\mu)}{\gamma^h(\xi)} q(\mu),$$

where $F(\xi, q, \gamma^h)$ is the *fundamental value* of money, and the second term in the right hand side is the monetary speculative component, also called *bubble*. We say that *debt constraints induce frictions over agent* h *in* $\tilde{D} \subset D$ if the plan of shadow prices $(\eta^h(\mu);\ \mu \in \tilde{D})$ that is defined implicitly, at each $\mu \in \tilde{D}$, by the conditions:

$$\begin{aligned} 0 &= \eta^h(\mu)\left(q(\mu)z^h(\mu) + p(\mu)M\right), \\ \gamma^h(\mu)q(\mu) &= \sum_{\nu\in\mu^+} \gamma^h(\nu)q(\nu) + \eta^h(\mu)q(\mu), \end{aligned}$$

is different from zero.

Just before our main result, we will show, in the following example, that a monetary equilibrium can arise in economies without uncertainty. Essentially, it is an extension of the monetary model of Bewley (1980) (for references over this example, see Mattalia (2003)) modifying the no short-sales constraint. However, we show that, in this monetary equilibrium, there is no bubble component, only a fundamental value component which is a consequence of the binding no short-sale constraints.

Example 1. Consider an infinite-horizon economy without uncertainty. There is only one asset, fiat money, and a single perishable good at each date. There are two agents $i \in \{1,2\}$ with identical utilities of the form

$$U^i(x_0^i, x_1^i, \ldots) = \sum_{t=0}^{\infty} \beta^t u(x_t^i),$$

where $\beta \in (0,1)$ denotes the intertemporal discounted factor and the function $u : \mathbb{R} \to \mathbb{R}$ is strictly increasing, continuous and strictly concave in $\mathbb{R}_+$.

Each agent i will receive an endowment of money, $e^i \geq 0$, only at the initial date $t = 0$, where $e_0^1 = 0$ and $e_0^2 = 1$. Moreover, the physical endowments of agents at date t are strictly positive and given by $(w_t^1, w_t^2) := (\overline{w}, \underline{w})$, when t is even; and by $(w_t^1, w_t^2) := (\underline{w}, \overline{w})$, when t is odd. Note that $\underline{w} < \overline{w}$.

Finally, the amount of debt each agent can take is bounded, at each date t, by a constraint $M^i(t, p(t)) := p(t)M$, where $p(t)$ denotes the price of the single commodity at t and $M = 0$ is the borrowing limit[3]. The price of money at date t will be denoted by $q(t)$.

We affirm that the prices $q(t) = q^* > 0$ (and $p(t) = 1$) constitute an equilibrium monetary price. Also, the consumption and portfolio allocations

[3]It should be noted that this example could be extended to a fixed $M > 0$. The objective here, however, is to keep the example as simple as possible.

$$x_t^i = \begin{cases} \overline{w} - q^* & \text{if } i = 1 \text{ and t is even or } i = 2 \text{ and t is odd;} \\ \underline{w} + q^* & i = 1 \text{ and t is odd or } i = 2 \text{ and t is even.} \end{cases}$$

$$z_t^i = \begin{cases} 0 & \text{if } i = 1 \text{ and t is even or } i = 2 \text{ and t is odd;} \\ 1 & i = 1 \text{ and t is odd or } i = 2 \text{ and t is even.} \end{cases}$$

where x_t^i is the consumption choice and z_t^i is the portfolio choice. They are budget and market feasible.

Also, they are optimal choices for the agents as the following Euler conditions are satisfied (see Definition A1 and Proposition A2 in the Appendix A),

$$(\gamma_t^i, \gamma_t^i q^*) = (\beta^t u'(x_t^i), \gamma_{t+1}^i q^* + \lambda_t^i q^*),$$

where γ_t^i is the candidate for Kuhn-Tucker multiplier and λ_t^i is the candidate for shadow price associated to the debt constraint.

Of course, as $x_t^i > 0$, we know that $\gamma_t^i = \beta^t u'(x_t^i)$. Then, it can be shown that, under Kuhn-Tucker multipliers, transversality condition is satisfied and that there is no bubble component. Under Kuhn-Tucker multipliers, the discounted value of future wealth is also finite.

Note that at each t, debt constraint is binding for one of the agents: when t is even, agent 1 binds his constraint. When t is odd, agent 2 portfolio choice is $z_t^2 = 0$. As a consequence, there is a monetary equilibrium that is actually a fundamental value. Money has a positive value as a consequence of the positive shadow values associated to the binding debt constraints.

Theorem. *Under Assumption A, for any equilibrium* $\left[(p, q); ((x^h, z^h); h \in H)\right]$ *we have that,*

(1) If $q(\xi) > 0$ *then debt constraints induce frictions over each agent in* $D(\xi)$.

(2) If $M \neq 0$ *and some* $h \in H$ *has a binding debt constraint at a node* $\mu \in D(\xi)$, *then* $q(\xi) > 0$.

(3) If for each $\xi \in D$, $u^h(\xi, \cdot)$ *is differentiable in* $\mathbb{R}_{++}^L$ *and* $\lim_{\|x\|_{min} \to 0^+} \nabla u^h(\xi, x) = +\infty$, *then any monetary equilibrium is Pareto inefficient.*

Observation. Item (1) is related to Theorem 3.3 in Santos & Woodford (1997), that establishes that, under uniform impatience, assets in positive net supply are free of price bubbles for deflators that yield finite present value of wealth. However, in the frictionless framework used by these authors, absence of bubbles necessarily leads to a zero price of money. The converse, item (2) and item (3) are new in the literature.

Moreover, it follows from items (1) and (2) that binding debt constraints always induce frictions, i.e., positive shadow prices. Also, if an agent becomes borrower at some node in $D(\xi)$, then *all individuals* are borrowers at some node of $D(\xi)$. In other words, in a monetary equilibrium, all agents take a monetary loan (at some node).

Proof of the Theorem. (1) By definition, if for some $h \in H$, $(\eta^h(\mu); \mu \geq \xi) = 0$ then $F(\xi, q, \gamma^h) = 0$. Therefore, as in Santos & Woodford (1997), a monetary equilibrium is a *pure bubble*. However, uniform impatience implies that bubbles are ruled out in equilibrium.

Indeed, at each $\xi \in D$ there exists an agent $h = h(\xi)$ with $q(\xi)z^h(\xi) \geq 0$. Thus, by the impatience property, $0 \leq (1-\pi)q(\xi)z^h(\xi) \leq p(\xi)v(\xi)$. Moreover, financial market feasibility allows us to find a lower bound for individual debt. Therefore, for each $h \in H$, the plan $\left(\frac{q(\xi)z^h(\xi)}{p(\xi)v(\xi)}\right)_{\xi \in D}$ is uniformly bounded. Furthermore, as money is in positive net supply, it follows that $\left(\frac{q(\xi)}{p(\xi)v(\xi)}\right)_{\xi \in D}$ is uniformly bounded too. As by Lemma A1 we know that, for any $h \in H$, $\sum_{\xi \in D} \gamma^h(\xi)p(\xi)w^h(\xi) < +\infty$, it follows from Assumption A2 that bubbles do not arise in equilibrium.

Therefore, we conclude that, if $q(\xi) > 0$ then $(\eta^h(\mu); \mu \geq \xi) \neq 0$, for all $h \in H$.

(2) Suppose that, for some $h \in H$, there exists $\mu \geq \xi$ such that $q(\mu)z^h(\mu) = -p(\mu)M$. Since monotonicity of preferences implies that $p(\xi) \gg 0$, if $M \neq 0$ then $q(\mu) > 0$. Also, Assumption A1 assures that Kuhn-Tucker multipliers, $(\gamma^h(\eta); \eta \in D)$, are strictly positive. Therefore, the equations that define shadow prices imply that $q(\xi) > 0$.

(3) Suppose that there exists an efficient monetary equilibrium, in the sense that individuals' marginal rates of substitution coincide. As $\lim_{\|x\|_{min} \to 0^+} \nabla u^h(\xi, x) = +\infty, \forall (h, \xi) \in H \times D$, all agents have interior consumption along the event-tree. Positive net supply of money implies that there exists, at each $\xi \in D$, at least one lender. Therefore, by the efficiency property, it follows that *all* individuals have zero shadow prices. A contradiction with item (1) above. □

Some remarks,

(1) It follows from the proof of the Theorem that under uniform impatience

monetary debt is uniformly bounded—in real terms—along the even-tree. Thus, it is easy to find a vector $M^* \in \mathbb{R}^L_+$ such that, in any equilibrium, and for each node ξ, the debt constraint $q(\xi)z^h(\xi) \geq -p(\xi)M^*$ is non-binding. Therefore, when $M > M^*$ monetary equilibria disappear. That is, contrary to what may be expected, frictions induced by debt constraints improve welfare.

(2) Given a monetary equilibrium, there always exists a non-arbitrage deflator, *incompatible with physical Euler conditions* (see Definition A1), for which the price of money is a pure bubble. Indeed, define $\nu := (\nu(\xi) : \xi \in D)$ by $\nu(\xi_0) = 1$, and

$$\begin{aligned} \nu(\xi) &= 1, && \forall \xi > \xi_0 : q(\xi) = 0, \\ \frac{\nu(\xi)}{\nu(\xi^-)} &= \frac{\gamma^h(\xi)}{\gamma^h(\xi^-) - \eta^h(\xi^-)}, && \forall \xi > \xi_0 : q(\xi) > 0. \end{aligned}$$

Euler conditions on $(\gamma^h(\xi); \xi \in D)$ imply that, for each $\xi \in D$, $\nu(\xi)q(\xi) = \sum_{\mu \in \xi^+} \nu(\mu)q(\mu)$. Therefore, using the plan of deflators ν, *financial Euler conditions* hold and the positive price of money is a bubble. Since under uniform impatience assumption the monetary debt is uniformly bounded along the event-tree, under these deflators the discounted value of future individual endowments has to be infinite.

We remark that the plan of state prices ν is compatible with the frictionless theory of bubbles developed by Santos & Woodford (1997) and, in that frictionless context, we recover a property that was previously found by them: a monetary bubble is possible only for deflators under which we have an infinite discounted value of future wealth.

2.4 About Uniform Impatience

To highlight the role that uniform impatience has in our Theorem, we adapt Example 1 in Araujo, Páscoa & Torres-Martínez (2007) in order to prove that without Assumption A2 money may have a pure bubble for Kuhn-Tucker multipliers. Moreover, bubbles on the price of money will be compatible with a finite discounted value of future wealth. Essentially because individuals will believe that, as time goes on, the probability that the economy may fall in a path in which endowments increase without an upper bound converges to zero fast enough.

Example 2. Assume that each $\xi \in D$ has two successors: $\xi^+ = \{\xi^u, \xi^d\}$. There are two agents $H = \{1, 2\}$ and only one commodity. Each $h \in H$

has physical endowments $(w^h_\xi)_{\xi\in D}$, receives financial endowments $e^h \geq 0$ only at the first node, and has preferences represented by the utility function $U^h(x) = \sum_{\xi\in D} \beta^{t(\xi)}\rho^h(\xi)\, x_\xi$, where $\beta \in (0,1)$ and the plan $(\rho^h(\xi))_{\xi\in D} \in (0,1)^D$ satisfies $\rho(\xi_0) = 1$, $\rho^h(\xi) = \rho^h(\xi^d) + \rho^h(\xi^u)$ and

$$\rho^1(\xi^u) = \frac{1}{2^{t(\xi)+1}}\rho^1(\xi), \qquad \rho^2(\xi^u) = \left(1 - \tfrac{1}{2^{t(\xi)+1}}\right)\rho^2(\xi).$$

Suppose that agent $h = 1$ is the only one endowed with the asset, i.e., $(e^1, e^2) = (1, 0)$ and that, for each $\xi \in D$,

$$w^1_\xi = \begin{cases} 1 + \beta^{-t(\xi)} & \text{if } \xi \in D^{du}, \\ 1 & \text{otherwise}; \end{cases} \qquad w^2_\xi = \begin{cases} 1 + \beta^{-t(\xi)} & \text{if } \xi \in \{\xi_0^d\} \cup D^{ud}, \\ 1 & \text{otherwise}; \end{cases}$$

where D^{du} is the set of nodes attained after going down followed by up, that is, $D^{du} = \{\eta \in D : \exists\xi,\ \eta = (\xi^d)^u\ \}$ and D^{ud} denotes the set of nodes reached by going up and then down, that is, $D^{ud} = \{\eta \in D : \exists\xi,\ \eta = (\xi^u)^d\ \}$.

Agents will use positive endowment shocks in low probability states to buy money and sell it later in states with higher probabilities. Let prices be $(p_\xi, q_\xi)_{\xi\in D} = (\,\beta^{t(\xi)},\, 1)_{\xi\in D}$ and suppose that consumption of agent h is given by $x^h_\xi = w^{h'}_\xi$, where $h \neq h'$. It follows from budget constraints that, at each ξ, the portfolio of agent h must satisfy $z^h_\xi = \beta^{t(\xi)}(w^h_\xi - w^{h'}_\xi) + z^h_{\xi^-}$, where $z^h_{\xi_0^-} := e^h$ and $h \neq h'$.

Thus, the consumption allocations above jointly with the portfolios $(z^1_{\xi_0},\, z^1_{\xi^u},\, z^1_{\xi^d}) = (1,\, 1,\, 0)$ and $(z^2_\xi)_{\xi\in D} = (1 - z^1_\xi)_{\xi\in D}$ are budget and market feasible. Finally, given $(h,\xi) \in H \times D$, let $\gamma^h_\xi = \rho^h(\xi)$ be the candidate for Kuhn-Tucker multiplier of agent h at node ξ. It follows that conditions below hold and they assure individual optimality (see Proposition A2 in the Appendix A),

$$\begin{aligned} (\gamma^h_\xi p_\xi, \gamma^h_\xi q_\xi) &= (\beta^{t(\xi)}\rho^h(\xi), \gamma^h_{\xi^u} q_{\xi^u} + \gamma^h_{\xi^d} q_{\xi^d}), \\ \sum_{\{\eta\in D:\, t(\eta)=T\}} \gamma^h_\eta p_\eta M &\longrightarrow 0, \qquad \text{as } T \to +\infty, \\ \sum_{\{\eta\in D:\, t(\eta)=T\}} \gamma^h_\eta q_\eta z^h_\eta &\longrightarrow 0, \qquad \text{as } T \to +\infty. \end{aligned}$$

Note that, by construction and independently of $M \geq 0$, the plan of shadow prices associated to debt constraints is zero. Therefore, for any M, money has a zero fundamental value and a bubble under Kuhn-Tucker multipliers. Also, the diversity of individuals beliefs about the uncertainty (probabilities $\rho^h(\xi)$) implies that both agents perceive a finite present value of aggregate wealth.[4] Finally, Assumption A2 is not satisfied, because aggregated physical endowments were unbounded along the event-tree.[5] □

Appendix A: Duality Theory of Individual Optimality

Under Assumption A1, we will use duality theory to determine necessary conditions for individual optimality. To attempt this objective, we restrict our attention, without loss of generality, to prices $(p,q) \in \mathbb{P} := \{(p,q) \in \mathbb{R}_+^{L\times D} \times \mathbb{R}_+^D : (p(\xi), q(\xi)) \in \Delta^{\#L+1},\ \forall \xi \in D\}$, where, for each $m > 0$, the *simplex* $\Delta^m := \{z = (z_1,\dots,z_m) \in \mathbb{R}_+^m : \sum_{k=1}^m z_k = 1\}$. Also, remember that the super-gradient of a concave function $f : X \subset \mathbb{R}^L \to \mathbb{R} \cup \{-\infty\}$ at point $x \in X$ is defined as the set of vectors $p \in \mathbb{R}^L$ such that, for all $x' \in X$, $f(\xi, x') - f(\xi, x) \leq p(x' - x)$.

For convenience of notations, let the set of nodes with date T in $D(\xi)$ be denoted by $D_T(\xi)$. Finally, let $D^T(\xi) = \bigcup_{k=t(\xi)}^T D_k(\xi)$ be the set of successors of ξ with date less than or equal to T. When $\xi = \xi_0$ notations above will be shorten to D_T and D^T.

[4]Using agent' h Kuhn-Tucker multipliers as deflators, the present value of aggregated wealth at $\xi \in D$, denoted by PV_ξ^h, satisfies,

$$\begin{aligned} PV_\xi^h &= \sum_{\mu \geq \xi} \frac{\gamma_\mu^h}{\gamma_\xi^h} p_\mu W_\mu = \frac{2}{\rho^h(\xi)} \sum_{\mu\geq\xi} \rho^h(\mu)\beta^{t(\mu)} + \frac{1}{\rho^h(\xi)} \sum_{\{\mu\geq\xi : \mu \in D^{ud}\cup D^{du} \cup \{\xi_0^d\}\}} \rho^h(\mu) \\ &= 2\frac{\beta^{t(\xi)}}{1-\beta} + \sum_{\{\mu\geq\xi:\mu\in D^{ud}\cup D^{du}\cup\{\xi_0^d\},\, t(\mu)\leq t(\xi)+1\}} \frac{\rho^h(\mu)}{\rho^h(\xi)} \\ &+ \sum_{s=t(\xi)+1}^{+\infty} \left[\frac{1}{2^{s+1}}\left(1-\frac{1}{2^s}\right) + \left(1 - \frac{1}{2^{s+1}}\right)\frac{1}{2^s}\right] \\ &= 2\frac{\beta^{t(\xi)}}{1-\beta} + \frac{3}{2}\frac{1}{2^{t(\xi)}} - \frac{1}{3}\frac{1}{4^{t(\xi)}} + \frac{1}{\rho^h(\xi)} \sum_{\{\mu\geq\xi:\mu\in D^{ud}\cup D^{du},\, t(\mu)\leq t(\xi)+1\}} \rho^h(\mu) < +\infty. \end{aligned}$$

[5]If Assumption A2 holds, there are $(\delta,\pi) \in \mathbb{R}_{++} \times (0,1)$ such that, for any $\xi \in D^{uu} := \{\mu \in D : \exists \eta \in D; \mu = (\eta^u)^u\}$,

$$\frac{1}{\delta} = \frac{w_\xi^h}{\delta} > \frac{1-\pi}{\beta^{t(\xi)}\rho^h(\xi)} \sum_{\mu>\xi} \rho^h(\mu)\beta^{t(\mu)} W_\mu, \quad \forall h \in H.$$

Thus, for all $(\xi, h) \in D^{uu} \times H$, $\beta^{t(\xi)}\left(\frac{1}{\delta(1-\pi)} + W_\xi\right) > PV_\xi^h$. On the other hand, given $\xi \in D^{uu}$,

$$PV_\xi^1 \geq \frac{1}{\rho^1(\xi)} \sum_{\{\mu\geq\xi:\mu\in D^{ud}\cup D^{du},\, t(\mu)\leq t(\xi)+1\}} \rho^1(\mu) = 1 - \frac{1}{2^{t(\xi)+1}}.$$

Therefore, as for any $T \in \mathbb{N}$ there exists $\xi \in D^{uu}$ with $t(\xi) = T$, we conclude that, $\beta^T\left(\frac{1}{\delta(1-\pi)} + 2\right) > 0.5$, for all $T > 0$. A contradiction.

DEFINITION A1. *Given* $(p,q) \in \mathbb{P}$ *and* $y^h = (x^h, z^h) \in B^h(p,q)$, *we say that* $(\gamma^h(\xi); \xi \in D) \in \mathbb{R}^D_+$ *constitutes a family of Kuhn-Tucker multipliers (associated to* y^h*) if there exist, for each* $\xi \in D$, *super-gradients* $u'(\xi) \in \partial u^h(\xi, x^h(\xi))$ *such that,*

(a) For every $\xi \in D$, $\gamma^h(\xi)\, g^h_\xi(y^h(\xi), y^h(\xi^-); p, q) = 0$.

(b) The following Euler conditions hold,

$$\begin{aligned} \gamma^h(\xi)p(\xi) &\geq u'(\xi), \\ \gamma^h(\xi)p(\xi)x^h(\xi) &= u'(\xi)\, x^h(\xi), \\ \gamma^h(\xi)q(\xi) &\geq \sum_{\mu \in \xi^+} \gamma^h(\mu)q(\mu), \end{aligned}$$

where the last inequality is strict only if the associated debt constraint is binding at ξ.

(c) The following transversality condition holds:

$$\limsup_{T \to +\infty} \sum_{\xi \in D_T} \gamma^h(\xi)q(\xi)z^h(\xi) \leq 0.$$

LEMMA A1. (FINITE DISCOUNTED VALUE OF INDIVIDUAL ENDOWMENTS)

Fix a plan $(p,q) \in \mathbb{P}$ *and* $y^h = (x^h, z^h) \in B^h(p,q)$ *such that* $U^h(x^h) < +\infty$. *If Assumption A1 holds then for any family of Kuhn-Tucker multipliers associated to* y^h, $(\gamma^h(\xi); \xi \in D)$, *we have* $\sum_{\xi \in D} \gamma^h(\xi)\, \big(p(\xi)\, w^h(\xi) + q(\xi)e^h(\xi)\big) < +\infty$.

PROOF. Let $\mathcal{L}^h_\xi : \mathbb{R}^{L+1} \times \mathbb{R}^{L+1} \to \mathbb{R} \cup \{-\infty\}$ be the function defined by $\mathcal{L}^h_\xi(y(\xi), y(\xi^-)) = v^h(\xi, y(\xi)) - \gamma^h(\xi)\, g^h_\xi(y(\xi), y(\xi^-); p, q)$, where $y(\xi) = (x(\xi), z(\xi))$ and $v^h(\xi, \cdot) : \mathbb{R}^L \times \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is given by

$$v^h(\xi, y(\xi)) = \begin{cases} u^h(\xi, x(\xi)) & \text{if } x(\xi) \geq 0; \\ -\infty & \text{otherwise.} \end{cases}$$

It follows from Assumption A1 and Euler conditions that, for each $T \geq 0$,

$$\sum_{\xi \in D^T} \mathcal{L}^h_\xi(0,0) - \sum_{\xi \in D^T} \mathcal{L}^h_\xi(y^h(\xi), y^h(\xi^-)) \leq - \sum_{\xi \in D_T} \gamma^h(\xi)q(\xi)(0 - z^h(\xi)).$$

Therefore, as for each $\xi \in D$, $\gamma^h(\xi)\, g^h_\xi(y^h(\xi), y^h(\xi^-); p, q) = 0$, we have that, for any $S \in \mathbb{N}$,

$$0 \leq \sum_{\xi \in D^S} \gamma^h(\xi) \left(p(\xi) w^h(\xi) + q(\xi) e^h(\xi) \right)$$

$$\begin{aligned} &\leq \limsup_{T \to +\infty} \sum_{\xi \in D^T} \gamma^h(\xi) \left(p(\xi) w^h(\xi) + q(\xi) e^h(\xi) \right) \\ &\leq U^h(x^h) + \limsup_{T} \sum_{\xi \in D_T} \gamma^h(\xi) q(\xi) z^h(\xi) \\ &\leq U^h(x^h) < +\infty, \end{aligned}$$

which concludes the proof. □

Proposition A1. (Necessary conditions for individual optimality) *Fix a plan $(p, q) \in \mathbb{P}$ and $y^h = (x^h, z^h) \in B^h(p, q)$ such that $U^h(x^h) < +\infty$. If Assumption A1 holds and y^h is an optimal allocation for agent $h \in H$ at prices (p, q), then there exists a family of Kuhn-Tucker multipliers associated to y^h.*

Proof. Suppose that $(y^h(\xi))_{\xi \in D}$ is optimal for agent $h \in H$ at prices (p, q). For each $T \in \mathbb{N}$, consider the truncated optimization problem,

$$(P^{h,T}) \qquad \max \sum_{\xi \in D^T} u^h(\xi, x(\xi)) \quad \text{s.t.} \begin{cases} g^h_\xi\left(y(\xi), y(\xi^-); p, q\right) \leq 0, \ \forall \xi \in D^T, \\ q(\xi) z(\xi) \geq -p(\xi) M, \forall \xi \in D^T \setminus D_T, \\ (x(\xi), z(\eta)) \geq 0, \ \forall (\xi, \eta) \in D^T \times D_T. \end{cases}$$

It follows that, under Assumption A1 each truncated problem $P^{h,T}$ has a solution $(y^{h,T}(\xi))_{\xi \in D^T}$.[6]

[6] In fact, as $(y^h(\xi))_{\xi \in D}$ is optimal and $U^h(x^h) < +\infty$, it follows that there exists a solution for $P^{h,T}$ if and only if there exists a solution for the problem,

$$(\tilde{P}^{h,T}) \qquad \max \sum_{\xi \in D^T} u^h(\xi, x(\xi)) \quad \text{s.t.} \begin{cases} g^h_\xi\left(y(\xi), y(\xi^-); p, q\right) \leq 0, \ \forall \xi \in D^T, \text{ where } y(\xi) = (x(\xi), z(\xi)), \\ z(\xi) \geq -\frac{p(\xi) M}{q(\xi)}, \ \forall \xi \in D^{T-1} \text{ such that } q(\xi) > 0 \\ z(\xi) = 0, \text{if } \left[\xi \in D^{T-1} \text{ and } q(\xi) = 0\right] \text{or } \xi \in D_T, \\ x(\xi) \geq 0, \ \forall \xi \in D^T. \end{cases}$$

Indeed, it follows from the existence of an optimal plan which gives finite utility that if $q(\xi) = 0$ for some $\xi \in D$, then $q(\mu) = 0$ for each successor $\mu > \xi$. Now, budget feasibility assures that,

$$z(\xi) \leq \frac{p(\xi) w^h(\xi)}{q(\xi)} + z(\xi^-), \ \forall \xi \in D^{T-1} \text{ such that } q(\xi) > 0.$$

As $z(\xi_0^-) = 0$, the set of feasible financial positions is bounded in the problem $(\tilde{P}^{h,T})$. Thus, budget feasible consumption allocations are also bounded and, therefore, the set of admissible strategies is compact. As the objective function is continuous, there is a solution for $(\tilde{P}^{h,T})$. Moreover, the optimality of $(y^h(\xi))_{\xi \in D}$ in the original problem implies that

Given a multiplier $\gamma \in \mathbb{R}$, let $\mathcal{L}^h_\xi(\cdot, \gamma; p, q) : \mathbb{R}^{L+1} \times \mathbb{R}^{L+1} \to \mathbb{R} \cup \{-\infty\}$ be the Lagrangian at node ξ, i.e.,

$$\mathcal{L}^h_\xi(y(\xi), y(\xi^-), \gamma;\, p, q) = v^h(\xi, y(\xi)) - \gamma\, g^h_\xi(y(\xi), y(\xi^-); p, q).$$

It follows from Rockafellar (1997, Theorem 28.3) that there exist non-negative multipliers $(\gamma^{h,T}(\xi))_{\xi \in D^T}$ such that the following saddle point property

$$\sum_{\xi \in D^T} \mathcal{L}^h_\xi(y(\xi), y(\xi^-), \gamma^{h,T}(\xi);\, p, q) \leq \sum_{\xi \in D^T} \mathcal{L}^h_\xi(y^{h,T}(\xi), y^{h,T}(\xi^-), \gamma^{h,T}(\xi);\, p, q),$$

is satisfied, for each plan $(y(\xi))_{\xi \in D^T} = (x(\xi), z(\xi))_{\xi \in D^T}$ for which

$$\begin{aligned} (x(\xi), z(\eta)) &\geq 0, && \forall (\xi, \eta) \in D^T \times D_T, \\ q(\xi) z(\xi) &\geq -p(\xi) M, && \forall \xi \in D^T \setminus D_T. \end{aligned}$$

Moreover, at each $\xi \in D^T$, multipliers satisfy $\gamma^{h,T}(\xi)\, g^h_\xi(y^{h,T}(\xi), y^{h,T}(\xi^-); p, q) = 0$.

Analogous arguments to those made in Claims A1-A3 in Araujo, Páscoa & Torres-Martínez (2007) imply that,

Claim. *Under Assumption A1, the following conditions hold:*
(i) For each $t < T$,

$$0 \leq \sum_{\xi \in D^t} \gamma^{h,T}(\xi) \left(p(\xi) w^h(\xi) + q(\xi) e^h(\xi) \right) \leq U^h(x^h).$$

(ii) For each $0 < t < T$,

$$\sum_{\xi \in D_t} \gamma^{h,T}(\xi) q(\xi) z^h(\xi^-) \leq \sum_{\xi \in D \setminus D^{t-1}} u^h(\xi, x^h(\xi)).$$

(iii) For each $\xi \in D^{T-1}$ and for any $y(\xi) = (x(\xi), z(\xi))$, with $x(\xi) \geq 0$ and $q(\xi) z(\xi) \geq -p(\xi) M$,

$U^h(x^h)$ is greater than or equal to $\sum_{\xi \in D^T} u^h(\xi, x^{h,T}(\xi))$. In fact, the plan $(\tilde{y}_\xi)_{\xi \in D}$ defined by $\tilde{y}_\xi = y^{h,T}_\xi$, for each $\xi \in D^T$, and by $\tilde{y}_\xi = 0$ otherwise, is budget feasible in the original economy and, therefore, the allocation $(y^{h,T}(\xi))_{\xi \in D^T}$ cannot improve the utility level of agent h.

$$u^h(\xi, x(\xi)) - u^h(\xi, x^h(\xi)) \leq$$

$$\left(\gamma^{h,T}(\xi)p(\xi); \gamma^{h,T}(\xi)q(\xi) - \sum_{\mu \in \xi^+} \gamma^{h,T}(\mu)q(\mu)\right) \cdot (y(\xi) - y^h(\xi)) + \sum_{\eta \in D \setminus D^T} u^h(\eta, x^h(\eta)).$$

Now, at each $\xi \in D$, $\underline{w}^h(\xi) := \min_{l \in L} w^h(\xi, l) > 0$. Also, as a consequence of monotonicity of $u^h(\xi)$, $||p(\xi)||_\Sigma > 0$. Thus, item (i) above guarantees that, for each $\xi \in D$,

$$0 \leq \gamma^{h,T}(\xi) \leq \frac{U^h(x^h)}{\underline{w}^h(\xi)\, ||p(\xi)||_\Sigma}, \quad \forall T > t(\xi).$$

Therefore, the sequence $(\gamma^{h,T}(\xi))_{T \geq t(\xi)}$ is bounded, node by node. As the event-tree is countable, there is a common subsequence $(T_k)_{k \in \mathbb{N}} \subset \mathbb{N}$ and non-negative multipliers $(\gamma^h(\xi))_{\xi \in D}$ such that, for each $\xi \in D$, $\gamma^{h,T_k}(\xi) \rightarrow_{k \to +\infty} \gamma^h(\xi)$, and

$$\gamma^h(\xi) g^h_\xi(p, q, y^h(\xi), y^h(\xi^-)) \;=\; 0\,; \tag{2-1}$$

$$\limsup_{t \to +\infty} \sum_{\xi \in D_t} \gamma^h(\xi) q(\xi) z^h(\xi^-) \;\leq\; 0\,, \tag{2-2}$$

where equation (2-1) follows from the strictly monotonicity of $u^h(\xi)$, and equation (2-2) is a consequence of item (ii) (taking the limit as T goes to infinity and, afterwards, the limit in t).

Moreover, using item (iii), and taking the limit as T goes to infinity, we obtain that, for each $y(\xi) = (x(\xi), z(\xi))$, with $x(\xi) \geq 0$ and $q(\xi)z(\xi) \geq -p(\xi)M$,

$$\begin{aligned} u^h(\xi, x(\xi)) - u^h(\xi, x^h(\xi)) \;&\leq \\ (\gamma^h(\xi)p(\xi)\,;\; \gamma^h(\xi)q(\xi) \;&-\; \sum_{\mu \in \xi^+} \gamma^h(\mu)q(\mu)) \cdot (y(\xi) - y^h(\xi)). \end{aligned}$$

Let $\mathcal{F}^h(\xi, p, q) = \{(x, z) \in \mathbb{R}^L \times \mathbb{R} : x \geq 0 \;\wedge\; q(\xi)z \geq -p(\xi)M\}$.

It follows that $\left(\gamma^h(\xi)p(\xi)\,;\, \gamma^h(\xi)q(\xi) - \sum_{\mu \in \xi^+} \gamma^h(\mu)q(\mu)\right)$ belongs to the super-differential set of the function $v^h(\xi, \cdot) + \delta(\cdot, \mathcal{F}^h(\xi, p, q))$ at point $y^h(\xi)$, where $\delta(y, \mathcal{F}^h(\xi, p, q)) = 0$, when $y \in \mathcal{F}^h(\xi, p, q)$ and $\delta(y, \mathcal{F}^h(\xi, p, q)) = -\infty$, otherwise. Notice that, for each $y \in \mathcal{F}^h(\xi,p,q)$, $\kappa \in \partial\delta(y,\mathcal{F}^h(\xi,p,q)) \Leftrightarrow 0 \leq k(y'-y), \ \forall y' \in \mathcal{F}^h(\xi,p,q)$.

Now, by Theorem 23.8 in Rockafellar (1997), for all $y \in \mathcal{F}^h(\xi,p,q)$, if $v'(\xi)$ belongs to $\partial\left[v^h(\xi,y) + \delta(y,\mathcal{F}^h(\xi,p,q))\right]$ then there exists $\tilde{v}'(\xi) \in \partial v^h(\xi,y)$ such that both $v'(\xi) \geq \tilde{v}'(\xi)$ and $(v'(\xi) - \tilde{v}'(\xi)) \cdot (x, q(\xi)z + p(\xi)M) = 0$, where $y = (x,z)$. Therefore, it follows that there exists, for each $\xi \in D$, a super-gradient $\tilde{v}'(\xi) \in \partial v^h(\xi, y^h(\xi))$ such that,

$$\left(\gamma^h(\xi)p(\xi)\,;\ \gamma^h(\xi)q(\xi) - \sum_{\mu\in\xi^+}\gamma^h(\mu)q(\mu)\right) - \tilde{v}'(\xi) \geq 0,$$

$$\left[\left(\gamma^h(\xi)p(\xi)\,;\ \gamma^h(\xi)q(\xi) - \sum_{\mu\in\xi^+}\gamma^h(\mu)q(\mu)\right) - \tilde{v}'(\xi)\right] \cdot (x^h(\xi), q(\xi)z^h(\xi) + p(\xi)M) = 0.$$

As $\tilde{v}'(\xi) \in \partial v^h(\xi, y^h(\xi))$ if and only if there is $u'(\xi) \in \partial u^h(\xi, x^h(\xi))$ such that $\tilde{v}'(\xi) = (u'(\xi), 0)$, it follows from last inequalities that Euler conditions hold.

On the other side, item (i) in claim above guarantees that, $\sum_{\xi\in D}\gamma^h(\xi)(p(\xi)w^h(\xi) + q(\xi)e^h(\xi)) < +\infty$ and, therefore, equations (2-1) and (2-2) assure that, $\limsup_{t\to+\infty}\sum_{\xi\in D_t}\gamma^h(\xi)q(\xi)z^h(\xi)$

$$\begin{aligned}
&\leq \limsup_{t\to+\infty}\sum_{\xi\in D_t}\gamma^h(\xi)\left(p(\xi)w^h(\xi) + q(\xi)e^h(\xi) + q(\xi)z^h(\xi^-)\right) \\
&\leq \limsup_{t\to+\infty}\sum_{\xi\in D_t}\gamma^h(\xi)q(\xi)z^h(\xi^-) \ \leq \ 0,
\end{aligned}$$

which imply that transversality condition holds. □

Note that we could prove, alternatively, the existence of a state price deflator that satisfies the *financial* Euler equation only using, as Santos & Woodford (1997), non-arbitrage conditions. However, to attempt our objectives we need to assure that Kuhn-Tucker deflators exist, in the sense of Definition A1, and also that the discounted value of endowments, using these deflators, is finite.

On the other hand, as under Kuhn-Tucker multipliers the deflated value of individual endowments is finite, our transversality condition is equivalent to the requirement imposed by Magill & Quinzii (1996), provided that either short sales were avoided or individual endowments were uniformly bounded away from zero.

Corollary.

Fix $(p,q) \in \mathbb{P}$. *Under Assumption A1, given* $h \in H$ *suppose that either* $M = 0$ *or there exists* $\underline{w} \in \mathbb{R}^L_{++}$ *such that, at any* $\xi \in D$, $w^h(\xi) \geq \underline{w}$. *If* y^h *is an optimal allocation for agent* h *at prices* (p,q), *then for any plan of Kuhn-Tucker multipliers associated to* y^h, $(\gamma^h(\xi))_{\xi \in D}$, *we have,*

$$\lim_{T \to +\infty} \sum_{\xi \in D_T} \gamma^h(\xi) q(\xi) z^h(\xi) = 0.$$

Proof. Let $(\gamma^h(\xi))_{\xi \in D}$ be a plan of Kuhn-Tucker multipliers associated to y^h. We know that the transversality condition of Definition A1 holds. On the other hand, it follows directly from the debt constraint that,

$$\sum_{\xi \in D_T} \gamma^h(\xi) q(\xi) z^h(\xi) \geq -\sum_{\xi \in D_T} \gamma^h(\xi) p(\xi) M \geq -\left(\max_{l \in L} M_l\right) \sum_{\xi \in D_T} \gamma^h(\xi) \|p(\xi)\|_\Sigma.$$

Therefore, when $M = 0$ we obtain the result. Alternatively, assume that for any $\xi \in D$, $w^h(\xi) \geq \underline{w}$. As by Lemma A1 the sum $\sum_{\xi \in D} \gamma^h(\xi) p(\xi) w^h(\xi)$ is well defined and finite we have that

$$\sum_{\xi \in D} \gamma^h(\xi) \|p(\xi)\|_\Sigma < +\infty.$$

Thus, $\liminf_{T \to +\infty} \sum_{\xi \in D_T} \gamma^h(\xi) q(\xi) z^h(\xi) \geq 0$ which implies, using the transversality condition of Definition A1, that $\lim_{T \to +\infty} \sum_{\xi \in D_T} \gamma^h(\xi) q(\xi) z^h(\xi) = 0$. □

We end this Appendix with a result that determines sufficient requirements to assure that a plan of consumption and portfolio allocations is individually optimal. Note that the result below will assure that, when either short-sales were avoided, i.e., $M = 0$, or individual endowments were uniformly bounded away from zero, a budget feasible plan is individually optimal *if and only if* there exists a family of Kuhn-Tucker multipliers associated to it.

Proposition A2. (Sufficient conditions for individual optimality)

Fix a plan $(p,q) \in \mathbb{P}$. *Under Assumption A1, suppose that given* $y^h = (x^h, z^h) \in B^h(p,q)$, *there exists a family of Kuhn-Tucker multipliers* $(\gamma^h(\xi); \xi \in$

D) associated to y^h*. Then, if*

$$\lim_{T\to+\infty} \sum_{\xi\in D_T} \gamma^h(\xi)p(\xi)M \quad = \quad 0,$$

then y^h *is an optimal allocation for agent* h *at prices* (p,q)*.*

Proof. Note that, under the conditions above

$$\lim_{T\to+\infty} \sum_{\xi\in D_T} \gamma^h(\xi)q(\xi)z^h(\xi) = 0.$$

On the other hand, it follows from Euler conditions that, for each $T \geq 0$,

$$\sum_{\xi\in D^T} \mathcal{L}^h_\xi(y(\xi), y(\xi^-), \gamma^h(\xi);\, p,q) - \sum_{\xi\in D^T} \mathcal{L}^h_\xi(y^h(\xi), y^h(\xi^-), \gamma^h_\xi;\, p,q)$$

$$\leq - \sum_{\xi\in D_T} \gamma^h(\xi)q(\xi)(z(\xi) - z^h(\xi)).$$

Moreover, as at each node $\xi \in D$ we have that $\gamma^h(\xi)g^h_\xi(y^h(\xi), y^h(\xi^-); p, q) = 0$, each budget feasible allocation $y = ((x(\xi), z(\xi)); \xi \in D)$ must satisfy

$$\sum_{\xi\in D^T} u^h(\xi, x(\xi)) - \sum_{\xi\in D^T} u^h(\xi, x^h(\xi)) \leq - \sum_{\xi\in D_T} \gamma^h(\xi)q(\xi)(z(\xi) - z^h(\xi)).$$

Now, as the sequence $\left(\sum_{\xi\in D_T} \gamma^h(\xi)q(\xi)z^h(\xi)\right)_{T\in\mathbb{N}}$ converges, it is bounded. Thus,

$$\begin{aligned}\limsup_{T\to+\infty}\left(- \sum_{\xi\in D_T} \gamma^h(\xi)q(\xi)(z(\xi) - z^h(\xi))\right) &\leq \limsup_{T\to+\infty}\left(- \sum_{\xi\in D_T} \gamma^h(\xi)q(\xi)z(\xi)\right)\\ &\leq \lim_{T\to+\infty} \sum_{\xi\in D_T} \gamma^h(\xi)p(\xi)M = 0.\end{aligned}$$

Therefore,

$$U^h(x) = \limsup_{T\to+\infty} \sum_{\xi\in D^T} u^h(\xi, x(\xi)) \leq U^h(x^h),$$

which guarantees that the allocation $(x^h(\xi), z^h(\xi))_{\xi\in D}$ is optimal. □

Appendix B: On the Fundamental Value of Money

In the frictionless theory developed by Santos & Woodford (1997), that is, where debt constraints are non saturated, two (equivalent) definitions of the fundamental value of money make economic sense. The fundamental value is either (1) equal to the discounted value of future deliveries that an agent will receive for one unit of money that he buys and keeps forever; (2) equal to the discounted value of rental services, that coincides with deliveries, given the absence of any friction associated to debt constraint.

These concepts do not coincide when frictions are allowed. Thus, we adopt the second definition, that internalizes the role that money has: it allows for intertemporal transfers, although its deliveries are zero.

Proposition B1. (Non-existence of negative bubbles)

Under Assumption A1, given an equilibrium $\left[(p,q);((x^h,z^h);h\in H)\right]$, *at each node* $\xi \in D$, $q(\xi) \geq F(\xi,q,\gamma^h)$, *where* $(\gamma^h(\xi);\xi\in D)$ *denotes the agent's h plan of Kuhn-Tucker multipliers and*

$$F(\xi,q,\gamma^h) := \frac{1}{\gamma^h(\xi)} \sum_{\mu\in D(\xi)} \left(\gamma^h(\mu)q(\mu) - \sum_{\nu\in\mu^+} \gamma^h(\nu)q(\nu)\right),$$

is the fundamental value of money at $\xi \in D$.

Proof. By Proposition A1, there are, for each agent $h \in H$, non-negative shadow prices $(\eta^h(\xi);\xi\in D)$, satisfying for each $\xi \in D$,

$$\begin{aligned} 0 &= \eta^h(\xi)\left(q(\xi)z^h(\xi)+p(\xi)M\right); \\ \gamma^h(\xi)q(\xi) &= \sum_{\mu\in\xi^+} \gamma^h(\mu)q(\mu) + \eta^h(\xi)q(\xi). \end{aligned}$$

Therefore,

$$\gamma^h(\xi)q(\xi) = \sum_{\mu\geq\xi} \eta^h(\mu)q(\mu) + \lim_{T\to+\infty} \sum_{\mu\in D_T(\xi)} \gamma^h(\mu)q(\mu).$$

As multipliers and monetary prices are non-negative, the infinite sum in the right hand side of equation above is well defined, because its partial sums are increasing and bounded by $\gamma^h(\xi)q(\xi)$. This also implies that the limit of the (discounted) asset price exists. □

Note that the rental services that one unit of money gives at $\mu \in D$ are equal to $q(\mu) - \sum_{\nu \in \mu^+} \frac{\gamma^h(\nu)}{\gamma^h(\mu)} q(\mu)$. Thus, the *fundamental value* of money at a node ξ, as was defined in Proposition B1, coincides with the discounted value of (unitary) future rental services.

3
Intertemporal Discounting and Uniform Impatience

The uniform impatience assumption – see Magill & Quinzii (1994, Hypotheses A2 and A4), Hernandez & Santos (1996, Assumption C.3) or Magill & Quinzii (1996, Assumptions B2 and B4) – is a usual requirement for the existence of equilibrium in economies with infinite-lived debt-constrained agents. In this note, we fully characterize uniform impatience in terms of intertemporal discount factors. As an interesting side result, we obtain that the uniform impatience assumption does not hold for agents with hyperbolic intertemporal discounting.[1]

We follow the notation of Magill & Quinzzi (1994). Consider a framework where an infinite-lived price-taker agent demands L different commodities at any node ξ of an infinite countable event-tree D. This agent may trade financial assets to implement intertemporal transfers of wealth. At any $\xi \in D$, she receives a physical endowment $w(\xi) \in \mathbb{R}^L_+$ and makes contingent consumption plans, $x(\xi)$, to maximize her preferences, which are represented by a function $U : \mathbb{R}^{L\times D}_+ \to \mathbb{R}_+ \cup \{+\infty\}$. Aggregated physical endowments in the economy at node ξ are given by $W_\xi \in \mathbb{R}^L_{++}$. The date associated with a node $\xi \in D$ is denoted by $t(\xi)$.

Assumption A. Let $U^h(x) := \sum_{\xi\in D} u^h(\xi, x(\xi))$, where for any $\xi \in D$, $u^h(\xi,\cdot) : \mathbb{R}^L_+ \to \mathbb{R}_+$ is a continuous, concave and strictly increasing function. Also, $\sum_{\xi\in D} u^h(\xi, W_\xi)$ is finite.

Uniform impatience. *There are $\pi \in [0, 1)$ and $(v(\mu); \mu \in D) \in \mathbb{R}^{D\times L}_+$ such that, given a consumption plan $(x(\mu); \mu \in D)$, with $0 \le x(\mu) \le W_\mu$, for any $h \in H$, we have*

$$u^h(\xi, x(\xi) + v(\xi)) + \sum_{\mu>\xi} u^h(\mu, \pi' x(\mu)) > \sum_{\mu\ge\xi} u^h(\mu, x(\mu)), \qquad \forall \xi \in D,\ \forall \pi' \ge \pi.$$

Moreover, there is $\delta^h > 0$ such that, $w^h(\xi) \ge \delta^h v(\xi)$, $\forall \xi \in D$.

[1] Models of intertemporal choice in which agents have hyperbolic preferences have been widely studied recently (for some of these models, see, for example, Laibson (1998)).

The requirements of impatience above depend on both preferences and physical endowments. The assumptions imposed by Hernandez & Santos (1996) and Magill & Quinzii (1994, 1996) are particular instances of such requirements. Indeed, in Hernandez & Santos (1996), for any $\mu \in D$, $v(\mu) = W_\mu$. Also, since in Magill & Quinzii (1994, 1996) initial endowments are uniformly bounded away from zero by an interior bundle $\underline{w} \in \mathbb{R}^L_+$, they assume that $v(\mu) = (1, 0, \ldots, 0)$, $\forall \mu \in D$.

Our main result is,

PROPOSITION 1. *Suppose that Assumption A holds, that* $(W_\xi; \xi \in D)$ *is a bounded plan and that there is* $\underline{w}^h \in \mathbb{R}^L_+ \setminus \{0\}$ *such that,* $w^h(\xi) \geq \underline{w}^h$, $\forall \xi \in D$. *Moreover, there exists a function* $u^h : \mathbb{R}^L_+ \to \mathbb{R}_+$ *such that, for any* $\xi \in D$, $u^h(\xi, \cdot) \equiv \beta^h_{t(\xi)} \rho^h(\xi) u^h(\cdot)$, *where* $\beta^h_{t(\xi)} > 0$, $\rho^h(\xi) = \sum_{\mu \in \xi^+} \rho^h(\mu)$ *and* $\rho^h(\xi_0) = 1$. *For each* $t \geq 0$, *let* $s^h_t = \frac{1}{\beta^h_t} \sum_{r=t+1}^{+\infty} \beta^h_r$. *Then, the function* U^h *satisfies uniform impatience if and only if* $(s^h_t)_{t \geq 0}$ *is bounded.*

PROOF. Assume that $(W_\xi; \xi \in D)$ is a bounded plan. That is, there is $\overline{W} \in \mathbb{R}^L_+$ such that, $W_\xi \leq \overline{W}$, $\forall \xi \in D$. If $(s^h_t)_{t \geq 0}$ is bounded, then there exists $\overline{s}^h > 0$ such that, $s^h_t \leq \overline{s}^h$, for each $t \geq 0$. Also, since $\mathbb{F} := \{x \in \mathbb{R}^L_+ : x \leq \overline{W}\}$ is compact, the continuity of u^h assures that there is $\pi \in (0, 1)$ such that $u^h(x) - u^h(\pi' x) \leq \frac{u^h(\overline{W} + \underline{w}^h) - u^h(\overline{W})}{2\overline{s}^h}$, $\forall x \in \mathbb{F}$, $\forall \pi' \geq \pi$. Thus, uniform impatience follows by choosing $\delta = 1$ and $v(\xi) = \underline{w}^h$, $\forall \xi \in D$. Indeed, given a plan $(x(\mu); \mu \in D) \in \mathbb{R}^{L \times D}_+$ such that, $x(\mu) \leq W_\mu$ $\forall \mu \in D$, the concavity of u^h assures that, for any $\xi \in D$ and $\pi' \geq \pi$,

$$\begin{aligned}
\sum_{\mu > \xi} \beta^h_{t(\mu)} \rho^h(\mu) u^h(x(\mu)) \quad &- \quad \sum_{\mu > \xi} \beta^h_{t(\mu)} \rho^h(\mu) u^h(\pi' x(\mu)) \\
&\leq \quad \frac{\beta^h_{t(\xi)} s_t}{2\overline{s}^h} \rho^h(\xi) \left(u^h(\overline{W} + \underline{w}^h) - u^h(\overline{W}) \right) \\
&< \quad \beta^h_{t(\xi)} \rho^h(\xi) u^h(x(\xi) + v(\xi)) - \beta^h_{t(\xi)} \rho^h(\xi) u^h(x(\xi)).
\end{aligned}$$

Reciprocally, suppose that uniform impatience property holds. Then, given $(x(\mu); \mu \in D) \in \mathbb{R}^{L \times D}_+$ such that, $x(\mu) \leq W_\mu$, for all $\mu \in D$, there are $(\pi, \delta^h) \in [0, 1) \times \mathbb{R}_{++}$ and $(v(\mu); \mu \in D)$ $\mathbb{R}^{D \times L}_+$ satisfying, for any $\xi \in D$, $w^h(\xi) \geq \delta^h v(\xi)$, such that, for any $\xi \in D$,

$$\frac{1}{\beta^h_{t(\xi)}\rho^h(\xi)}\left[\sum_{\mu>\xi}\beta^h_{t(\mu)}\rho^h(\mu)u^h(x(\mu))-\sum_{\mu>\xi}\beta^h_{t(\mu)}\rho^h(\mu)u^h(\pi x(\mu))\right]$$

$$< u^h(x(\xi)+v(\xi))-u^h(x(\xi)).$$

It follows that, for any node ξ,

$$\frac{1}{\beta^h_{t(\xi)}\rho^h(\xi)}\left[\sum_{\mu>\xi}\beta^h_{t(\mu)}\rho^h(\mu)u^h(\underline{w})-\sum_{\mu>\xi}\beta^h_{t(\mu)}\rho^h(\mu)u^h(\pi\underline{w})\right]< u^h\left(\left(1+\frac{1}{\delta^h}\right)\overline{W}\right).$$

Therefore, we conclude that, for any $\xi \in D$,

$$\frac{1}{\beta^h_{t(\xi)}}\left(u^h(\underline{w}^h)-u^h(\pi\underline{w}^h)\right)\sum_{t=t(\xi)+1}^{+\infty}\beta^h_t \;<\; u^h\left(\left(1+\frac{1}{\delta^h}\right)\overline{W}\right),$$

which implies that the sequence $(s^h_t)_{t\geq 0}$ is bounded. □

Under the conditions of Proposition 1, if intertemporal discount factors are constant, i.e. $\exists c^h \in \mathbb{R}_{++} : \frac{\beta^h_{t(\xi)+1}}{\beta^h_{t(\xi)}} = c^h$, $\forall \xi \in D$, then $c^h < 1$ and $s^h_t = \frac{c^h}{1-c^h}$, for each $t \geq 0$. In this case, the utility function U^h satisfies the uniform impatience condition.

However, even with bounded plans of endowments, uniform impatience is a restrictive condition when intertemporal discount factors are time varying. For instance, if we consider *hyperbolic intertemporal discount factors*, that is, $\beta^h_t = (1+at)^{-\frac{b}{a}}$, where $b > a > 0$, then the function U^h, as defined in the statement of Proposition 1, satisfies Assumption A and the sequence s^h_t goes to infinity as t increases. Therefore, in this case, uniform impatience does not hold.

4
Collateralized Assets and Asymmetric Information

4.1
Introduction

In a two-period incomplete markets economy with nominal assets, Cornet & de Boisdeffre (2002) contribute to the theory of asymmetric information extending the classical non-arbitrage asset pricing procedure. They propose a decentralized mechanism in which agents anticipate asset prices and make, before the trade of commodities and assets, a refinement of their signals by precluding arbitrage opportunities.

In this context, a vector of asset prices is implementable as equilibrium only if the pooling information, obtained after the exclusion of arbitrage opportunities, is non-empty (see de Boisdeffre (2007)). In particular, there are financial structures for which only asset prices that fully reveal information are equilibria.

To illustrate this point, consider a two-period economy with two states of nature in the second period, $\{u, d\}$. There is only one commodity and only one asset, an Arrow security contingent to $s = u$. There are two types of agents, $\{A, B\}$. Individuals of type A are uninformed about the realization of the uncertainty, while individuals of type B know that the state $s = d$ will occur with certainty. Then, applying the refinement mechanism of Cornet & de Boisdeffre (2002), only when the asset has zero price the pooling information (obtained by the elimination of arbitrage opportunities) will be non-empty. Furthermore, in the unique equilibrium, there is no trade and uninformed agents become fully informed.

Note that when an agent anticipates an asset price, she believes that it is a non-arbitrage price. Thus, the individual never anticipates asset prices that give unbounded gains today without any risk tomorrow. For this reason, in the preceding example, type B individuals will never anticipate a positive price for the asset.

Moreover, the existence of a financial position that gives unbounded gains tomorrow without any cost today will be interpreted by any agent as a signal that some states of nature will not occur. This will induce agents to refine their private information. For instance, in our example, uninformed individuals will become fully informed when they anticipate a zero price.

A natural question arises: why will uninformed individuals anticipate a zero price (becoming fully informed)? For Cornet & de Boisdeffre (2002) this happens because, even without rational expectations, agents will coincide in their forecasted prices. In fact, given that credit markets are frictionless, informed individual always will anticipate a zero price.

On the other hand, when borrowers can default, financial markets need to implement mechanisms to protect lenders of excessive losses. Usually, these mechanisms will set limits on the amount of debt and, therefore, may also preclude (endogenously) the unbounded gains associated to an arbitrage opportunity. For instance, in the previous example, if we burden borrowers to constitute collateral requirements that will be seized in case of default, then a positive price for the Arrow security may emerge. In fact, the financial frictions induced by the collateral constraints will prevent type B agents from obtaining unbounded gains when the security has a positive price. Therefore, new equilibria will appear when default is allowed.

In this paper, we extend the model of Cornet & de Boisdeffre (2002) to allow for default and collateral, as in Geanakoplos & Zame (2002). We prove that equilibrium always exists, with no need to update information through a predefined (centralized or decentralized) mechanism. In this sense, the set of common non-arbitrage prices will increase when default is allowed.

Essentially, Cornet & de Boisdeffre (2002) point out that in the absence of default, a non-arbitrage price (common to every agent) may no longer exist, thus agents may need to update information to preclude arbitrage opportunities. When default is allowed and assets are collateralized, the existence of a margin between the market value of the collateral and the asset price will bound the amount of wealth that more informed borrowers extract from less informed lenders. Thus, endogenously, markets only allow for limited arbitrage opportunities and physical-financial trade becomes possible. Alternatively, from the perspective of an environment that allows agents to update information before commodities and assets can be traded, we focus on the second stage. Thus, we assure the existence of equilibrium independently of either the nature of the mechanism that was used to update information or the final distribution of information.

4.2 Collateralized Assets in an Economy with Asymmetric Information

4.2.1 Model

We consider a discrete time economy with two periods, $t \in \{0, 1\}$. There is no uncertainty at $t = 0$ and we denote by $s = 0$ the unique state of nature at this date. At $t = 1$, there is a finite set S of states of nature that can be reached. Let $\Sigma = \{0\} \bigcup S$.

There is a finite set of commodities, L, that can be traded at each $s \in \Sigma$. Commodities may suffer depreciation contingent on the state of nature. Thus, if one unit of good $l \in L$ is consumed in $t = 0$, an amount $Y_s(l, l')$ of the good $l' \in L$ is obtained at $s \in S$. Given $s \in S$, the matrix $Y_s = (Y_s(l, l'))_{(l,l') \in L \times L}$ has non-negative entries. Note that we allow for perishable and perfect durable goods in our economy.

As in Geanakoplos & Zame (2002), the financial sector is characterized by a finite set J of collateralized assets that can be negotiated at $t = 0$ and that make promises contingent on the states $s \in S$. More formally, each $j \in J$ is characterized by a process $(A(s, j);\, s \in S) \in \mathbb{R}_+^{L \times S}$ of state contingent real promises, and by its physical unitary collateral requirements, $C_j \in \mathbb{R}_+^L$. Collateral guarantees are always held by the borrowers.

Following Cornet & de Boisdeffre (2002), there is a finite number of agents, $h \in H$, that have a private information $S_h \subset S$ about the states of nature that will occur in $t = 1$. This private information is correct in the sense that the true state will belong to S_h. Therefore, the state of nature that actually will occur in $t = 1$ will belong into the *pooled information set*: $\underline{S} := \bigcap_h S_h$.

Each $h \in H$ may trade commodity and assets at $t = 0$, and make plans for consumption at each $s \in S_h$. Let $\Sigma_h = \{0\} \cup S_h$. We assume that consumer $h \in H$ is also characterized by his endowments, $w^h = (w_s^h; s \in \Sigma_h) \in \mathbb{R}_+^{\Sigma_h \times L}$, and by his preferences over consumption, that are represented by a function $U^h : \mathbb{R}_+^{\Sigma_h \times L} \to \mathbb{R}_+$.

The consumption allocation of $h \in H$, which includes the physical collateral requirements, is denoted by $x^h = (x_s^h;\, s \in \Sigma_h)$. We denote by θ_j^h (resp. φ_j^h) the quantity of asset j that agent h buys (resp. short-sells). Let $\theta^h = (\theta_j^h;\, j \in J)$ and $\varphi^h = (\varphi_j^h;\, j \in J)$.

We assume that each $h \in H$ observes commodity prices at $t = 0$, $p_0 \in \mathbb{R}_+^L$. Moreover, future (state contingent) commodity prices will be anticipated by h, denominated $p_s^h \in \mathbb{R}_+^L$ for each $s \in S_h$. Let $p^h = (p_s^h; s \in S_h)$. Also, each agent will form expectations about the unitary asset price, $q^h \in \mathbb{R}_+^J$.

Individuals trade assets and demand commodities after the anticipation of prices. As the only penalty in case of default is the seizure of the collateral guarantees, borrowers will pay the minimum between the depreciated value of the collateral and the market value of the original debt. Thus, as any agent h believes that her forecasted prices are correct, she will anticipate that the (unitary) nominal payment made by $j \in J$ at the state of nature $s \in S_h$ is given by $D_{s,j}(p^h) := \min\{p^h_s A(s,j), p^h_s Y_s C_j\}$.

Therefore, given prices (p_0, p^h, q^h), each agent $h \in H$ chooses an allocation $(x^h, \theta^h, \varphi^h)$ in his budget set $B^h(p_0, p^h, q^h) \subset \mathbb{E}^h := \mathbb{R}_+^{\Sigma_h \times L} \times \mathbb{R}_+^J \times \mathbb{R}_+^J$, which is given by the collection of vectors $(x, \theta, \varphi) = ((x_s)_{s\in\Sigma_h}, (\theta_j, \varphi_j)_{j\in J}) \in \mathbb{E}^h$ such that, $x_0 \geq \sum_{j\in J} C_j \varphi_j$ and

$$\begin{aligned} p_0 x_0 + q^h(\theta - \varphi) &\leq p_0 w_0^h; \\ p_s^h(x_s - w_s^h) &\leq p_s^h Y_s x_0 + \sum_{j\in J} D_{s,j}(p^h)\,(\theta_j - \varphi_j), \quad \forall s \in S_h. \end{aligned}$$

Note that the restriction $x_0 \geq \sum_{j\in J} C_j \varphi_j$ assures that first period consumption is compatible with financial promises, as the physical collateral required by the market is effectively constituted and held by the borrowers.

Definition. An equilibrium for our economy is given by a vector of prices $(\overline{p}_0, (\overline{p}^h, \overline{q}^h)_{h\in H})$ jointly with individual allocations $(\overline{x}^h, \overline{\theta}^h, \overline{\varphi}^h)_{h\in H} \in \prod_{h\in H} \mathbb{E}^h$, such that,

(a) For each $h \in H$, $(\overline{x}^h, \overline{\theta}^h, \overline{\varphi}^h) \in \mathrm{argmax}_{(x,\theta,\varphi)\in B^h(\overline{p}_0, \overline{p}^h, \overline{q}^h)}\, U^h(x)$.

(b) Physical and asset markets clear at states of nature that may occur, i.e.,

$$\sum_{h\in H}(\overline{x}_0^h, \overline{\theta}^h) = \sum_{h\in H}(w_0^h, \overline{\varphi}^h); \qquad \sum_{h\in H}\overline{x}_s^h = \sum_{h\in H} w_s^h + Y_s \sum_{h\in H} w_0^h, \;\; \forall s \in \underline{S}.$$

(c) For each $(h, h') \in H \times H$, $\overline{q}^h = \overline{q}^{h'}$. Moreover, at each $s \in \underline{S}$, $\overline{p}_s^h = \overline{p}_s^{h'}$, for all $(h, h') \in H \times H$.

Note that, as in de Boisdeffre (2007), we assume that in equilibrium forecasted prices coincide in the states of nature that may occur, although agents do not know the characteristics of the other individuals and information is private.[1] However, individuals do not need to coincide on the expected

[1]Without imposing rational expectations hypotheses, in a contemporaneous working paper, Daher, Martins-da-Rocha, Páscoa & Vailakis (2006) analyze a temporary equilibrium model with collateralized asset, in which agents are allowed to have beliefs about the characteristics of the others. In this context, the perfect foresight behaviour described in our model may appear, for some readers, as more natural.

commodity prices at states of nature $s \notin \underline{S}$.

4.2.2 Equilibrium Existence

As we point out earlier, the collateralization of financial contracts assures a natural mechanism to protect lenders in case of default and will induce endogenous bounds on short-sales. Thus, as in Geanakoplos & Zame (2002), equilibrium will always exist, even when agents do not make any refinement of their private information.

Theorem. Assume that,

A. For each $h \in H$, $(w_0^h, (w_s^h + Y_s w_0^h)_{s \in S_h}) \gg 0$.

B. For each $j \in J$, $C_j \neq 0$.

C. For each $h \in H$, $U^h : \mathbb{R}_+^{\Sigma_h \times L} \to \mathbb{R}_+$ is continuous, strictly increasing and quasi-concave.

D. For each $j \in J$, there exists $s \in \bigcup_{h \in H} S_h$ such that $(A(s,j), Y_s C_j) \gg 0$.

Then, given any vector of commodity prices forecasts $(\hat{p}_s^h,\ h \in H,\ s \in S_h \setminus \underline{S}) \gg 0$, there exists an equilibrium $\left[(\overline{p}_0, (\overline{p}^h, \overline{q}^h)_{h \in H});\ (\overline{x}^h, \overline{\theta}^h, \overline{\varphi}^h)_{h \in H}\right]$ for our economy in which $\overline{p}_s^h = \hat{p}_s^h$ for any $h \in H$ and $s \in S_h \setminus \underline{S}$.

Proof. Let $\underline{\Sigma} = \{0\} \cup \underline{S}$ and, for each $h \in H$, define $\eta^h = (\hat{p}_s^h)_{s \in (S_h \setminus \underline{S})}$. As in Geanakoplos & Zame (2002), collateral constraints jointly with feasibility conditions (item (b) of equilibrium definition) will assure that, under Assumption B, equilibrium individual allocations, if there exists, are uniformly bounded at the states $s \in \underline{\Sigma}$, independently of the price level. Now, as $\eta^h \gg 0$, budget restrictions will imply that consumption allocations of agent h at nodes $s \in S_h \setminus \underline{S}$ are also uniformly bounded.

Thus, for each $h \in H$ there exists a non-empty, compact and convex set $K^h \subset \mathbb{E}^h$ such that, to find an equilibrium we can restrict, without loss of generality, individual h to choose budget feasible allocations in K^h. Moreover, sets K^h are constructed in such way that feasible allocations are in the interior of $\prod_{h \in H} K^h$. Let

$$\mathbb{P} = \{(p_0, (\mu_s)_{s \in \underline{S}}, q) \in \mathbb{R}_+^L \times \mathbb{R}_+^{L \times \underline{S}} \times \mathbb{R}_+^J :\ \|(p_0, q)\|_1 = 1;\ \|\mu_s\|_1 = 1,\ \forall s \in \underline{S}\},$$

where $\|\cdot\|_1$ denotes the norm of the sum.

We will find an equilibrium for our economy as a fixed point of a set-valued mapping. To attempt this objective we define, for each $h \in H$, a correspondence $\Psi^h : \mathbb{P} \twoheadrightarrow K^h$ by

$$\Psi^h(p_0, (\mu_s)_{s\in\underline{S}}, q) = Argmax_{(x,\theta,\varphi)\in B^h(p_0,p^h,q)\cap K^h} U^h(x),$$

where $p^h = ((\mu_s)_{s\in\underline{S}}, \eta^h)$.

Moreover, if $\Delta_+^{L+J} := \{z \in \mathbb{R}_+^L \times \mathbb{R}_+^J : \|z\|_1 = 1\}$ and $\Delta_+^L := \{z \in \mathbb{R}_+^L : \|z\|_1 = 1\}$, let $\Psi_0 : \prod_{h\in H} K^h \twoheadrightarrow \Delta_+^{L+J}$ be the correspondence,

$$\Psi_0((x^h, \theta^h, \varphi^h)_{h\in H}) = Argmax_{(p_0,q)\in\Delta_+^{L+J}} \ p_0 \sum_{h\in H}(x_0^h - w_0^h) + q \sum_{h\in H}(\theta^h - \varphi^h).$$

and, for each $s \in \underline{S}$, define $\Psi_s : \prod_{h\in H} K^h \twoheadrightarrow \Delta_+^L$ by,

$$\Psi_s((x^h, \theta^h, \varphi^h)_{h\in H}) = Argmax_{\mu_s\in\Delta_+^L} \ \mu_s \sum_{h\in H}(x_s^h - w_s^h - Y_s w_0^h).$$

Now, it is not difficult to see that, under Assumptions A-C and as a consequence of Berge Maximum Theorem (see Aliprantis & Border (1999), Theorem 16.31) each one of the correspondences above is upper hemicontinuous and has non-empty, convex and compact values. Therefore, as the set $\mathbb{P} \times \prod_{h\in H} K^h$ is non-empty, convex and compact, it follows from Kakutani Fixed Point Theorem (see Aliprantis & Border (1999), Corollary 16.51) that there exists a fixed point, $[(\overline{p}_0, (\overline{\mu}_s)_{s\in\underline{S}}, \overline{q}); (\overline{x}^h, \overline{\theta}^h, \overline{\varphi}^h)_{h\in H}]$, for the set-valued mapping $\Omega : \mathbb{P} \times \prod_{h\in H} K^h \twoheadrightarrow \mathbb{P} \times \prod_{h\in H} K^h$ defined by,

$$\Omega((p_0, (\mu_s)_{s\in\underline{S}}, q); (x^h, \theta^h, \varphi^h)_{h\in H}) =$$

$$\prod_{s\in\underline{\Sigma}} \Psi_s((x^h, \theta^h, \varphi^h)_{h\in H}) \times \prod_{h\in H} \Psi^h(p_0, (\mu_s)_{s\in\underline{S}}, q).$$

Let, for each $h \in H$, $\overline{p}^h := ((\overline{\mu}_s)_{s\in\underline{S}}, \eta^h)$. Under Assumption C and D, analogous arguments to those made in Araujo, Páscoa & Torres-Martínez (2002, Lemma 2) will guarantee that

$$[(\overline{p}_0, (\overline{p}^h)_{h\in H}, \overline{q}); (\overline{x}^h, \overline{\theta}^h, \overline{\varphi}^h)_{h\in H}]$$

is an equilibrium for our economy. Finally, by construction, we have that, for each $h \in H$, equilibrium prices $\overline{p}_s^h$ coincide with forecasted prices $\hat{p}_s^h$, for any $s \in S_h \setminus \underline{S}$. $\square$

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