Mixing properties of a mechanical model of Brownian motion

Ph.D. thesis in mathematics
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1 Summary: Goals and Results.

In this work we study one dimensional mechanical system of infinitely many point particles interacting through elastic collisions with tagged particle, subject to a constant force. All point particles are field neutral and the mass of the tagged particle and field neutral particles are equal (the last condition can be removed, and we can consider the case of "heavy" tagged particle). A special feature of our model is that all neutral particles are equipped with a lifetime, which starts to discount after the first collision with the tagged particle. When lifetime expires, the point particle is removed from the system, while the tagged particle has infinite lifetime and remains in the system forever. The principal question is to determine long time behavior of the tagged particle. Our main goal in this work is to generalize results (and extend techniques) of [12] to as broad as possible class of distributions of lifetimes of the neutral particles in mechanical models of the Brownian motion. It is believed that in the 'original model', i.e. no lifetimes, during the evolution each neutral particle interacts with the tagged particle only finitely many times, and then 'flies away'. However, the tail of the distribution of the last collision is expected to decay polynomially, thus producing long term memory in the system. Our motivation comes from the fact, that the understanding of behavior of models with more general class of lifetime distribution might serve as another step forward in developing new (stochastic) tools which permit to analyze this long standing problem.

Applying an approach, which relies on imploring line covering techniques by random intervals, proposed in our collaboration with V. Beffara, V. Sidoravicius and M.E. Vares, we succeed to show that the Law of Large Numbers and the Invariance Principle for the rescaled position of the tracer particle holds as long as the lifetimes of the neutral particles are integrable random variables. Moreover, we are able to show that for the class of "physically relevant" distributions of lifetimes, such as inverse Gaussian (nonintegrable case!), the mechanical system at low density of neutral particles still undergoes periods of clustering (against the predictions in physics literature), and, in fact, is Bernoulli system (see for the detailed description of the results the next session).

The key element of the proof is to show that the mechanical systems under our assumptions undergo so called 'clustering process', *i.e.* has infinitely many regeneration instants, when the system looses completely influence of its 'past' on its 'future', and then to establish the tail asymptotic for the clustering event to occur. The control on tail decay determines the decay of correlations for the system. This is the hardest part of the approach. Once this is achieved, there is a number of available "standard techniques" which one applies in this case in a routine way to obtain the Law of Large Numbers and the Invariance Principle. It is important to notice that in this work we prove the LLN and the CLT only for the discrete dynamics, obtained by observing our system at the times of the collisions of the tagged particle with freshly coming neutral particles (*i.e.* at the moment of the first collision between the tagged particle and each neutral particle).

Future perspective: we believe that the idea to implore interval covering techniques is potentially very robust in the force driven systems, where one expects ballistic behaviour of a tracer particle at large time scales. Currently we are working on the extensions of these methods to the systems with neutral particles moving with Maxwellian velocities.

2 Introduction.

In this work we are concerned with the asymptotic behaviour of one dimensional mechanical systems, in particular with the motion of a tracer particle (t.p.) subject to a constant electric field in a random environment of neutral gas particles (n.p.s). This is one of fundamental questions in non-equilibrium statistical mechanics.

Our main model of interest, we call it *Model 1*, from "mechanical point of view" is exactly the same as in [12], and informally can be described as follows: we consider semi-infinite segment $[0, +\infty)$, with neutral particles initially located at positions $x_i > 0$, and the charged tracer particle is located at the origin. All particles including the t.p. have equal mass one. The constant force $f > 0$ acts only on the charged particle, while the neutral particles do not feel the force. At the moment of collision the tracer particle exchanges velocities with the neutral particle elastically. Neutral particles are initially standing, and interdistances between any two neighbouring particles are independent identically distributed exponential random variables with the parameter λ. However differently from [12], where the lifetimes of particles were taken as i.i.d. exponential random variables with parameter 1, in the present work we will assume that lifetimes χ_i of neutral particles are i.i.d. random variables which are integrable, $\mathbb{E}\chi_1 < +\infty$.

To obtain control on Model 1 we will consider auxiliary *Model 2* : a one-dimensional particle system in $\mathbb{R}_+ = [0, \infty]$ consisting of the t.p. interacting through elastic collisions with infinitely many point-like particles of an ideal gas and, as before, we suppose that all particles including the t.p. have equal mass one. Randomness enters through a measure under which the t.p. initially is at rest, located at the origin, and which governs "injection" of n.p.s in to the system in the following way: the n.p.s collide with the t.p. for the first time at Poisson times, i.e. the times between consecutive first (fresh) collisions of the t.p. with the n.p.s are i.i.d. exponential random variables with intensity $\varrho > 0$. In other words, differently from Model 1, where neutral particles are initially assumed to be standing at exponential interdistances, in Model 2 we will assume that the n.p.s arrive (are injected in to the system) at Poisson times at the position of the t.p. with zero incoming velocity. Then they remain in the system. Between collisions, a constant force $f > 0$ acts only on the t.p. while the n.p.s, as in Model 1, do not feel the force and do not interact among each other either. At collisions, the t.p. exchanges velocities with the n.p.s elastically. For convenience neutral particles are thought as undistinguishable pulses which only exchange velocities at collisions with each other and are re-labeled afterwards, i.e. we may think they cross each other. Multiple arrivals at the same first collision time with equal velocities are excluded by our construction. In this way, the proof of the fact that the dynamics of the system seen from the position of the t.p. is well-defined and governed by a uniform motion plus elastic collisions which obey the rules of classical mechanics follows the same line as the proof of the main theorem in [13]. In the case of Model 1 the existence of the dynamics is proved in [13].

In general situation (no lifetimes) the system is expected to be asymptotically free, it is expected that the velocity of the t.p. does not approach to equilibrium with an exponential decay in general. The reason for this behaviour lies in the appearance of multiple recollisions between the t.p. and the same neutral particles of the environment. When the t.p. accelerates, it can collide with a n.p. many times which influences the friction force and affects the limiting velocity of the t.p. In particular, a n.p. which has collided earlier with the t.p., can recollide after an arbitrarily large time. This potentially can create a long tail memory which is responsible for a power law behaviour of correlations. So far, there is unfortunately no satisfactory way of treating fully Newtonian systems without any stochastic dynamics. One alternative approach proposed by [12] is the introduction of lifetimes for the n.p.s. The notion of lifetimes had already appeared indirectly in the models of [8], [9] or [5] for instance, where geometric restrictions and conditions on the velocity of the n.p.s lead (explicitly or not) to uniformly bounded lifetimes. Explicit exponential lifetimes of the n.p.s in this context were introduced for the first time in [12]. In this work the authors were concerned with relaxing the condition of lower uniform boundedness of the interdistances in [4]. The model of [12] together with ours is some kind of asymptotic version of the virtually one-dimensional model of [5], the so-called modified Rayleigh gas with only horizontaly moving stick of height 1 subject to a constant force and collisions with n.p.s. In [5] the second dimension is only available to the n.p.s. Indeed, the horizontal initial velocity component v_1 and the vertical initial velocity component v_2 of the n.p.s in [5] determine their 'lifetimes' $\chi = \frac{1}{v_2}$, due to the assumption that the vertical velocity components of the stick and the n.p.s do not change at collision times. In other words, χ is the time each of the n.p. remains inside the strip available to the stick, and once a n.p. leaves the strip, it can be considered 'extinct' since it has become out of reach for the moving stick. In this way, our model can be interpreted as having zero horizontal velocity component and stick length going to zero, with the difference that our model is initially a Poisson system in time and not in space, and secondly the n.p.s in [5] enter the strip available to the stick in a Poissonian manner and therefore do not necessarily collide for the first time at exponential interdistances with the stick as well. On the other hand, allowing both v_1 and v_2 to be normal distributed at the same time, χ becomes inverse Gaussian (one-sided $\frac{1}{2}$-stable), a heavy tailed distribution, known to be the distribution of the first time a Brownian excursion hits some given level. This case is excluded in [5] where a uniform lower bound of the vertical velocity distribution is imposed to exclude 'long living' n.p.s and control recollisions. In view of this motivation, we suppose therefore that the t.p. has an infinite lifetime and the lifetimes of the n.p.s are i.i.d. with absolutely continuous distribution, independent of the first collision (arriving) times at which they start to be discounted.

In [4] and [11], the mechanical motion is 'delayed' with respect to a Markovian evolution where n.p.s are annihilated immediately after collisions (and thus neglecting recollisions at all), i.e. $t_n(\omega) \geq \widetilde{t}_n(\omega)$ where $\widetilde{t}_n(\omega)$ (resp. $t_n(\omega)$) is the hitting time of the t.p. of the position of the n-th n.p. at first collision in the Markovian (resp. interaction) dynamics in some configuration ω. In our case, it follows directly from the definition of the model that the Markovian velocity is an upper bound for the velocity of the t.p. in the true dynamics for any time, since here $t_n(\omega) = t_n(\widetilde{\omega})$ for the suitable Markovian configuration $\widetilde{\omega}$, since in contrast to the above models, fresh n.p.s can arrive during the interaction of the t.p. with a block of already moving particles and the t.p. does not have to 'go through' moving n.p.s in front of it first to reach the next fresh particle. In particular, the times when the n.p.s become extinct coincide in these two dynamics for our model. Observe also that in [4] and [11], beeing specified initially in space, intercollision times of the t.p. with standing n.p.s in the Markovian evolution are proportional to the square root of the interdistances, making them Weibull distributed, in contrast to our case, where the intercollision times in both Markovian and true dynamics coincide and are exponential. By symmetry between these models, in our case the interdistances in the Markovian evolution are Weibull with possibly different parameters. Still, since our model is truly one-dimensional, the t.p. cannot 'overtake' n.p.s during the evolution.

As all other models mentioned above, either directly or indirectly, our analysis relies essentially on the somewhat artificial notion of the so-called cluster times, i.e. first collision times at which the t.p. will not interact in the future with any n.p. it had collided with before including, the n.p. it collides with at this time. These times will then determine the mixing properties of the system. To construct a specific subset of cluster times, due to lack of mechanical arguments, we recurr first to the finding of conditions for the lifetime distribution which guarantee the existence of times of total extinction, i.e. stopping times with respect to the dynamics at which all previously moving n.p.s become extinct. The only memory of the past is then contained in the own velocity of the t.p. Upon this, since all particles have equal mass, cluster times are constructed by a simple mechanical argument and the general cluster times are then stochastically dominated by these special ones. This interpretation is indeed very close to the classical concepts of random covering problems in some different context and which allows us to interpret the set of moments of total extinction of the n.p.s as the so-called 'uncovered set' which is the closed image of an associated subordinator, i.e. an increasing Lévy process which represents here the continuous time analogue of a renewal process. Cluster times are then stochastically bounded by the characteristics of this process. One natural way of generalization is to allow other distributions for the interarrival times of the n.p.s. One might think that if the interarrival time distribution were substituted by some heavier tailed distribution like Weibull, the most natural one in a Markovian (annihilation) version of the dynamics in the original one-dimensional model in [11] or [4] as already noted above, one may expect that such heavy-tailed interarrival distributions (still with finite mean) favour the non-covering of $\mathbb{R}_+$ more than the light-tailed exponent ial distribution, that is the 'heaviness' of lifetime distributions which caused covering in the Poisson case might be weakend in the non-Poisson arrival case.

2.1 The mechanical models.

For convenience, we begin first with the formal description of Model 2. The state space of this system seen from the position of the t.p. as described in the introduction is given by

$$\Omega = \mathbb{R}_+ \times X = \{\omega = (V, x) : V \in \mathbb{R}_+, x \in X\}$$

where for any bounded $A \in \mathcal{B}(\mathbb{R}_+)$,

$$X = \{x \subseteq \mathbb{M} \times (0, +\infty) : \ \text{card}(x \cap (A \times \mathbb{R}_+) \times (0, +\infty)) < \infty \text{ and } \text{card}(x \cap (q, v) \times (0, +\infty)) \leq 1\}$$

is the (marked) environment of the n.p.s and $\mathbb{M} = \mathbb{R}_+ \times \mathbb{R}_+$ is the one-particle state space consisting of the (relative w.r.t. the position of t.p.) position q and (absolute) velocity v of one n.p. Here V stands for the velocity of the t.p. (the first particle) and x is the point process of all locally finite subsets (in space) of $\mathbb{M}$, marked by the lifetimes, whose projection $x : \Omega \to X$ is given by

$$x(\omega) = x_m(\omega)$$

where x_m are the moving n.p.s in the configuartion ω. As for the main quantities, we write $(q_n(t))_{n\in\mathbb{N}}$ for the positions of the n.p.s relative to the t.p. at time t, $(v_n(t))_{n\in\mathbb{N}}$ for the absolute velocities of the n.p.s at time t, $(\sigma_n)_{n\in\mathbb{N}}$ denote the interarrival times of the n.p.s and $(\chi_n)_{n\in\mathbb{N}}$ the lifetimes of the n.p.s, with the convention that the t.p. has an infinite lifetime, $q_n \leq q_{n+1}$ and if $q_n = q_{n+1}$, then $v_n < v_{n+1}$ and $\chi_n < \chi_{n+1}$. $q_0(t)$ denotes the position of the t.p. at time t. The topology of X is the one for which a fundamental system of neighbourhoods of a point $x \in X$ is given by

$$G_{A,B,C} = \{x' \in X : \mathrm{card}(x \cap (A \times B) \times C) = \mathrm{card}(x' \cap (A \times B) \times C)\}$$

with A, B and C open sets in $\mathbb{R}_+$ resp. $(0, +\infty)$ such that A is bounded with $x \cap (\partial A \times B) \times C = \emptyset$ where ∂A is the boundary of a set A. With this topology, X is a Polish space and we denote by $\mathcal{B}(X)$ its Borel σ-algebra resp. $\mathcal{B}(\Omega) = \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(X)$ the Borel σ-algebra of Ω. Initially, the t.p. is at rest and we endow $(\Omega, \mathcal{B}(\Omega))$ with the probability measure P, concentrated on the space of initial configurations

$$\Omega_0 = \{\omega \in \Omega : V(\omega) = 0,\ x(\omega) = \emptyset\},$$

under which the interarrival times $(\sigma_n)_{n\in\mathbb{N}}$ of the fresh n.p.s are i.i.d. exponential distributed with intensity $\varrho > 0$ and the lifetimes $(\chi_n)_{n\in\mathbb{N}}$ are i.i.d. and independent of $(\sigma_n)_{n\in\mathbb{N}}$, with common absolute continuous distribution function which we denote by $F_{\chi_1}(t) = \int_0^t f_{\chi_1}(y)dy$ for some density f_{χ_1}. When we speak of arrivals of n.p.s, we mean the times at which n.p.s appear (are injected) at the position of the t.p. with incoming velocity zero. The lifetime of the n-th n.p. starts to be discounted at the n-th arrival time $t_n = \sum_{i=1}^n \sigma_i$ with $t_0 = 0$, whereas the t.p. has an infinite lifetime. Thus, the initial configuration can be described by the point process $(t_n, \chi_n)_{n\in\mathbb{N}}$ on the upper right plane $\mathbb{H} = \mathbb{R}_+ \times (0, \infty)$ with intensity measure $n : \mathcal{B}(\mathbb{H}) \to \mathbb{R}_+$ given by

$$n(B) = \varrho \int_B F_{\chi_1}(dy)dt = \varrho \int_B f_{\chi_1}(y)dydt$$

for $B \in \mathcal{B}(\mathbb{H})$. One then associates to each n.p. its lifetime interval, i.e. the interval $I_n = (t_n, t_n + \chi_n)$ for the n-th. n.p. The dynamics, which we will denote by $(T^t)_{t\in\mathbb{R}_+}$, is then such that the t.p. is uniformly accelerated by the force $f > 0$ between consecutive collisions and at these collisions, it exchanges its velocity elastically with the n.p.s according to the mechanical rule

$$\Delta V = -\Delta v$$

where $\Delta V = V^+ - V^-$ and $\Delta v = v^+ - v^-$ are the velocity jumps and V^+ (V^-) and v^+ (v^-) are the outgoing (incoming) velocities of the t.p. and the n.p.s. The dynamics is right-continuous in the sense that at collision times the velocities are the outgoing ones, i.e. $V^+ = V$ and $v^+ = v$. In this way, the dynamics is P-a.s. well-defined on Ω (the same argument as in [4], Proposition A.1, works here as well). All statements about the dynamics will be understood such that they hold for those ω for which the dynamics is well-defined. If convenient, we may write as well for $\omega \in \Omega$ and $t > 0$,

$$\omega(t) = T^t\omega = (V_t(\omega), x(\omega(t))) = (V_t(\omega), x_m(\omega(t)) \cup x_0(\omega(t)))$$

where $V_t(\omega) = V(T^t\omega) = V(\omega(t))$ is the velocity of the t.p. at time $t \geq 0$ given by

$$V_t(\omega) = ft + \sum_{s\in J(V)\cap[0,t]} \Delta V_s(\omega)$$

with $V_0 = V_{t_n} = 0$ and $J(V) = \{t : \Delta V_t < 0\}$ is the set of jump times of the process $V = (V_t)_{t\in\mathbb{R}_+}$. The position $Q_t(\omega) = Q(T^t\omega) = Q(\omega(t))$ of the t.p. at time $t \geq 0$ for $\omega \in \Omega$ is then

$$Q_t(\omega) = \int_0^t V_s(\omega)ds$$

with $Q_0 = 0$. Note also that for $t \in J(V)$, the jumps are $\Delta V_t = V_t - V_{t-} = -ft$ with $V_{t-} = V_t^-$. The dynamics at the particular moments of arrivals of the n.p.s is called the discrete dynamics and is (well-)defined by

$$\omega(n) = T^n\omega = T^{t_n}\omega = (0, x_m(\omega(n)) \cup x_0(\omega(n)))$$

for any $n \geq 1$ on the associated configuration space

$$\Omega_1 = \{\omega \in \Omega : V(\omega) = 0\}.$$

Finally, the evolution of the initial measure under the discrete dynamics $(T^n)_{n\in\mathbb{N}}$ is denoted by

$$P_n(C) = P(T^{-n}C)$$

for $C \in \mathcal{B}(\Omega_1)$. As for Model 1 (cf. [11]), we make the suitable modifications and recall the description in the introduction and the notation for Model 2. The topology is analogous to the one of Model 2 with the difference that a configuration consists now of a sequence of positions, velocities and (residual) lifetimes instead of arrival times, velocities and (residual) lifetimes. We will maintain the nomenclatura of Model 2 and denote all related quantities of the dynamics of Model 1 by the same letters, e.g. $V_t = V_t(\omega)$ for $\omega \in \Omega$, making it clear at confusing places, about which model we are talking. Formally, all particles are initially at rest, i.e. $V = v_n = 0$ for any $n \geq 1$ and the initial measure P on $(\Omega, \mathcal{B}(\Omega))$ is such that the sequence of interparticle distances $\xi_n = q_n - q_{n-1}, n \geq 1$, is i.i.d. exponential with density $\lambda > 0$. The lifetimes $\chi_n, n \geq 1$, begin to be discounted at the times of first collisions $t_n, n \geq 1$, are i.i.d. with distribution function $F_{\chi_1} = F_{\chi_1}$ and independent of the whole sequence $(\xi_n)_{n\in\mathbb{N}}$. The initial configuration can then be described by the point process $(q_n, \chi_n)_{n\in\mathbb{N}}$ on $\mathbb{H} = \mathbb{R}_+ \times (0, \infty)$ with intensity measure $n : \mathcal{B}(\mathbb{H}) \to \mathbb{R}_+$ given by

$$n(A) = \lambda \int_A F_{\chi_1}(dy)dq$$

for $A \in \mathcal{B}(\mathbb{H})$. Analogous to Model 2, the dynamics $(T^t)_{t\in\mathbb{R}_+}$ is such that the t.p. is uniformly accelerated by the force $f > 0$ between consecutive collisions and at these collisions, it exchanges its velocity elastically with the n.p.s according to $\Delta V = -\Delta v$. By the same conventions as in Model 2, the dynamics is well-defined by [11], Remark 2, since the proof is independent of the lifetime (distribution). At difference to Model 2, here we have a initial Poissonian system in space and the first hitting times t_n depend heavily on possible recollisions, whereas in Model 2, the freshly arriving n.p.s 'rain down' on the t.p. at exponential interarrival times independent of recollisions.

In the sequel, we write generically $F_Z(z) = P(Z \leq z)$ resp. $F_Z(z) = P(Z \leq z)$ for the distribution function under the initial measure P resp. P for some random element Z on Ω or Ω and $\bar{F}_Z = 1 - F_Z$ resp. $\bar{F}_Z = 1 - F_Z$ for its tail. For a stochastic process we often also write the common abbreviation $Z = (Z_t)_{t\in\mathbb{R}_+}$.

We may state now our main results which concerns Model 1 (initially standing n.p.s). Let $\mu^{(n)}$ denotes the measure on the phase space as seen from the t.p. at the moment of the first collision with the n-th n.p.

Theorem (1.1) Under assumptions that the lifetimes $\{\chi_j\}_j$ of n.p.s are i.i.d. integrable random variables, $\mathbb{E}\chi_1 < +\infty$, and initial density of the particles is small enough, there exists a probability measure μ on X so that

$$\mu^{(n)} \;\to\; \mu \quad \text{weakly, as} \quad n \;\to\; +\infty.$$

The proof of the above theorem is based on the construction of so-called cluster index, and using this construction we in fact will get the following:

Corollary (1.2) Under assumptions of the Theorem (1.1) the following holds:
I) Let $\mathcal{M}_m^n$ denote the σ-field generated by the variables $\tau_i : m \leq i \leq n$, where τ_i is the interarrival time between $(i-1)$-th and i-th particle. Then under assumptions $\mathbb{E}\chi_1 < +\infty$, and initial density of the particles is small enough, there exist positive constants a, b such that

$$\sup_{A \in \mathcal{M}_0^k\,,\, B \in \mathcal{M}_{k+n}^{+\infty}} |\mu(B|A) - \mu(B)| \leq ce^{-c'n}$$

for all $k, n \geq 1$.
II) there exists a positive constant v_d (the drift velocity), so that

$$\frac{q_0(t)}{t} \;\to\; v_d, \qquad \mu_0 \;-\; \text{a.s.};$$

III) There exist a positive constant σ such that the process

$$\Big(\frac{q_0(ut) - v_d ut}{\sigma\sqrt{u}}\Big)_{0 \leq t \leq 1}$$

converge in law, on $D([0,1], \mathbb{R})$, to a standard Brownian motion, as $u \to +\infty$.

As we just mentioned above, the proof of the above theorem is based on the construction of a cluster index. This is the key part of the work, and is contained in Section 3.5. We will achieve this by doing coupling between three models, which in some stochastic sense dominate each other. First we will show that if lifetimes of n.p.s are integrable, then the Markovian (anihilation) version of Model 2 has infinitely many regeneration times with any density of the injected n.p.s. This will imply that the same property holds for the original version of Model 2. Finally, using comparisons and couplings, we will show that this implies that the Model 1 with small enough density of particles also posses the same property. Once this is achieved, we briefly outline consequences and give references to the all necessary steps, which are at this point rather standard, to achieve the proof of the Theorem and of the Corollary.

2.2 Annihilation dynamics.

We begin again with Model 2. Due to possible recollisions, without enlarging the underlying probability space, the velocity process $V = (V_t)_{t \in \mathbb{R}_+}$ is a non-Markovian càdlàg process which increases linearly in time proportional to the constant field $f > 0$ between successive collisions and at these collision times has negative jumps. A simple auxiliary Markovian dynamics can be achieved by annihilating the n.p.s immediately at each first collision with the t.p., which makes the corresponding velocity process Markovian due to the exclusion of recollisions. At each collision (which is then always a first one) the environment of the n.p.s is recreated according to the initial measure and the t.p. is the only moving particle in the system. The state space of this dynamics is denoted by $\widetilde{\Omega}$. Since the velocity change in the interval (t_{n-1}, t_n) is only due to the constant field $f > 0$, the corresponding velocity process, denoted by $\widetilde{V} = (\widetilde{V}_t)_{t \in \mathbb{R}_+}$, becomes

$$\widetilde{V}_t = f(t - t_{\vartheta_t})$$

for any $t \geq 0$, $\widetilde{V}_0 = 0$, where $\vartheta_t = \sum_{n \in \mathbb{N}} 1_{\{t_n \leq t\}}$ is the Poisson process of the number of freshly arriving n.p.s in the interval $[0, t]$ with mean $\frac{1}{\varrho}$ (this number coincides here with the number of collisions due to the exclusion of recollisions) and $t_{\vartheta_t} = \sum_{n=1}^{\vartheta_t} \sigma_n$ the time of the last arrival of a n.p. before time t. We also write $(\widetilde{T}^t)_{t \in \mathbb{R}_+}$ for the corresponding dynamics. It follows that $\widetilde{V}$ is a strong Markov process with infinitesimal generator

$$A^{\widetilde{V}} \varphi(v) = f \varphi'(v) + \varrho(\varphi(0) - \varphi(v))$$

with $v \in \mathbb{R}_+$ and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ bounded and continuous, and $\widetilde{V}$ hits the zero on the set of its jump times $J(\widetilde{V}) = \{t_n : n \in \mathbb{N}\}$. Note also that $\lim_{t \to \infty} \vartheta_t = +\infty$, and $\vartheta_t > n$ if and only if $t_n < t$. By the renewal theorem (cf. [6]) the law of the velocity process converges as $t \to \infty$ to the stationary distribution $\widetilde{\nu}$ given by

$$\widetilde{\nu}(B) = \varrho E \int_0^{\sigma_1} 1_B(\widetilde{V}_s) ds$$

for $B \in \mathcal{B}(\mathbb{R}_+)$, in particular $E_{\widetilde{\nu}} \widetilde{V}_. = \frac{\varrho f}{2} E \sigma_1^2 = \frac{f}{\varrho}$. As for the strong law of large numbers (SLLN) for the position process $\widetilde{Q} = (\widetilde{Q}_t)_{t \in \mathbb{R}_+}$ in the annihilation dynamics, it is sufficient to consider moments of first arrivals only, since $t_{\vartheta_t} \leq t < t_{\vartheta_t + 1}$ and $\widetilde{Q}_{t_{\vartheta_t}} \leq \widetilde{Q}_t \leq \widetilde{Q}_{t_{\vartheta_t + 1}}$ imply

$$\frac{\widetilde{Q}_{t_{\vartheta_t}}}{t_{\vartheta_t + 1}} \leq \frac{\widetilde{Q}_t}{t} \leq \frac{\widetilde{Q}_{t_{\vartheta_t + 1}}}{t_{\vartheta_t}}.$$

Since $\widetilde{Q}_{t_{\vartheta_t}} = \sum_{k=1}^{\vartheta_t} \widetilde{\xi}_k$ with $\widetilde{\xi}_k = \int_{t_{k-1}}^{t_k} \widetilde{V}_s ds$ is a sum of i.i.d. random variables with $E\widetilde{\xi}_1 = \frac{f}{2} E \sigma_1^2 = \frac{f}{\varrho^2}$, the existence of the Markovian drift $\widetilde{V}_D = \lim_{t \to \infty} \frac{\widetilde{Q}_t}{t}$

follows at once from the SLLN for the sequence of i.i.d. random variables $(\widetilde{\xi}_n)_{n\in\mathbb{N}}$ and from $\lim_{t\to\infty}\frac{\vartheta_t}{t}=\varrho$ P-a.s., namely

$$\widetilde{V}_D=\lim_{t\to\infty}\frac{\vartheta_t}{t}\Big(\frac{\sum_{n=1}^{\vartheta_t}\widetilde{\xi}_n}{\vartheta_t}\Big)=\frac{f}{\varrho}\quad P\text{-a.s. } (\widetilde{\nu}\text{-a.s.})$$

with $\widetilde{V}_D=\widetilde{V}_D(f)\in(0,+\infty)$ for $f>0$. Thus in the limit the force can be interpreted as being balanced by the friction exercited by the environment. Observe that due to possible recollisions, the velocity in the annihilation dynamics $(\widetilde{T}^t)_{t\in\mathbb{R}_+}$ is an upper bound for the velocity in the interacting dynamics $(T^t)_{t\in\mathbb{R}_+}$ in the sense that if we consider the original Model 2 dynamics and its annihilating version, with the same inter-arrival times of n.p.s, then for any $t\geq 0$ one has

$$V_t\leq\widetilde{V}_t.$$

In particular, $\limsup_{t\to\infty}\frac{Q_t}{t}\leq\widetilde{V}_D$ P-a.s ($\widetilde{\nu}$-a.s.) and $\xi_n<\infty$ P-a.s. since $\xi_n\leq\widetilde{\xi}_n$ for any $n\geq 1$ where as above, $\xi_n=\int_{t_{n-1}}^{t_n}V_s ds$ and $\widetilde{\xi}_n=\int_{t_{n-1}}^{t_n}\widetilde{V}_s ds$ are the travelled distances of the t.p. in the interacting dynamics $(T^t)_{t\in\mathbb{R}_+}$ resp. annihilation dynamics $(\widetilde{T}^t)_{t\in\mathbb{R}_+}$ between successive arrivals of fresh n.p.s.

Remark. As for the SLLN for the displacement of the t.p. in Model 1, one has

$$\frac{\widetilde{Q_n}}{\widetilde{t_n}}=\frac{n^{-1}\sum_{k=1}^{n}\xi_k}{n^{-1}\sum_{k=1}^{n}\sqrt{\frac{2\xi_k}{f}}}$$

and hence similarily for the asymptotic drift

$$\widetilde{V_D}=\lim_{t\to\infty}\frac{\widetilde{Q_t}}{t}=\frac{E\xi_1}{E\widetilde{\tau}_1}\quad P\text{-a.s.}$$

As for the comparison with the original Model 1 dynamics, this reads now for two initial configurations ω and $\widetilde{\omega}$ such that $x_0(\omega)=x_0(\widetilde{\omega})$,

$$V_{t_n}\leq\widetilde{V_{\widetilde{t_n}}}$$

for any $n\geq 1$. For the formal proof of that fact we refer to [14], Proposition 3.3.

3 Renewal structure of the dynamics.

Coming back to the mechanical model on the probability space $(\Omega, \mathcal{B}(\Omega), P)$, we make the following definitions using the same notation as in sections 1.1 and 1.2.

Definition 2.1.

1. A moment $t > 0$ is called a *time of total annihilation (extinction)* for the initial configuration $\omega \in \Omega$ iff $\omega(t) \in \widetilde{\Omega}$. In words, at time $t > 0$, all n.p.s such that the t.p. had met before time t are annihilated from the system (extinct, lifetimes of all of them expired) and the remaining 'memory of the past' is contained only in the velocity of the t.p. at that time. We denote by $D(\omega) \subseteq \mathbb{R}_+$ the set of all times of total annihilation for the configuration ω.

2. An arrival time $t_k > 0$ is called a *cluster time (regeneration time)* for the initial configuration $\omega \in \Omega$ iff the t.p. will never collide for any $t > t_k$ with the n.p.s it had collided with for $t \leq t_k$, including the freshly arriving n.p. it collides with at t_k. If t_k is a cluster time for ω, we call the integer k a *cluster index* for ω.

3. A cluster time t_k is called *double cluster time* for ω iff t_{k-1} is as well a cluster time for ω. The index k is then called *double cluster index* for ω.

Remark 2.2. If there is an infinite sequence of cluster indices $k_1 < k_2 < ...$ for the configuration ω, then $\mathbb{R}_+ = \bigcup_{n\in\mathbb{N}} J_{k_n}$ where on each of the intervals $J_{k_n} = [t_{k_n}, t_{k_{n+1}})$ with $J_{k_n} \cap J_{k_m} = \emptyset$ for any $n \neq m$, the t.p. can interact only with n.p.s born in such an J_{k_n} and only with them. One then might say that the dynamics is regenerative or 'splits into independent clusters' $C_n(\omega) = \{\omega(k_n + t) : 0 \leq t \leq t_{k_{n+1}}\}, n \geq 1$, since

$$P(T^{k_n} A \cap B) = P(A)P(B)$$

for any $A \in \mathcal{B}(\Omega)$, $B \in \{T^{-k_n} A : A \in \mathcal{B}(\Omega)\}$ and $n \geq 1$. Note also that D can be written as $D = \mathbb{R}_+ \setminus \bigcup_{n\in\mathbb{N}} I_n$, the so-called *uncovered set* of $\mathbb{R}_+$ (cf. [7],[12]) where $I_n = (t_n, t_n + \chi_n)$ are the lifetimes intervals. The zero is always contained in D if there are no initially moving n.p.s. Denoting by $\mathcal{C}$ the σ-algebra generated by the cluster times $(t_{k_n})_{n\in\mathbb{N}}$, then under the conditional measure $P^{\mathcal{C}} = P(.|\mathcal{C})$, the travelled distances $(\xi_{k_n+1})_{n\in\mathbb{N}}$ are i.i.d. Weibull with form parameter $\frac{1}{2}$ and scale parameter $\frac{f}{2\varrho^2}$ whose tail is given by

$$\bar{F}_{\xi_{k_1+1}}(\xi) = \exp(-\varrho\sqrt{\frac{2\xi}{f}})$$

resp. its density takes the form

$$f_{\xi_{k_1+1}}(\xi) = \frac{\varrho}{\sqrt{2f\xi}}\exp(-\varrho\sqrt{\frac{2\xi}{f}})$$

with variance $\Gamma = \frac{5f^2}{\varrho^4} > 0$. This distribution is also known as the Rayleigh distribution. Furthermore, under $P^{\mathcal{C}}$, the vectors $(\xi_{k_n+2}, ..., \xi_{k_{n+1}})_{n\in\mathbb{N}}$ are independent and for any $m \neq n$, the m-vector $(\xi_{k_m+2}, ..., \xi_{k_{m+1}})$ is independent of ξ_{k_n+1}. To construct a specific subset of cluster times of those as defined in Definition 2.1.2, as already indicated in the introduction, upon a condition which guarantees that the set D is non-trivial, the existence of cluster times is then shown by a simple mechanical argument.

3.1 Random covering interpretation.

In the light of the previous subsection, we consider the maximal residual lifetime (survival) process $R = (R_t)_{t\in\mathbb{R}_+}$ starting at $R_0 = r \geq 0$, i.e. the process which decreases linearly with slope one between consecutive arrivals and jumps upward at arrival times if the lifetimes of these n.p.s are larger than the incoming value of R. The magnitude of the jump is given by the difference between such lifetimes and the incoming values of R, hence the absolute value of R at jump times is the lifetime of the arriving n.p. at that time. If this process becomes zero at some (random) time, all n.p.s which where alive before this time are extinct (annihilated). Formally, the linear part of R in $t_{\vartheta_t} < t < t_{\vartheta_t+1}$ is given by

$$R_t = (R_{t_{\vartheta_t}} - \alpha_t)^+ = (R_{t_{\vartheta_t}} - \alpha_t) \vee 0 = (R_{t_{\vartheta_t}} - \alpha_t)1_{\{R_{t_{\vartheta_t}} > \alpha_t\}}.$$

This is decreasing linearly with slope one since α is increasing in $(t_{\vartheta_t}, t_{\vartheta_t+1})$. The jump of R at $t = t_n$ for some $n \geq 1$ is then

$$\Delta R_{t_n} = (\chi_n - R_{t_n-})^+,$$

in particular

$$R_{t_n} = \begin{cases} \chi_n & \text{if} \quad \chi_n > R_{t_n-} \\ R_{t_n-} & \text{if} \quad \chi_n \leq R_{t_n-}. \end{cases}$$

Altogether, R can be written as

$$R_t = (r - t + \sum_{k=1}^{\vartheta_t} (\chi_k - R_{t_k-})^+) \vee 0 \tag{1}$$

for any $t \geq 0$. As usual, P_r refers to distribution of R starting at the point $R_0 = r$, and we consider the naturally filtered complete probability space $(\Omega, \mathcal{F}^R, \mathcal{F}_t^R, P)$. Since the interarrival times of the n.p.s are i.i.d. exponential with intensity $\varrho > 0$, R forms a strong Markov process with inifinitesimal generator given by

$$\begin{aligned} A^R\varphi(r) &= -\varphi'(r) + \varrho E(\varphi(\chi_1 \vee r) - \varphi(r)) \\ &= \varrho \int_r^\infty \varphi'(s)\bar{F}_{\chi_1}(s)ds - \varphi'(r), \end{aligned}$$

where $\bar{F}_{\chi_1}(s) = P(\chi_1 > s)$ is the tail of the lifetime distribution and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ bounded, continuous with bounded derivatives. The process is strong Markov since

$$R_{\tau+t} - R_\tau = (\sum_{\vartheta_\tau+1}^{\vartheta_{\tau+t}} (\chi_k - R_{t_k-})^+ - t) \vee 0$$

starts at zero and is independent of $\mathcal{F}_\tau^R$ where τ is a finite $\mathcal{F}_t^R$-stopping time since $(\vartheta_{\tau+t} - \vartheta_\tau)_{t\in\mathbb{R}_+}$ is a Poisson process independent of $\mathcal{F}_\tau^R$ by the lack of memory property of the exponential distribution of the interarrival times. One reads the first equality in the form of the generator directly from the representation (1), whereas the second equality follows from the calculation

$$\begin{aligned} E\varphi(\chi_1 \vee r) &= \int_0^\infty \varphi(s \vee r)F_{\chi_1}(ds) \\ &= \varphi(r)F_{\chi_1}(r) + \int_r^\infty \varphi(s)F_{\chi_1}(ds) \\ &= \varphi(\infty) - \int_r^\infty \varphi'(s)F_{\chi_1}(s)ds \\ &= \int_r^\infty \varphi'(s)\bar{F}_{\chi_1}(s)ds + \varphi(r), \end{aligned}$$

using integration by parts in the third equality. Define now the survival probability and the first time of total extinction as

$$\begin{aligned} \pi(r) &= P(R_t > 0 \text{ for any } t \geq 0 | R_0 = r) \\ H(r) &= \inf\{t > 0 : R_t = 0 | R_0 = r\} \end{aligned}$$

with $\pi(0) = 0$ and $\pi(\infty) = 1$ resp., and $H(\infty) = \infty$ P-a.s. The crucial observation is that by the Markov property of R, π is a bounded invariant function, i.e. $E_r\pi(R_t) = \pi(r)$ and hence $A^R\pi(r) = 0$ if π is in the domain of A^R (which will indeed follow by the verification argument below). Note that $(\pi(R_t))_{t\in\mathbb{R}_+}$ is a martingale iff $A^R\pi(r) = 0$ by standard argument. Indeed, both directions follow by Dynkin's formula since $A^R\pi(r) = 0$ on the one hand, and on the other hand, $(\int_0^t A^R\pi(R_s)ds)_{t\in\mathbb{R}_+}$ is a continuous martingale of finite variation with mean zero, hence it must be zero. Now plugging in differentiation to get rid of the integral term yields

$$\pi''(r) = -\varrho \bar{F}_{\chi_1}(r)\pi'(r).$$

Solving this equation and using integration, subject to the initial condition $\pi(0) = 0$, gives the expression for the survival probability

$$\pi(r) = \beta \int_0^r \exp(-\varrho \int_0^s \bar{F}_{\chi_1}(u)du)ds$$

for some constant $\beta \geq 0$. In particular π is continuous and differentiable. Since $0 \leq \pi \leq 1$ and $\lim_{r\to\infty}\pi(r) = \pi(\infty) = 1$, if the right hand side above does not converge, it follows that one must have $\beta = 0$ and hence $\pi(r) = 0$ resp. $H(r) < \infty$ P-a.s. for any $r > 0$. In other words, the zero is a recurrent state for R. The case $R_0 = r > 0$ corresponds to initially moving n.p.s with r as the maximum residual lifetime of all these already moving n.p.s at time zero. On the other hand, if the above integral converges as $r \to \infty$, again from $\lim_{r\to\infty}\pi(r) = \pi(\infty) = 1$ it follows that

$$\beta^{-1} = \int_0^\infty \exp(-\varrho \int_0^s \bar{F}_{\chi_1}(u)du)ds < \infty \tag{2}$$

and $\pi(r) > 0$ for any $r > 0$. Since π is a positive bounded invariant function, $(M^\pi_{t\wedge H(r)})_{t\in\mathbb{R}_+}$ is a stopped martingale where we have defined

$$M^\pi_t = \pi(R_t) = P_{R_t}(R_s > 0 \text{ for any } s \geq 0)$$

with $M^\pi_0 = \pi(r) > 0$ for any $r > 0$ and $0 < M^\pi_t \leq 1$. By optimal stopping, $EM^\pi_t = EM^\pi_0 = \pi > 0$ for any $t \geq 0$. By continuity, $\pi(\infty) = 1$ and the martingale convergence theorem, the limit $M^\pi_\infty = \lim_{t\to\infty} M^\pi_t$ exists and equals $1_A = M^\pi_\infty = 1$ P-a.s. for the invariant set $A = \{R_t > 0 \text{ for any } t \geq 0\}$. Taking expectation and bounded convergence yields $\pi(r) = E1_A = 1$ for any $r > 0$ which entails $\lim_{t\to\infty} R_t = \infty$ resp. $H(r) = \infty$ P-a.s. for any $r > 0$. If $r = 0$, we set $\widetilde{M^\pi_t} = M^\pi_{t_1\vee t}$ starting at $\widetilde{M^\pi_0} = \pi(\chi_1) > 0$ instead of M^π_t and proceed as before. Hence if $r = 0$, then $D = \{0\}$ P-a.s. resp. if $r > 0$, then $D = \emptyset$ P-a.s. in this case.

3.2 Distribution of the first time of total extinction.

The above martingale method is not adequate to determine the distribution of the first time of total extinction. But using well-known concepts from excursion theory, one can show that this equals to the first passage time of an associated subordinator above some fixed level, and this can be calculated explicitly in terms of the characteristics of the subordinator. More precisely, we suppose that $R_0 = 0$ and set for $H = H_0$ for brevity. Define the local time of R in $[0, t]$ as

$$\tau_t = \int_0^t 1_{\{R_s=0\}} ds$$

which equals to the amount of time less or equal to t which R spents at zero. It is constant on time intervals where R is away from zero (excursion intervals) and increases on time intervals where R hits the zero, hence if we denote $Y_t = 1_{\{R_t=0\}}$ for short, then $Y_t = 1$ iff $\tau_{t+s} > \tau_t$ for any $s > 0$. From the construction of the strong Markov process R and the fact that the residual arrival time $\delta_t = t_{\vartheta_t+1} - t$ at time $t \geq 0$ is again exponential and has no atom at zero, one sees that $P_0(H = 0) = P(\delta_. > 0) > 0$ which entails by Blumenthal's zero-one law that $P_0(H = 0) = 1$, i.e. the process R will hit the zero a.s. infinitely often during any initial time interval and thus the zero is regular for itself. It then follows from [3], Chap. V, that the right-continuous inverse of the local time of R denoted by

$$S_t = \tau_t^{-1} = \inf\{s \geq 0 : \tau_s > t\}$$

is a subordinator, i.e. an increasing Lévy process starting at zero whose Laplace exponent $\Phi(\theta) = -\log Ee^{-\theta S_1}$, $\theta \geq 0$, is given by the Lévy-Khintchin formula

$$\Phi(\theta) = \eta\theta + \int_0^\infty (1 - e^{-\theta s})\Pi(ds) = \theta(\eta + \int_0^\infty e^{-\theta s}\bar{\Pi}(s)ds)$$

where Π is a Borel measure on $(0, \infty)$ (Lévy measure) such that $\int_0^1 s\Pi(ds) = \int_0^1 \bar{\Pi}(s)ds < \infty$ where $\bar{\Pi}$ denotes its tail, and $\eta = \lim_{\theta\to\infty} \frac{\Phi(\theta)}{\theta} > 0$ the drift. Equivalently, the Lévy-Itô decomposition reads

$$S_t = \eta t + \int_0^\infty sN_t(ds),$$

where N is the Poisson random measure $N_t(\Lambda) = \sum_{0<s\leq t} 1_\Lambda(\Delta Z_s)$, $t > 0$, $\Lambda \in \mathcal{B}((0, \infty))$, on $(0, \infty)^2$ associated to the jumps of Z with $EN_1(\Lambda) =$

$\Pi(\Lambda)$ as its intensity measure. The domain (time axis) of S is $\mathbb{R}_+$ and its image (range) the closure of $D = \{t \geq 0 : R_t = 0\}$. Observe that $D^c = \bigcup_{t \in J(S)} (S_{t-}, S_t)$, so that the 'gaps' in $\mathbb{R}_+$ ('pre-clusters' in the interpretation of the mechanical model) are the jumps of S where $J(S)$ is the set of its jump times. Following ideas from excursion theory, one has here two time scales (domains), the natural time scale of the Markov process R (of the dynamics $(T^t)_{t \in \mathbb{R}_+}$) and the time scale of the subordinator S. Furthermore,

$$\tau_t = \inf\{s \geq 0 : S_s > t\},$$

i.e. the local time of R coincides with the local time of S, and the potential (renewal) measure is given by

$$U(B) = \int_B P(R_t = 0)dt = \int_0^\infty P_{S_t}(B)dt$$

for $B \in \mathcal{B}(\mathbb{R}_+)$ resp. $U([0,t]) = E\tau_t$ where P_{S_t} denotes the distribution of S_t under P. As one sees from the Lévy-Khintchin formula, if S' is another subordinator with image the closure of D, there is a constant $c > 0$ such that $S'_t = S_{ct}$ P-a.s. for any $t \geq 0$. In this way, the subordinator is uniquely characterized up to a constant. Note also that $\tau_\infty = \infty$ P-a.s. iff the zero is recurrent for R. Indeed, Since ϑ is a Poisson process on $\mathbb{R}_+$, the number $N(A)$ of coordinates which fall into the set $A \in \mathcal{B}(\mathbb{H})$ is Poisson distributed with mean $n(A)$ where n is the intensity measure as in section 1.1. Hence for $A_t = \{(s,u) : 0 < t - s < u\}$ and $t > 0$ fixed, the probability that at time t all n.p.s born before t are extinct is given by

$$P(R_t = 0) = P(N(A_t) = 0) = \exp(-\varrho \int_0^t \bar{F}_{\chi_1}(t-s)ds)$$

with the mean number of alive n.p.s at time $t > 0$ given by

$$n(A_t) = EN(A_t) = \varrho \int_0^t \bar{F}_{\chi_1}(t-s)ds.$$

On the other hand, by symmetry and since $\bar{F}_{\chi_1}(t-s) = \bar{F}_{\chi_1}(s) - P(s < \chi_1 \leq t-s)$ on $(0, \frac{t}{2}]$, one has

$$\int_0^t \bar{F}_{\chi_1}(t-s)ds = 2\int_0^{t/2} \bar{F}_{\chi_1}(t-s)ds \leq 2\int_0^t \bar{F}_{\chi_1}(s)ds.$$

Hence in terms of the subordinator S, if $\beta^{-1} = \infty$, it follows that

$$E\tau_\infty = \int_0^\infty P(R_t = 0)dt \geq \int_0^\infty \exp(-2\varrho \int_0^t \bar{F}_{\chi_1}(s)ds)dt = \widetilde{c}\beta^{-1} = \infty$$

for some constant $\widetilde{c} > 0$, thus $\tau_\infty = \infty$ P-a.s. and D is unbounded. In particular, the renewal measure of S is a Radon measure with density

$$u(t) = \exp(-\varrho \int_0^t \bar{F}_{\chi_1}(t-s)ds).$$

To determine the distribution of H, note that taking the marginal distribution of the jump of S at its local time τ_t in Proposition 2, Chap. III, in [2], we can write for the fourth equality below

$$F_H(t) = P(H \leq t) = P(\exists s > 0 : S_s > t) = P(S_{\tau_t} > t) = \int_0^t \bar{\Pi}(t-s)u(s)ds.$$

In view of Definiton 2.1.1, we have therefore

$$P(\exists s \leq t : \omega(s) \in \widetilde{\Omega}) = \int_0^t \bar{\Pi}(t-s)u(s)ds.$$

Remark 2.2.1. To recover the characteristics of S, observe that on the one hand, the Laplace transform of the renewal measure can be expressed as

$$\frac{1}{\Phi(\theta)} = \int_0^\infty \exp(-\theta t - \varrho \int_0^t \bar{F}_{\chi_1}(t-s)ds)dt,$$

whereas by the Lévy-Khintchin formula we have the drift $\eta = \lim_{\theta\to\infty} \frac{\Phi(\theta)}{\theta}$ and the Laplace transform $\int_0^\infty e^{-\theta t}\bar{\Pi}(s)ds = \frac{\Phi(\theta)}{\theta} - \eta$. Inverting the last equation, one can recover the tail of the Lévy measure.

3.3 Construction of cluster times.

The divergence of β^{-1} is a sufficient condition to guarantee the existence of cluster times as defined in Definition 2.1.2. Conversely though, from the convergence of (2) one cannot deduce the absence of cluster times as defined in 2.1.2 since it does not exclude a more mechanical 'continuous loss of memory' as already indicated in section 1 and Remark 2.2. To see sufficiency, we apply a simple mechanical argument. Suppose now that that $\beta^{-1} = \infty$ and $r = 0$, in particular $H < \infty$ P-a.s. Given $H = s$, fix a constant $a > 0$ and denote by $V_\star = V_s > 0$ the velocity of the t.p. which is positive since the subordinator has positive drift. Then the outgoing velocity of the next arriving n.p. is the constant $v^+_{\widehat{\vartheta}} > V_\star + fa$ on the set $\{\delta_s > a\}$ due to the exclusion of recollisions where we have set $\widehat{\vartheta} = \vartheta_s + 1$ and $\delta_s = t_{\widehat{\vartheta}} - s$ is the residual arrival time as usual. Choosing now the lifetime of the n.p. smaller than $b = \widehat{t}$ such that

$$\widehat{t}v_{\widehat{\vartheta}}^{+} > \widehat{Q}_{\widehat{t}} = \int_0^{\widehat{t}} \widehat{V}_s ds$$

where $\widehat{V}_t = ft$ is the 'free' velocity, gives $b = \frac{2}{f}(V_\star + fa)$ and the lifetime of the n.p. is smaller than the time the t.p. would need to catch it up in the 'free' dynamics, i.e. with no further n.p.s arriving in $(t_{\widehat{\vartheta}}, \infty]$. Together with the distribution function $F_H(t) = (\bar{\nu} \star u)(t)$ of H as determined in the previous section and choosing the appropriate constants $a > 0$ and $b > 0$ as above, one gets with the event $A_H = \{\delta_H > a\} \cap \{\chi_{\vartheta_H+1} \leq b\}$ that

$$\begin{aligned} &P(\exists s \leq t : t_{\vartheta_s+1} \text{ is a cluster time}) \\ &\quad \geq P(\{H \leq t\} \cap A_H) = \int_0^t P(A_s|H = s) F_H(ds) \\ &\quad = \int_0^t e^{-\varrho a} F_{\chi_1}(b) F_H(ds) = e^{-\varrho a} F_{\chi_1}(b) \int_0^t \bar{\Pi}(t-s) u(s) ds \end{aligned}$$

using the conditional independence of the interarrival times $(\sigma_n)_{n\in\mathbb{N}}$ and the i.i.d. lifetimes $(\chi_n)_{n\in\mathbb{N}}$, the lack of memory of the residual arrival time δ_s and the absolute continuity of the lifetime distribution. This amounts to say that the distribution of the cluster times is dominated stochastically by F_H which on its side is determined by the tail of the Lévy measure and the renewal density of the associated subordinator. It will be convenient later to consider also double cluster times as in Definition 2.1.3 which is indeed analogous to the notion of 'good cluster indices' in [4]. To produce them in our case, replace the event $A_H = \{\delta_H > a\} \cap \{\chi_{\vartheta_H+1} \leq b\}$ by $A'_H = A_H \cap B_H$ where $B_H = \{H_{\vartheta_H+1} > b\} \cap \{\chi_{\vartheta_H+2} \leq 2b\}$ and then one has as above

$$P(\exists s \leq t : t_{\vartheta_s+1} \text{ is a double cluster time}) \geq c' F_H(t)$$

for some constant $c' = c'(a, V_\star, f, \varrho)$ with $0 < c' < \widehat{c}$. In particular, if $H_1 = H$, $H_k = \inf\{t > H_{k-1} : R_t = 0\}$ and $Y'_k = 1_{\{t_{\vartheta_{H_k}+1} \text{ is a double cluster time}\}}$ for $k \geq 1$, then $P(Y'_1 = 1) \geq c' > 0$ and $P(Y'_k = 1|Y'_1, ..., Y'_{k-1}) \geq c' > 0$ for $k \geq 2$. It follows that there is a $\gamma' > 0$ such that $\liminf_{n\to\infty} \frac{1}{n} \sum_{k=1}^n Y'_k \geq \gamma' > 0$ P-a.s.

3.4 Remark on renewal arrivals.

In most of "realistic" models interarrival times of the n.p.s are not exponential, and, therefore, one would like to drop this condition, i.e. we do not impose necessarily exponential times. As already observed, a natural choice in the context of the one-dimensional models of [4] and [11] would be a Weibull distribution. But one immediate consequence of dropping the exponential distribution is the loss of the Markov property of the associated process R of maximal survival, defined in the same way as in section 2.1, since the counting process ϑ is a Markov process if and only if the interarrival times are i.i.d. exponential due to the lack of memory of the exponential distribution. If ϑ is not Poisson, the future of the process R depends through the distribution of the time to the next arrival on the past via the time spent since the last arrival of a particle. If ϑ is a renewal process with finite mean $\frac{1}{\varrho}$ and existing second moment of the interarrival times, with absolute continuous interarrival distribution with density f_{σ_1} and corresponding distribution function F_{σ_1}, which is independent of the lifetimes, then one can make R being Markovian without enlarging the underlying probability space $(\Omega, \mathcal{B}(\Omega), P)$ by considering the bivariate process $(R, \alpha) = (R_t, \alpha_t)_{t\in\mathbb{R}_+}$, where $\alpha_t = t - t_{\vartheta_t}$ is the spent time since the last arrival before time $t > 0$. Note that the annihilation dynamics in the renewal case is analogous to the Poisson case in 1.2 where the Markovian drift becomes now $\widetilde{V}_D = \widetilde{V}_D(f) = \frac{\varrho f}{2} E\sigma_1^2$ by invoking the renewal theorem. Coming back to (R, α), for some $t > 0$ fixed, this process evolves deterministically for $s \in (t, t + \alpha_t)$ as $(R_s, \alpha_s) = (R_t - (s - t), \alpha_t - (s - t))$ and for $s = t_{\vartheta_t+1}$, $(R_s, \alpha_s) = (\chi_{\vartheta_t+1} \vee R_{t_{\vartheta_t+1}-}, 0)$. Since α is a strong Markov process and ϑ a counting process, (R, α) is a strong Markov process with $R_{t+s}(\omega) - R_s(\omega) = r + R_t(T^s\omega)$. The infinitesimal generator of (R, α) can be calculated from the same arguments as the one in the Poisson case as

$$A^{(R,\alpha)}\varphi(r, a) = \frac{f_{\sigma_1}(a)}{\bar{F}_{\sigma_1}(a)} \int_r^\infty \partial_y\varphi(y, a)\bar{F}_{\chi_1}(y)dy - \partial_r\varphi(r, a) + \partial_a\varphi(r, a)$$

for $\varphi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ in the domain of the generator, noting that α increases deterministically between two consecutive arrivals and if $\varrho(a)da + o(da)$ denotes the probability that there is an arrival in the interval $(a, a+da)$ conditionally that up to time $a > 0$ there is no arrival, then $\varrho(a) = \frac{f_{\sigma_1}(a)}{\bar{F}_{\sigma_1}(a)}$ is the hazard (failure) rate for the distribution of the interarrival times for which the well-known relations $\varrho(a) = \lim_{h\to 0} \frac{1}{h} P(a < \sigma_1 < a + h|\sigma_1 > a)$ and $\bar{F}_{\sigma_1}(a) = \exp(-\int_0^a \varrho(s)ds)$ hold. Note also that $(\vartheta_t - \int_0^t \varrho(\alpha_s)ds)_{t\in\mathbb{R}_+}$ is a martingale, i.e. $(\int_0^t \varrho(\alpha_s)ds)_{t\in\mathbb{R}_+}$ is the compensator of $(\vartheta_t)_{t\in\mathbb{R}_+}$. Analogously to the Poisson case, $(\varphi(R_t, \alpha_t))_{t\in\mathbb{R}_+}$ is a martingale iff

$$\partial_r\varphi(r, a) = \varrho(a) \int_r^\infty \partial_y\varphi(y, a)\bar{F}_{\chi_1}(y)dy + \partial_a\varphi(r, a).$$

3.5 Comparison and coupling.

The main goal of this subsection is to construct a suitable comparison between the original dynamics of the Model 1 and convenient interpretation as a line covering problem of the Markovian version of Model 2, which was described in previous subsections. The key difficulty is that in the original dynamics of Model 1, as well as of Model 2, we can say very little about the distribution of interarrival times of n.p.s, and to treat a line covering problem with such a complicated and dependent interdistances seems to be out of reach. We will take a different route: we will show that there is a stochastic comparison between the line covering problem with exponential interdistances (in the case of absence of covering) and low n.p.s density case in Model 1.

For this purpose we will introduce an auxiliary covering process, which we call ε-reinforced covering process. As in the original covering process associated with Markovian version of Model 2, we will assume that start-points where the covering intervals begin, are distributed according to the Poisson law with intensity $\varrho > 0$. However, from each such point now we will allow to start multiple number of covering intervals, which are all going rightwards. The probability that out of a given point we start k, $k \geq 1$, intervals is geometric with parameter ϵ, and equals to $\epsilon^{k-1}(1-\epsilon)$ for $0 < \epsilon < 1$, independently for each Poissonian point. The lengths of all intervals are chosen independently, and are distributed according to F_{χ_1}. Thus the only difference with the original model is, that in this reinforced model, we have multiple intervals starting from one point and the one with the maximal length is the one which is important. If $\{\chi_{nk} : n \in \mathbb{N}, 1 \leq k \leq \mathcal{N}_n^\epsilon\}$ is the infinite independent array of lifetimes associated to the ϵ-reinforced covering process, where $(\mathcal{N}_n^\epsilon)_{n\in\mathbb{N}}$ are i.i.d. geometric with $P(\mathcal{N}_1^\epsilon = k) = \epsilon^{k-1}(1-\epsilon)$, then $\chi_n^\epsilon = \max_{1\leq k\leq \mathcal{N}_n^\epsilon} \chi_{nk}, n \geq 1$, is i.i.d. with tail $\bar{F}_{\chi_1^\epsilon} = (1-\epsilon)\sum_{k\in\mathbb{N}} \epsilon^{k-1}\bar{F}_{\chi_1}^k$ by conditional independence. Note that the condition for non-covering for the reinforced model is the same as for the non-reinforced one with $(\chi_n)_{n\in\mathbb{N}}$ replaced by $(\chi_n^\epsilon)_{n\in\mathbb{N}}$. Since $\int_0^\infty \exp(-\varrho \int_0^s \bar{F}_{\chi_1^\epsilon}(u)du)ds \geq \int_0^\infty \exp(-\varrho \int_0^s \bar{F}_{\chi_1}(u)du)ds$ for any $0 < \epsilon < 1$ by $\bar{F}_{\chi_1} > \bar{F}_{\chi_1}^k$, for any $k \geq 1$ we have the following simple property comparing these two processes.

Proposition 2.4.1. If F_{χ_1} and ϱ are such that the original covering process with these parameters does not cover semi-infinite interval eventually a.s., then the ϵ-reinforced covering process with parameters $F_{\chi_1^\epsilon}$ and ϱ does not cover semi-infinite interval eventually a.s. ■

Before stating the next Lemma, we recall that by $\tau_n = \tau_n(F_{\chi_1}, \varrho)$ we denote the time between consecutive fresh collisions of the t.p. with the initially standing n.p.s in the original mechanical Model 1, which are heavily dependent due to recollisions, and where we assume that at time 0 the interdistances between particles are chosen as exponential random variables with parameter $\varrho > 0$ and their lifetimes distributed according to the law of χ_1.

Lemma 2.4.2. If χ (*the distribution of lifetimes of n.p.s in the mechanical model*) is such that there exists $\lambda > 0$ such that the associated line covering process with the parameters χ and λ does not cover semi-infinite interval eventually a.s., then there exist $0 < \lambda' < \lambda$, such that the *modified* line covering process with the interdistances between left-endpoints of successive intervals being taken as $\tau_i(\chi, \lambda')$ and distribution of covering interval length being χ, also does not cover semi-infinite interval eventually a.s.

Proof. We split the proof in several steps. Recall that we are assuming that $M = m = 1$. Below we present the proof which relies on this fact. However the statement of the Lemma holds for the general case $M \geq m$, but proofs are considerably more involved.
Step 1. Observe first of all, that $\tau_i \geq \sqrt{2\xi'_i/f}$ (= the free flight time between two consecutive collisions). Thus, if χ (= the distribution of lifetimes of n.p.s in the mechanical model) satisfies the condition of the Lemma, i.e. there exists $\lambda > 0$ such that the associated line covering process with the parameters χ and λ does not cover semi-infinite interval eventually a.s., then for any $\epsilon > 0$, to be specified later, we can find $0 < \lambda' \equiv \lambda'(\epsilon) < \lambda$ such that if ξ and ξ' are two independent exponential random variables with parameters λ and λ' respectively, then the following holds:

$$P(\sqrt{2\xi'/f} < \xi) \leq \epsilon.$$

Step 2. We begin with the following simple observation, which we state as the Proposition below. Consider two line covering processes Ξ and Ξ'. Interdistances between consecutive start-points of the segments in process Ξ are taken as $\{\xi_i\}_i$, and in the process Ξ' as $\{\xi'_i\}_i$. The length of covering segments $\{\ell_i\}_i$ and $\{\ell'_i\}_i$ in both processes are taken equal, i.e. $\ell_i = \ell'_i$, $\forall i$.

Proposition 2.4.3. If the process Ξ does not cover semi-infinite segment eventually, and $\xi_j \leq \xi'_j$, $\forall j$, then there is no eventual covering of the semi-infinite line in the process Ξ'.
Proof. Since $\xi_j \leq \xi'_j$, $\forall j$, we can couple both realizations of the start-points of sticks $\{x_i\}_i$ and $\{x'_i\}_i$ in such a way, that $x_j \leq x'_j$, and $x_j - x_{j-1} \leq x'_j - x'_{j-1}$ $\forall j$. Since $\ell_i = \ell'_i$, $\forall i$, it immediately implies the claim of the proposition.

To complete the proof of the Lemma we will construct another yet auxiliary covering process associated with the mechanical system. Take two sequences $\{\xi_i\}_i$ and $\{\xi'_i\}_i$ of independent exponential random variables with parameters λ and λ' in each sequence respectively. Take line covering process with interdistances between consecutive left-endpoints x_{i-1} and x_i of segments, being taken as $\sqrt{2\xi'_i/f}$. Next we will perform the following "surgery": every segment $[x_{i-1}, x_i]$ such that $\sqrt{2\xi'_i/f} < \xi_i$ will be subtracted, by performing the left shift of the semi-infinite configuration lying to the right of the point x_i by distance $x_i - x_{i-1}$, and associating point x_i with x_{i-1}, and every segment $[x_{i-1}, x_i]$ such that $\sqrt{2\xi'_i/f} \geq \xi_i$ will be contracted, by performing the left shift of the semi-infinite configuration lying to the right of the point x_i by distance $\sqrt{2\xi'_i/f} - \xi_i$, and associating point x_i with $\tilde{x}_i$. As one can easily see, the new obtained process, call it $\tilde{\tilde{\Xi}}$ is ϵ-reinforced process of the process Ξ (which by our assumptions does not cover semi-infinite line). According to Proposition (2.4.1), one can choose $\epsilon > 0$ such that $\tilde{\tilde{\Xi}}$ does not cover semi-infinite interval either.
For notational reason we can assume that in the process $\tilde{\tilde{\Xi}}$ the intervals which were completely subtracted have lengths $\tilde{\xi}_j = 0$. With this in hands we immediately can apply proposition (2.4.3) to compare the process $\tilde{\tilde{\Xi}}$ and the line covering process with the interdistances between left-endpoints of successive intervals being taken as $\tau_i(\chi, \lambda')$ and distribution of covering interval length being χ, concluding that if there is no eventual covering of the semi-infinite line in the process $\tilde{\tilde{\Xi}}$, then there is no eventual covering of the semi-infinite line in the latter process. This completes the proof of the Lemma. ■

Remark 1. Previous Lemma immediately implies that the original dynamics of the Model 1 (as well of Model 2) has infinitely many cluster indices, and inter-cluster-indices distribution has exponentially decaying tails as long as $E\chi_1 < +\infty$.

Remark 2. If lifetime distribution is inverse Gaussian (nonintegrable), our considerations still imply that at the low enough density of the n.p.s the system has infinitely many cluster indices. This in particular implies that system is Bernoulli system.

4 Invariant measure.

4.1 Tightness.

The key Lemma and Corollary 1 of the previous section brings us to the point that most of remaining work is just routine following of the standard procedures [14], which we outline in the next few short subsections.

We have for the mean number of alive n.p.s at time $t > 0$ by section 2.2. that

$$EN(A_t) = \varrho \int_0^t \bar{F}_{\chi_1}(t-s)ds \leq 2\varrho \int_0^t \bar{F}_{\chi_1}(s)ds \leq 2\varrho E\chi_1.$$

By Markov's inequality $P(N(A_t) > k) \leq k^{-1}EN(A_t) \leq k^{-1}2\varrho E\chi_1$ for $k > 0$, denoting $A_n = A_{t_n}$ and since $\lim_{n\to\infty} t_n = \infty$, if the lifetimes have finite mean, it follows that $P(R_t = 0) > 0$ for any $t > 0$ and

$$\lim_{k\to\infty} \limsup_{n\to\infty} P(N(A_n) > k) = 0$$

which entails by [11], Lemma 4.5, the tightness of $\{N(A_n) : n \in \mathbb{N}\}$. If $P_n = P \circ T^{-n}$ is the law of the discrete dynamics as defined in section 1.1, the above result entails the tightness of the family $\{P_n : n \in \mathbb{N}\}$ and hence the convergence to a necessarily invariant measure μ say, i.e. $\mu = \mu \circ T^{-n}$ for any $n \geq 1$ such that $\mu(\Omega_1) = 1$.

4.2 Coupling of μ and P, mixing and SLLN.

Existence of some limiting invariant measure μ under the discrete dynamics $(T^n)_{n\in\mathbb{N}}$ follows by tightness of the family $\{P_n : n \in \mathbb{N}\}$ as shown above under the condition of finite first moment of the lifetime distribution. Uniqueness follows by successfully coupling μ with the initial measure P. In the light of the construction of the cluster times in 2.3 it is quite clear how to produce a successful coupling. For this, let $\omega \in \Omega$ be a configuration with decomposition $x(\omega) = x_0(\omega) \cup x_m(\omega)$ and distributed according to the measure μ. Take two initial configurations ω' and ω'' distributed according to P resp. μ such that $x_0(\omega) = x_0(\omega') = x_0(\omega'')$, $x_m(\omega') = \emptyset$, $x_m(\omega) = x_m(\omega'')$, $R_0(\omega') = 0$ and $R_0(\omega'') = r_0$ for some $r_0 > 0$. Note also that under μ, the freshly arriving n.p.s and the moving n.p.s are independent and the fresh n.p.s distributed according to the initial measure P, i.e. $\mu = P \otimes \mu_m$ with $P(\Omega_{X_0}) = \mu_m(\Omega^c_{X_0}) = 1$. In words, the configuration ω' consists only of the t.p. and the freshly arriving n.p.s distributed according to P and ω'' has the same arriving n.p.s as ω' and its moving n.p.s are distributed according to μ_m with initial maximal residual lifetime r_0, i.e. their joint distribution is the measure Q on $\Omega_0 \times \Omega_0$ given by

$$Q(d\omega' d\omega'') = \varepsilon_{\{x_m(\omega')=\emptyset\}}\varepsilon_{\{x_0(\omega')=x_0(\omega'')\}}P(d\omega')\mu(d\omega'')$$

where ε is the Dirac point measure. If convenient, we write the related quantities to the two configurations with the corresponding superscrupts like

H'_1 or H''_1 for $H_1(\omega')$ resp. $H_1(\omega'')$, for instance. With the same notation as in the previous sections, letting $H_{r_0}(\omega') = \inf\{s > r_0 : R_s(\omega') = 0\}$, one has $H_{r_0}(\omega') = Z_{L_{r_0}}(\omega') = H(\omega'') = H(\omega)$ where Z is the associated subordinator and L_{r_0} its local time at $r_0 > 0$. Set $H_k(\omega') = H_k(\omega'')$ and $\chi_k(\omega') = \chi_k(\omega'')$ for any $1 \leq k \leq [H(\omega)]$ where $[.]$ denotes the integer part and let $V'_\star = V_H(\omega') > 0$ and $V''_\star = V_H(\omega'') > 0$. Fix now some $a' > 0$, sample $\delta_H(\omega')$ according to the exponential distribution $P(\delta'_H \leq s|\delta'_H > a')$ (again by lack of memory) and then set $\delta_H(\omega'') = \delta_H(\omega')$ in the configuration ω''. Both n.p.s arriving at $t'_{\vartheta_H+1}(= t''_{\vartheta_H+1})$ have some minimal outgoing velocity v'_{min} resp. v''_{min}, depending on $V'_\star$ resp. $V''_\star$ (and of f and a'). If $\chi''_{\vartheta_H+1} \leq b'$ and $\chi''_{\vartheta_H+1} \leq b''$ for the corresponding constants b' and b'', determined as in section 2.4, then t'_{ϑ_H+1} is a cluster time for both configurations ω' and ω''. Setting the interdistances as $\xi'_{\vartheta_H+k} = \xi''_{\vartheta_H+k}$ for any $k \geq 2$ concludes the coupling in this case. Otherwise, repeat the above procedure now with $H(T^{r_0}\omega')$ instead of $H(\omega)$. Since $D = \mathbb{R}_+$ P-a.s. and all other events involved have strictly positive probability, a successfull coupling also in this case will be achieved in finite time.

5 Invariance principle.

The proof of the invariance principle for the displacement of the t.p. under mixing conditions is now classic and follows the lines of [1] and [4]. Define the random element on $(\Omega_1, \mathcal{B}(\Omega_1), \mu)$ by $S_{[nt]} = n^{-1/2} Z_{[nt]}$ where $Z_n = \sum_{1\leq k\leq n}(\xi_k - \varrho^{-1}V_D)$ and $V_D = \varrho E_\mu \xi_1$ is the drift. Note that $E_\mu \widetilde{\xi}_1 = E\widetilde{\xi}_1 = f\varrho^{-2}$, hence by section 1.2, $E_\mu\xi_1 \leq f\varrho^{-2}$ and $E_\mu \xi_1^2 \leq 6f^2\varrho^{-4}$. Suppose $\sum_{n\in\mathbb{N}} \psi(n)^{1/2} < \infty$ where $\psi(n)$ is the mixing coefficient as in section 3.2. Then $\sigma^2 = \lim_{n\to\infty} n^{-1} E_\mu Z_n^2 = E_\mu(\xi_1 - \varrho^{-1}V_D) + 2\sum_{k\geq 2} E_\mu(\xi_1 - \varrho^{-1}V_D)(\xi_k - \varrho^{-1}V_D) < \infty$ and if $\sigma^2 > 0$, $S_{[nt]}$ converges weakly on the Skorokhod space to σW_t where $W = (W_t)_{t\in\mathbb{R}_+}$ is standard one-dimensional Brownian motion. Analogous to [4], if $\mathcal{C}'$ is the σ-algebra generated by the double cluster times, by the property of conditional variance, $E_\mu Z_n^2 = E_\mu(E_\mu^{\mathcal{C}'} Z_n^2) + E_\mu(E_\mu^{\mathcal{C}'} Z_n)^2 \geq E_\mu(E_\mu^{\mathcal{C}'} Z_n^2)$ and by conditional independence as in Remark 2.2,

$$n^{-1} E_\mu^{\mathcal{C}'} Z_n^2 = n^{-1} \sum_{k=1}^{n} E_\mu^{\mathcal{C}'} \xi_k^2 \geq \Gamma n^{-1} \sum_{k=1}^{n} Y'_k$$

with $\Gamma > 0$ as in Remark 2.2 and Y'_k the indicator of the k-th double cluster time as in section 2.3. Hence

$$\liminf_{n\to\infty} n^{-1} E_\mu^{\mathcal{C}'} Z_n^2 \geq \Gamma\gamma' > 0 \quad \mu\text{-a.s.}$$

for some $\gamma' > 0$ by section 2.3 and therefore $\sigma^2 > 0$ by integration. Replacing n by n^κ for $0 < \kappa < 1/2$ in the coupling of section 3.2 guarantees that $\xi_k(\omega') = \xi_k(\omega'')$ a.s. for $k \geq n^\kappa$ and n large enough with the appropriate configurations ω' and ω''. Using Minkowski's inequality and $\lim_{n\to\infty} n^{\kappa-1/2} = 0$, one sees that the invariance principle for $S_{[nt]}$ is valid also on $(\Omega_1, \mathcal{B}(\Omega_1), P)$. By [1], Theorem 17.1, we have that $n^{-1/2} Z_{\vartheta_{nt}}$ converges weakly to $\sigma \varrho^{-1/2} W_t$. Noting that $Q_t = \sum_{k=1}^{\vartheta_t} \xi_k + \int_{t_{\vartheta_t}}^t V_s ds$ it follows that $|S_{[nt]} - S_{\vartheta_{nt}}| \leq n^{-1/2} |\widetilde{\xi}_{\vartheta_{nt}+1}|$ and by Chebyshev's inequality this converges in probability P (resp. μ) to zero as $n \to \infty$ uniformly in t yielding the invariance principle for $(Q_t)_{t\in\mathbb{R}_+}$, i.e. $n^{-1/2} Q_{[nt]}$ converges weakly on Skorokhod space to $\widetilde{\sigma} W_t$ with $\widetilde{\sigma} = \sigma \varrho^{-1/2} > 0$.

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