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TESE DE DOUTORAMENTO IFT–T.007/08

Superstring Scattering Amplitudes with the Pure Spinor Formalism

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Setembro de 2008

Agradecimentos

Primeiramente eu gostaria de agradecer ao Nathan pela orientação *sui generis* ao longo dos últimos cinco anos. Agora que as coisas se aproximam do seu final há uma tendência natural de se recapitular a experiência de tê-lo como orientador. Ainda não consegui assimilar as nuances que definem essa situação, mas posso atestar para o fato de que elas realmente mudaram minha maneira de aprender. Agradeço sinceramente por todos os momentos em que estive em sua sala e pude aprender através da observação como é que se deve resolver um problema. Agora eu sei que envolve olhar através da janela, mas depois de já ter tentado muitas vezes acho que nunca conseguirei reproduzir o método completo. Para piorar a situação, a minha sala no IFT não tinha janela.

Agradeço também ao meu amigo Oscar Bedoya por ser meu companheiro científico. Eu poderia tentar enumerar os motivos que tornaram a nossa convivência uma experiência engrandecedora para mim, mas estou triste demais por saber que nossos dias sob o mesmo instituto estão prestes a terminar e não talvez não seria saudável ter uma listagem completa do que vou perder quando partirmos. Señor Bedoya, muchas gracias por haberme enseñado tantas cosas y en especial por su amistad! Nunca me olvidaré que usted siempre estuvo listo para ayudarme con cualquier cosa.

Agradeço também à Dáfni e ao Clóvis, pelos companheiros que são. Com eles aprendi coisas e foi um prazer enorme tê-los como amigos, pois nossas conversas nem sempre versaram sobre Física.

Ao Geová por suas perguntas, que me ensinaram mais do que ele imagina. E ao Anderson por sempre ter ido a minha sala me chamar para almoçar, mesmo sabendo que escutaria meus lamentos ou progressos sobre a conta em dois loops. Não duvidaria que ele soubesse fazer a conta em sua cabeça, pelo tanto que escutou e pelo inteligente que é.

Gostaria também de agradecer ao ICTP, pela incrível disposição de pagar para complementar minha formação. E à FAPESP por ter sido a agência que financiou tanto meu mestrado quanto doutorado.

Um agradecimento canino também tem seu lugar aqui. Quero agradecer à Mariposa por ter me acompanhado, debaixo da mesa, enquanto estava sozinho em Cali escrevendo a tese e pensando na conta de 5-pontos ao mesmo tempo. Não sei se ela sabia o motivo pelo qual ficávamos acordados até as quatro da manhã, mas ela nunca saiu dos meus pés.

Ao meu irmão, por nossas enumeráveis experiências que são testemunhos indeléveis do quão felizes somos quando estamos juntos. E também pela constante lembrança de nunca abandonar os gibis.

À minha mãe, pelo amor, compreensão, estímulo e pela pessoa maravilhosa que é.

Ao meu pai, pelo exemplo de retidão e honestidade. Foi ele também quem despertou meu interesse pela ciência, quando há mais de vinte anos me contou o que tinha lido sobre a velocidade da luz. Sua personalidade analítica e jeito matemático de pensar são exemplos diários de como gostaria de ser. A ele dedico essa tese.

Finalmente agradeço a minha adorável esposa Maria del Pilar, pelo amor incondicional. Seu apoio constante tem sido de fundamental importância. Além das nossas discussões científicas, ela nunca me disse que estava perdendo tempo quando escrevia *cleanup patches* para o kernel do linux. Mas também soube me estimular a continuar estudando no momento certo, talvez por me conhecer melhor do que eu mesmo.

Resumo

Esta tese discute como o formalismo de espinores puros pode ser utilizado para calcular amplitudes de espalhamento eficientemente. A enfâse recai sobre as expressões dos fatores cinemáticos no superespaço de espinores puros, onde as características simplificadoras inerentes dessa linguagem nos permitiram relacionar explicitamente as amplitudes de quatro-pontos em nível de árvore, um-loop e dois-loops. Enfatizamos como essas identidades simplificam de maneira elegante a tarefa de calcular as amplitudes de quatro-pontos para todas as possíveis combinações de partículas externas. Em particular, as amplitudes envolvendo férmions em dois-loops nunca antes haviam sido calculadas.

Também demonstramos a equivalência das amplitudes de um e dois-loops entre os formalismos mínimo e não-mínimo. Além disso calculamos a variação de gauge da amplitude de seis-pontos dos glúons, e obtemos o fator cinemático relacionado com o cancelamento da anomalia. Também mostramos como uma modificação *ad-hoc* da prescrição em um-loop nos leva facilmente ao resultado correto para a amplitude de cinco-pontos dos glúons.

Finalmente mostramos como calcular explicitamente as correlações que envolvem três espinores puros e cinco thetas.

Palavras Chaves: Supercordas; Supersimetria; Formalismo de Green-Schwarz; Espinores Puros

Áreas do conhecimento: Supersimetria; Teoria de Campos

Abstract

This thesis discusses how the pure spinor formalism can be used to efficiently compute superstring scattering amplitudes. We emphasize the pure spinor superspace representation for the kinematic factors, where the simplifying features of that language have allowed an explicit relation among the massless four-point amplitudes at tree-level, one- and two-loops to be found. We show how these identities elegantly simplify the task of computing the amplitudes for all possible external state combination related by supersymmetry. In particular, the two-loop amplitudes involving fermionic states had never been computed before the pure spinor formalism era.

We also prove the equivalence of one- and two-loop amplitudes computed with the minimal and non-minimal formalisms. Besides that we compute the gauge variation of the massless six-point open string amplitude and obtain the kinematic factor related to the anomaly cancellation. We also discuss how an ad-hoc modification of the one-loop prescription leads us quickly to the correct massless five-point amplitude for the gluons.

And finally we show how to explicitly compute the pure spinor superspace expressions containing three pure spinors and five thetas.

Praeterea, non debet poni superfluum aut aliqua distinctio sine causa, quia frustra fit per plura quod potest fieri per pauciora.

Petrus Aureolus, *Scriptum super primum Sententiarum*

At some point, "performance" is just more than a question of how fast things are, it becomes a big part of usability.

Linus Torvalds, *linux-kernel mailing list*

Contents

Chapter 1

Introduction

In his later years Einstein struggled to find a unified theory describing both gravity and electromagnetism and met failure. Nowadays superstring theory is the most promising candidate to fulfill what Einstein envisioned in the last century. It unifies in a quantum framework not only gravity and electromagnetism but also the electroweak and strong forces. It is even more than a simple construction which is able to handle all interactions together, as it *requires* them to be parts of a whole setup which breaks down if one of its parts is absent.

Among other things, superstring theory has provided us with a consistent quantum description of the gravitational force. One particular oscillation mode of the closed string has the right properties to be the quantum messenger of the gravitational force, the graviton. And its interactions are described precisely by the Einstein-Hilbert action [10],

$$S_{\rm EH} = \frac{1}{16\pi G_N}\int d^{10}x\sqrt{-g}R \tag{1.1}$$

plus quantum and superstring corrections to be described below.

Also present in the open superstring spectrum is a massless string with spin one which describes the Yang-Mills gluons (or photons), whose interactions in the low energy limit are described by the standard Yang-Mills action,

$$S_{\rm YM} = \frac{1}{g^2_{YM}}\int d^{10}x{\rm Tr}(F_{mn}F^{mn}) \tag{1.2}$$

together with other quantum or superstring corrections.

One of the most fundamental questions which naturally arise when studying the low energy properties of the superstring interactions is to understand what are the perturbative corrections to these two actions predicted by the theory. That question automatically leads us to contemplate the fact that superstring perturbation theory is finite to all loop orders [11]. Therefore besides unifying all forces of nature, superstring theory does it in such a way as to be *finite*. No renormalization is ever needed when deriving quantum corrections to the effective action.

One of the standard procedures to obtain these quantum corrections is through the computation of scattering amplitudes. For example, the information needed to derive higher-derivative terms in the Yang-Mills action (1.2) is encoded in the scattering of the string counterparts of the gluons, *i.e.*, the massless open strings with spin one. Analogously, quantum corrections to the Einstein-Hilbert action are determined by the scattering of massless closed strings with spin two.

The tree-level scattering of three gluons, for example, can be used to find the three point vertex in the expansion of the Yang-Mills action (1.2). Higher-point scatterings in string theory probe higher-order vertices in the low energy effective action and so forth. But the first true superstring corrections are obtained from the massless four-point scattering at tree-level [12], and are of quartic order in the field-strenght F_{mn} or Riemann tensor R_{mnpq},

$$S \propto \frac{\alpha'^2}{g^2_{YM}} \int d^{10}x\, \mathcal{F}^4, \quad S \propto \frac{\alpha'^3}{16\pi G_N} \int d^{10}x \sqrt{-g}\, \mathcal{R}^4 \tag{1.3}$$

where $\mathcal{F}^4$ and $\mathcal{R}^4$ are abbreviations for

$$\mathcal{F}^4 = t_8^{mnpqrstu} F_{mn} F_{pq} F_{rs} F_{tu},$$

$$\mathcal{R}^4 = t_8^{mnpqrstu} t_8^{abcdefgh} R_{mnab} R_{pqcd} R_{rsef} R_{tugh},$$

where the t_8 tensor is described in the Appendix C.

But superstring theory is in fact supersymmetric, so there are many more interactions in the effective actions for the gravitons and gluons than those of (1.1) and (1.2). In fact their actions in the low energy limit are given by the ten-dimensional supergravity and super-Yang-Mills actions, describing also their fermionic superpartners; the gravitino and gluino. Furthermore, all these extra terms are related by supersymmetry and also receive quantum and superstring corrections. Computing these corrections to all those terms has proven to be a challenging task over the years.

The computation of these various scattering amplitudes have been traditionally done using two different prescriptions, encompassed in the so-called Ramond-Neveu-Schwarz or Green-Schwarz formalisms.

The Ramond-Neveu-Schwarz formalism

The Ramond-Neveu-Schwarz formalism is based on spacetime vectors $X^m(\sigma,\tau)$ and $\psi^m(\sigma,\tau)$ which are scalars (the X^m's) or spinors (the ψ^m's) of the two-dimensional wordlsheet with coordinates σ,τ. The lack of spacetime spinors is the major source of difficulty in this formalism, as the computation of scattering amplitudes for fermionic strings is not natural in this framework. It has to be done using a clever construction of vertex operators for the spacetime spinors which uses *spin fields* Σ_α [23]

$$\Sigma_\alpha = \mathrm{e}^{\pm i\phi_1}\mathrm{e}^{\pm i\phi_2}\mathrm{e}^{\pm i\phi_3}\mathrm{e}^{\pm i\phi_4}\mathrm{e}^{\pm i\phi_5}$$

and the bosonization of the ψ^m's

$$\psi^1 \pm i\psi^2 = \mathrm{e}^{\pm i\phi_1}, \quad \psi^3 \pm i\psi^4 = \mathrm{e}^{\pm i\phi_2}, \quad \psi^5 \pm i\psi^6 = \mathrm{e}^{\pm i\phi_3},$$
$$\psi^7 \pm i\psi^8 = \mathrm{e}^{\pm i\phi_4}, \quad \psi^9 \pm i\psi^{10} = \mathrm{e}^{\pm i\phi_5}.$$

Furthermore, because the ψ^m's are spinors in the worldsheet, the computation of higher-loop scattering amplitudes require a sum over different *spin structures*. The fact that each term can have divergencies which are cancelled only after the sum is performed also leads to difficulties.

So if one uses the scattering amplitude prescription of the Ramond-Neveu-Schwarz formulation each scattering involving fermionic partners has to be computed in isolation, and the computation of the fermionic state is much more difficult due to the complicated nature of the vertex operator. The formalism is said to lack manifest supersymmetry.

The Green-Schwarz formalism

In contrast the Green-Schwarz formulation is manifestly supersymmetric. It is based on the worldsheet fields X^m and θ^α, which are spacetime vectors and spinors, respectively. The drawback in this formalism comes from the fact that it has a complicated action,

$$S = \frac{1}{\pi}\int d^2z\,\Big[\frac{1}{2}\partial X^m\overline{\partial}X_m - i\partial X^m\theta_L\gamma_m\overline{\partial}\theta_L - i\overline{\partial}X^m\theta_R\gamma_m\partial\theta_R$$
$$- \frac{1}{2}(\theta_L\gamma^m\overline{\partial}\theta_L)(\theta_L\gamma_m\partial\theta_L + \theta_R\gamma_m\partial\theta_R) - \frac{1}{2}(\theta_R\gamma^m\partial\theta_R)(\theta_L\gamma_m\overline{\partial}\theta_L + \theta_R\gamma_m\overline{\partial}\theta_R)\Big],$$

which is impossible to quantize preserving manifest Lorentz covariance. By breaking $SO(1,9)$ covariance to $SO(8)$ with the light cone gauge choice the action simplifies

$$S = \frac{1}{4\pi}\int d^2z\,\left(\partial X^i\overline{\partial}X^i + S^a_L\overline{\partial}S^a_L + S^b_R\partial S^b_R\right).$$

In this gauge the construction of vertex operators is possible and the computation of scattering amplitudes can be done. For example, the gluon and gluino vertices are given, in a Lorentz frame where $k^+ = 0$, $\zeta^+ = \zeta^- = 0$, by

$$V_B(\zeta, k) = \zeta^i(\dot{X}^i - \frac{1}{4}S^a S^b k^j \gamma^{ij}_{ab})\mathrm{e}^{ikX},$$

$$V_F(\zeta, k) = (u^a F^a + u^{\dot{a}} F^{\dot{a}})\mathrm{e}^{ikX},$$

where

$$F^a = \sqrt{\frac{p^+}{2}} S^a, \quad F^{\dot{a}} = (2p^+)^{-1/2}\Big[(\gamma \cdot \dot{X} S)^{\dot{a}} + \frac{1}{3} : (\gamma^i S)^{\dot{a}}(S\gamma^{ij}S) : k^j\Big].$$

However, the need of a non-covariant gauge and restricted kinematics are features which reduce the power of this manifestly supersymmetric approach. For example, in the light cone gauge one looses the conformal symmetry of the original theory and therefore can not use the powerful methods of conformal field theory. Furthermore it is not always possible to impose those restrictions simultaneously.

So, up to the year 2000 the computations of superstring scattering amplitudes could be done using these two different formalisms. The results were equivalent but required different amounts of work to be performed. Furthermore, either spacetime supersymmetry or Lorentz covariance was hidden in the middle steps. Nevertheless both symmetries are fundamental requirements of superstring theory and as such the results must respect them. The fact that the end result has all these symmetries while they are not obvious in the middle steps means that the formalisms were introducing spurious difficulties were there should be none.

The Pure Spinor formalism

The pure spinor formalism was born at the dawn of the new millenium [1], as a successful attempt to solve this long-standing problem of finding a manifestly supersymmetric and covariant superstring formalism. The focus of this thesis is to show how it can be used in the computations of scattering amplitudes, highlighting the virtues and elegance of manifest Lorentz covariance and spacetime supersymmetry in the results obtained. The amplitudes computed so far in the pure spinor formalism turned out to be immensely easier to obtain. As a sounding example of how simpler computations can be, a good measure is to compare the hundred-pages long calculation of the four-point amplitude at two-loops in the RNS formalism [13] versus the ten-pages-long computation using pure spinors [8][2]. Of course the results were shown to be equivalent [2], as well as for all other amplitudes computed so far [3][4][5] (see [6] for a general tree-level proof), proving that the pure spinor formalism produces the same results while being simpler.

Right after the formalism came into light, the tree-level amplitudes were shown to be equivalent with the RNS computations in [6], for amplitudes containing any number of bosons and up to four fermions. Years later, Berkovits spelled out the multiloop prescription [7][8] and paved the way to show the equivalence of his formalism up to the two-loop level, which is the state-of-the-art situation as of 2008.

In the computations of massless four-point amplitudes the results can be written down in terms of a supersymmetric kinematic factor in pure spinor superspace times a function which is manifestly equal to their RNS and GS counterparts. So the comparison of the results require the evaluation of the pure spinor superspace integrals appearing in the kinematic factors. For example, the supersymmetric kinematic factors in the massless four-point amplitudes at one- and two-loop order were originally written as [7][8]

$$K_{\rm one-loop} = \tag{1.4}$$
$$\int d^{16}\theta (\epsilon T^{-1})^{((\alpha\beta\gamma))}_{[\rho_1\ldots\rho_{11}]}\theta^{\rho_1}\ldots\theta^{\rho_{11}}(\gamma_{mnpqr})_{\beta\gamma}\Big[A_{1\alpha}(\theta)(W_2(\theta)\gamma^{mnp}W^3(\theta))\mathcal{F}^{qr}_4(\theta) + {\rm cycl}(234)\Big]$$

$$K_{\rm two-loop} = \tag{1.5}$$
$$\int d^{16}\theta(\epsilon T^{-1})^{((\alpha\beta\gamma))}_{[\rho_1\ldots\rho_{11}]}\theta^{\rho_1}\ldots\theta^{\rho_{11}}(\gamma^{mnpqr})_{\alpha\beta}\gamma^s_{\gamma\delta}\Big[\mathcal{F}^1_{mn}(\theta)\mathcal{F}^2_{pq}(\theta)\mathcal{F}^3_{rs}(\theta)W^{4\delta}(\theta)\Delta(z_1,z_3)\Delta(z_2,z_4)$$
$$+{\rm perm}(1234)\Big]$$

where $\Delta(y,z) = \epsilon^{CD}\omega_C(y)\omega_D(z)$, ω_C are the two holomorphic one-forms defined in [13], $A^I_\alpha(\theta)$, $W^{I\alpha}(\theta)$ and $\mathcal{F}^I_{mn}(\theta)$ are the super-Yang-Mills connection and the linearized spinor and vector superfield-strengths for the I^{th} external state with momentum k^m_I satisfying $k_I\cdot k_I = 0$, and $(T^{-1})^{\alpha\beta\gamma}_{\rho_1\ldots\rho_{11}}$ is a Lorentz-invariant tensor which is antisymmetric in $[\rho_1 ... \rho_{11}]$ and symmetric and γ-matrix traceless in $(\alpha\beta\gamma)$. Up to an overall normalization constant,

$$(T^{-1})^{\alpha\beta\gamma}_{\rho_1\ldots\rho_{11}} = \epsilon_{\rho_1\ldots\rho_{16}}(\gamma^m)^{\kappa\rho_{12}}(\gamma^n)^{\sigma\rho_{13}}(\gamma^p)^{\tau\rho_{14}}(\gamma_{mnp})^{\rho_{15}\rho_{16}}(\delta^{(\alpha}_\kappa\delta^\beta_\sigma\delta^{\gamma)}_\tau - \frac{1}{40}\gamma^{(\alpha\beta}_q\delta^{\gamma)}_\kappa\gamma^q_{\sigma\tau}).$$

To finally get the final result for these pure spinor amplitudes one should use the θ-expansions of the super-Yang-Mills superfields listed in (B.24), plug them back into the above expressions, compute the traces of a multitude of gamma matrix arrays (some of them containing as much as twenty gamma matrices) and finally perform the $d^{16}\theta$ superspace integration. Looking at the multitude of vector and spinor indices of (1.4) and (1.5) one would conclude that the manifest Lorentz covariance and supersymmetry of the pure spinor formalism were making those kinematic factors expressions look awkward and laborious to be evaluated. Fortunately that is not the case, in fact quite the opposite is true. With the observation that, up to an overall coefficient, [2]

$$\int d^{16}\theta(\epsilon T^{-1})^{((\alpha\beta\gamma))}_{[\rho_1\ldots\rho_{11}]}\theta^{\rho_1}\ldots\theta^{\rho_{11}}f_{\alpha\beta\gamma}(\theta) = \langle\lambda^\alpha\lambda^\beta\lambda^\gamma f_{\alpha\beta\gamma}(\theta)\rangle \tag{1.6}$$

those scary-looking kinematic factors (1.4) and (1.5) simply become[1]

$$K_1 = \langle(\lambda A^1)(\lambda\gamma^m W^2)(\lambda\gamma^n W^3)\mathcal{F}^4_{mn}\rangle + \text{cycl.}(234), \tag{1.7}$$

$$K_2 = \langle(\lambda\gamma^{mnpqr}\lambda)\mathcal{F}^1_{mn}\mathcal{F}^2_{pq}\mathcal{F}^3_{rs}(\lambda\gamma^s W^4)\rangle\Delta(1,3)\Delta(2,4) + \text{perm.}(1234). \tag{1.8}$$

The pure spinor correlator in the right-hand side of (1.6) was defined since day one in [1] and allows a tremendous simplification in the computations, which in fact become trivial to perform. As explained in [1], the pure spinor expression $\langle\lambda^\alpha\lambda^\beta\lambda^\gamma f_{\alpha\beta\gamma}(\theta)\rangle$ is to be evaluated by selecting the terms which contain five θ's proportional to the (normalized) pure spinor measure,

$$\langle(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)\rangle = 1. \tag{1.9}$$

In Appendix A we show that the computation of pure spinor expressions containing an arbitrarily complicated combination of three λ's and five θ's is uniquely determined by symmetry alone. Using the method described in the appendix, many pure spinor superspace expressions were explicitly evaluated with not much effort. For example, in [3] and [2] the kinematic factors for the bosonic components of (1.7) and (1.8) were shown

[1] The biggest problem with the brute-force approach of computing thousand of gamma matrix traces which follow from expressions like (1.4) is that one misses various identities which become clear in their pure spinor superspace representation of (1.7) and (1.8). The identities (1.11) and (1.12) are simple examples of what can be accomplished. Furthermore, as the usual tool to compute traces of gamma matrices at my disposal at that time was Mathematica with the package GAMMA, which are so inefficient at this specific task, computations along those lines could take more than 24 hours of run-time, which I considered unacceptable. With the method of Appendix A together with the efficiency of FORM [14][15], those computations don't take longer than 10 seconds. And following Linus Torvalds' citation in this thesis, "performance is a big part of usability". In handsight, it was the performance requirements which I set as a goal in the begining of this enterprise which allowed the quick verification of superspace identities, making further progress much faster than otherwise it would be.

to reproduce the t_8-tensorial structure appearing in the low energy effective action for superstrings (1.3). That provided the proof that the pure spinor formalism reproduces the same results as the RNS formalism [13] up to the two-loop level. And as the pure spinor expressions for the kinematic factors are supersymmetric, the computation of the fermionic terms pose no further difficulties and were also evaluated [5][9]. This situation is in deep contrast to the need of computing each amplitude separatedly for all the external superpatners as is the case in the RNS and GS formalisms. Furthermore, as summarized below, the simple nature of the pure spinor representation for the kinematic factors allowed for an explicit proof that the massless four-point amplitudes were all related to one another at different orders in perturbation theory, namely at tree-, one- and two-loops.

Firstly, the idea was to obtain a pure spinor superspace expression for the massless four-point kinematic factor at tree-level [5]

$$K_0 = \frac{1}{2}k_1^m k_2^n \langle(\lambda A^1)(\lambda A^2)(\lambda A^3)\mathcal{F}^4_{mn}\rangle - (k^1\cdot k^3)\langle A^1_n(\lambda A^2)(\lambda A^3)(\lambda\gamma^n W^4)\rangle + (1\leftrightarrow 2). \quad (1.10)$$

Then it was shown through manipulations in pure spinor superspace that (1.10) was proportional to the masslees four-point kinematic factor at one-loop (1.7)

$$K_0 = -\langle(\lambda A^1)(\lambda\gamma^m W^2)(\lambda\gamma^n W^3)\mathcal{F}^4_{mn}\rangle = -\frac{1}{3}K_1. \quad (1.11)$$

After that, using a proof based on BRST-equivalence of some pure spinor expressions, the two-loop kinematic factor (1.8) was related to the tree-level factor as follows

$$K_2 = -32K_0\left[(u-t)\Delta(1,2)\Delta(3,4) + (s-t)\Delta(1,3)\Delta(2,4) + (s-u)\Delta(1,4)\Delta(2,3)\right]. \quad (1.12)$$

That was the first time ever that these kinematic factors were shown to be related as a whole without having to compute every possible scattering of bosonic and/or fermionic states[2] individually, showing case by case their proportionality to each other. Of course once one obtains the same kinematic factor for the computation of four-point gluon (or graviton) scattering at different loop orders, supersymmetry can be used to argue that the kinematic factors for the superpartners are also the same, as can be seen in [34]:

> *Having discovered this result for bosons, it becomes plausible that supersymmetry ensures that the one-loop four-particle amplitudes involving fermions*

[2]And here we note that the scattering computation of fermionic states at two loops has never been done using the RNS formalism (and nothing at all with the GS formalism).

also have the same kinematic factors as the tree diagrams. In fact, this must be the case, because the various K factors given in § 7.4.2 can be related to one another by supersymmetry transformations.

Nevertheless, it is worth having an explicit simple proof that the kinematic factors (1.10), (1.7) and (1.8) satisfy the identities (1.11) and (1.12). Note that explicit two-loop computations involving fermionic external states have never been done before the rather easy pure spinor computations of [5][9]. Furthermore, with identities like (1.11) and (1.12) it is not even needed to compute the one- and two-loop kinematic factors explicitly in components anymore. That is truly a remarkable simplification compared to the standard RNS and GS formalisms.

So this thesis emphasises the study of pure spinor superspace expressions and their role in obtaining simple relations for seemingly complicated amplitudes. It is structured as follows.

In chapter 2 we review the pure spinor formalism and the prescriptions to compute scattering amplitudes in the minimal and non-minimal versions.

In chapter 3 the manifestly supersymmetric kinematic factors for massless massless four-point amplitudes at tree-, one- and two-loop levels are studied and explicitly evaluated in components. In section 3.7 we also compute the gauge variation of the massless six-point amplitude for open strings, which gives rise to a pure spinor superspace representation for the gauge anomaly kinematic factor

$$K_{\rm anom} = \langle (\lambda\gamma^m W^1)(\lambda\gamma^n W^2)(\lambda\gamma^p W^3)(W^4\gamma_{mnp}W^5)\rangle. \tag{1.13}$$

Furthermore, in section 3.8 we evaluate the bosonic components of the interesting pure spinor superspace expression

$$\langle (\lambda\gamma^r W^1)(\lambda\gamma^s W^2)(\lambda\gamma^t W^3)(\theta\gamma^m\gamma^n\gamma_{rst}W^4)\rangle, \tag{1.14}$$

from which the t_8 and ϵ_{10} tensors naturally emerge in a unified manner, in the form $\eta^{mn}t_8^{m_1n_1\ldots m_4n_4} - \frac{1}{2}\epsilon_{10}^{mnm_1n_1\ldots m_4n_4}$.

In section 3.3.5 we digress about the expression

$$\langle (\lambda\gamma^m\theta)(\lambda\gamma^n W)(\lambda\gamma^p W)(W\gamma_{mnp}W)\rangle, \tag{1.15}$$

which turns out to be proportional to the one-loop kinematic factor of (1.7) and consequently it is supersymmetric despite the explicit appearance of θ. We will show its supersymmetry by relating the bosonic components of (1.15) with the left hand side of the following identity

$$\langle \Big[(D\gamma_{mnp}A)\Big](\lambda\gamma^m W)(\lambda\gamma^n W)(\lambda\gamma^p W)\rangle = -8\langle(\lambda A)(\lambda\gamma^m W)(\lambda\gamma^n W)\mathcal{F}_{mn}\rangle, \tag{1.16}$$

in such a way as to finaly prove that

$$\langle(\lambda\gamma^m\theta)(\lambda\gamma^n W)(\lambda\gamma^p W)(W\gamma_{mnp}W)\rangle = 8\langle(\lambda A)(\lambda\gamma^m W)(\lambda\gamma^n W)\mathcal{F}_{mn}\rangle.$$

Finally, in section 3.9 we consider an intriguing pure spinor superspace expression

$$\langle(\lambda\gamma^m\theta)(\lambda\gamma^n\gamma^{rs}W^5)(\lambda\gamma^p W^1)(W^3\gamma_{mnp}W^4)\mathcal{F}^2_{rs}\rangle \tag{1.17}$$

whose *bosonic* component expansion reproduces the massless five-point amplitude of open strings. We will show that (1.17) is proportional to

$$\langle(D\gamma_{mnp}A^1)(\lambda\gamma^m\gamma^{rs}W^5)(\lambda\gamma^n W^3)(\lambda\gamma^p W^4)\mathcal{F}^2_{rs}\rangle - (2\leftrightarrow 5) \tag{1.18}$$

which is one of the terms produced in the evaluation of

$$\langle(\overline{\lambda}\gamma_{mnp}D)\Big[(\lambda A^1)(\lambda\gamma^m\gamma^{rs}W^5)(\lambda\gamma^n W^3)(\lambda\gamma^p W^4)\mathcal{F}^2_{rs}\Big]\rangle - (2\leftrightarrow 5), \tag{1.19}$$

which appears in the massless five-point computation with the non-minimal pure spinor formalism[3].

Chapter 4 contains some conclusions and possible directions for further inquiry along the lines of the study presented in this thesis.

In Appendix A we describe an efficient method to compute pure spinor superspace expressions in terms of the polarizations and momenta of the external particles. This is the method which was used in several papers to obtain the final component expression for various kinematic factors.

A brief review of $\mathcal{N}=1$ super-Yang-Mills theory in $D=10$ is given in Appendix B, together with the explicit θ-expansion of the superfields used in this thesis.

And finally the famous t_8-tensor is written down explicitly in Appendix C. This is done both in terms of explicit Kronecker deltas as well as in terms of its contraction with four field-strengths F_{mn}. We also present its (previously unknown) $U(5)$-covariant form which can be deduced from the pure spinor expression (1.15).

[3]This is work in progress with the colaboration of Christian Stahn.

Chapter 2

The Pure Spinor Formalism

The pure spinor formalism is an efficient tool to compute superstring scattering amplitudes in a covariant way, and this is the aspect which we will emphasize in this thesis.

Being manifestly supersymmetric and containing no wordsheet spinors, it does not require the summation over the spin structures which makes the evaluation of higher-loop amplitudes in the RNS formalism a difficult task. And as it can be covariantly quantized, one does not need to go to the light-cone gauge as in the Green-Schwarz formulation, avoiding the problems when one has to do so. We will now review the origins of the pure spinor formalism and how it was constructed, establishing our notation along the way. Then we will explain how amplitudes are to be computed using Berkovits' formalism.

2.1 Siegel's modification of the Green-Schwarz formalism

The main difficulty one faces when trying to quantize the Green-Schwarz action (written here in the conformal gauge)

$$S = \frac{1}{\pi}\int d^2z \left[\frac{1}{2}\partial X^m \overline{\partial} X_m - i\partial X^m \theta_L \gamma_m \overline{\partial}\theta_L - i\overline{\partial} X^m \theta_R \gamma_m \partial\theta_R \right.$$
$$\left. -\frac{1}{2}(\theta_L\gamma^m\overline{\partial}\theta_L)(\theta_L\gamma_m\partial\theta_L + \theta_R\gamma_m\partial\theta_R) - \frac{1}{2}(\theta_R\gamma^m\partial\theta_R)(\theta_L\gamma_m\overline{\partial}\theta_L + \theta_R\gamma_m\overline{\partial}\theta_R)\right],$$

is related to the complicated nature of the fermionic constraints d_α. To see this we compute the conjugate momentum to θ^α_L, denoted by p^L_α, to obtain

$$p^L_\alpha = \frac{i}{2}\left(\gamma_m\theta_L\right)_\alpha \left[\Pi^m + \frac{i}{2}\left(\theta_L\gamma^m\partial_1\theta_L\right)\right].$$

As it depends on θ_L^α, it defines a constraint $d_\alpha^L = p_\alpha^L - \frac{i}{2}\left(\theta_L\gamma^m\right)_\alpha \Pi_m + \frac{1}{4}\left(\theta_L\gamma^m\right)_\alpha \left(\theta_L\gamma_m\partial_1\theta_L\right)$ which satisfies the OPE

$$d_\alpha^L(z)d_\beta^L(w) \to -i\frac{\gamma_{\alpha\beta}^m\Pi_m}{z-w}. \tag{2.1}$$

Due to the Virasoro constraint $\Pi_m\Pi^m = 0$ the relation (2.1) mixes first and second class types of constraints in such a way that is very difficult to disantangle them covariantly. The standard way to deal with this situation is to go to the light-cone gauge, where the two types of constraints can be treated separatedly in (2.1).

In 1986 Warren Siegel [35] proposed a new approach to deal with this problem. His idea was to treat the conjugate momenta for θ^α as an independent variable, proposing the following action for the left-moving variables[1]

$$S = \frac{1}{2\pi}\int d^2z\left[\frac{1}{2}\partial X^m\overline{\partial}X_m + p_\alpha\overline{\partial}\theta^\alpha\right]. \tag{2.2}$$

Together with (2.2) one should add an appropriate set of first-class constraints to reproduce the superstring spectrum. The Virasoro constraint $T = -\frac{1}{2}\Pi^m\Pi_m - d_\alpha\partial\theta^\alpha$ and the kappa symmetry generators of the GS formalism, given by $G^\alpha = \Pi^m(\gamma_m d)^\alpha$, where

$$\Pi^m = \partial X^m + \frac{1}{2}(\theta\gamma^m\partial\theta) \tag{2.3}$$

should certainly be elements of that set of constraints. Furthermore, in his approach the variable d_α

$$d_\alpha = p_\alpha - \frac{1}{2}\left(\partial X^m + \frac{1}{4}(\theta\gamma^m\partial\theta)\right)(\gamma_m\theta)_\alpha$$

was not supposed to be a constraint.

Even though there was a succesfull description of the superparticle using Siegel's approach, the whole set of constraints was never found for the superstring case. However, as we shall see below, Siegel's idea was used by Berkovits in his proposal for the pure spinor formalism.

Note that the action (2.2) defines a CFT whose OPE's are given by

$$X^m(z,\overline{z})X^n(w,\overline{w}) \longrightarrow -\frac{\alpha'}{2}\eta^{mn}\ln|z-w|^2, \quad p_\alpha(z)\theta^\beta(w) \longrightarrow \frac{\delta_\alpha^\beta}{z-w}, \tag{2.4}$$

$$d_\alpha(z)d_\beta(w) \longrightarrow -\frac{\alpha'}{2}\frac{\gamma_{\alpha\beta}^m\Pi_m}{z-w}, \quad d_\alpha(z)\Pi^m(w) \longrightarrow \frac{\alpha'}{2}\frac{(\gamma^m\partial\theta)_\alpha}{z-w}. \tag{2.5}$$

[1]We will restrict our attention to the left-moving variables only, as it is straightforward to add the right-moving part.

Furthermore, if $V(y,\theta)$ is a generic superfield then its OPE's with d_α and Π^m are computed as follows

$$d_\alpha(z)V(y,\theta) \longrightarrow \frac{\alpha'}{2}\frac{D_\alpha V(y,\theta)}{z-y}, \quad \Pi^m(z)V(y,\theta) \longrightarrow \frac{\partial^m V(y,\theta)}{z-y}, \tag{2.6}$$

where the supersymmetric derivative D_α is given by

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + \frac{1}{2}(\gamma^m\theta)_\alpha\partial_m. \tag{2.7}$$

The energy momentum tensor for the action (2.2) is given by

$$T(z) = -\frac{1}{2}\partial X^m\partial X_m - p_\alpha\partial\theta^\alpha$$

as can be easily checked by using the known results of the bosonic string and the bc system with $\lambda = 1$, in the notation of [31]. Furthermore, the central charge is $c = +10-32 = -22$, where each pair of p_α and θ^β have $c = -3(2\lambda-1)^2+1 = -2$, for a total of -32.

The non-vanishing of the central charge leads to problems when quantizing the theory, so that was a major difficulty in Siegel's approach to the GS formalism.

Furthermore in [35] Siegel proposed that the supersymmetric integrated massless vertex operator in his approach should be

$$U = \int dz(\partial\theta^\alpha A_\alpha(x,\theta) + A_m(x,\theta)\Pi^m + d_\alpha W^\alpha(x,\theta)) \tag{2.8}$$

where the superfields appearing in (2.8) are the SYM superfields which are reviewed in the appendix. But there is a problem with this supposition if one wants it to be equivalent to the RNS formalism, where the vertex operator for a gluon is given by (see (7.3.25) in [33])

$$U^{\rm RNS}_{\rm gluon} = \int dz(A_m\partial X^m + \frac{1}{2}\psi^m\psi^n F_{mn}), \tag{2.9}$$

where the field-strength is $F_{mn} = \partial_m A_n - \partial_n A_m$. To see this one uses the superfield expansions of appendix B to conclude that the gluon vertex operator obtained from (2.8) is

$$U^{\rm Siegel}_{\rm gluon} = \int dz(A_m\partial X^m - \frac{1}{4}(p\gamma^{mn}\theta)F_{mn})$$

Comparing both expressions we see that the operator which multiplies $\frac{1}{2}F_{mn}$ is the Lorentz current for the fermionic variables in each formalism. To see this we use Noether's method to define the variation of (2.2) under the Lorentz transformation to be $\delta S = \frac{1}{2\pi}\int \frac{1}{2}\overline{\partial}\varepsilon_{mn}\Sigma^{mn}$, where

$$\delta p_\alpha = \frac{1}{4}\varepsilon_{mn}(\gamma^{mn})_\alpha{}^\beta p_\beta, \qquad \delta\theta^\alpha = \frac{1}{4}\varepsilon_{mn}(\gamma^{mn})^\alpha{}_\beta\theta^\beta.$$

Therefore the variation of (2.2) is

$$\begin{aligned}\frac{1}{2\pi}\int \delta(p_\alpha\overline{\partial}\theta^\alpha) &= \frac{1}{2\pi}\int\left[\frac{1}{4}\varepsilon_{mn}(\gamma^{mn})_\alpha{}^\beta p_\beta\overline{\partial}\theta^\alpha + \frac{1}{4}p_\alpha\overline{\partial}(\varepsilon_{mn}(\gamma^{mn}\theta)^\alpha)\right].\\ &= \frac{1}{2\pi}\int\left[+\frac{1}{4}\overline{\partial}\varepsilon_{mn}p_\alpha(\gamma^{mn})^\alpha{}_\beta\theta^\beta\right],\end{aligned}$$

so that

$$\Sigma^{mn} = \frac{1}{2}(p\gamma^{mn}\theta) \tag{2.10}$$

is the Lorentz currents of the fermionic variables. However the Lorentz currents of the fermionic variables in Siegel's approach had a double pole coefficient of +4 instead of +1 as in the RNS formalism. Using the OPE (2.4) we get

$$\begin{aligned}\Sigma^{mn}(w)\Sigma^{pq}(z) &= \frac{1}{4}\frac{p(\gamma^{mn}\gamma^{pq} - \gamma^{pq}\gamma^{mn})\theta}{w-z} + \frac{1}{4}\left(\frac{\mathrm{tr}(\gamma^{mn}\gamma^{pq})}{(w-z)^2}\right),\\ &= \frac{\eta^{p[n}\Sigma^{m]q} - \eta^{q[n}\Sigma^{m]p}}{w-z} + 4\frac{\eta^{m[q}\eta^{p]n}}{(w-z)^2}\end{aligned} \tag{2.11}$$

where we used that $\gamma^{mn}\gamma^{pq} - \gamma^{pq}\gamma^{mn} = 2\eta^{np}\gamma^{mq} - 2\eta^{nq}\gamma^{mp} + 2\eta^{mq}\gamma^{np} - 2\eta^{mp}\gamma^{nq}$ and $\mathrm{tr}(\gamma^{mn}\gamma^{pq}) = -32\delta^{mn}_{pq}$. Recalling that in the RNS formalism the OPE of the Lorentz currents for the fermionic variables $\Sigma_{\mathrm{RNS}} = \psi^m\psi^n$ satisfies

$$\Sigma^{mn}_{RNS}(w)\Sigma^{pq}_{RNS}(z) \to \frac{\eta^{p[n}\Sigma^{m]q}_{RNS} - \eta^{q[n}\Sigma^{m]p}_{RNS}}{w-z} + \frac{\eta^{m[q}\eta^{p]n}}{(w-z)^2} \tag{2.12}$$

the discrepancy in the coefficient for the double between (2.11) and (2.12) would make the computations of scattering amplitudes using (2.9) and (2.8) not equivalent between both approaches.

2.2 The elements which led to the pure spinor formalism

The modification of Siegel's approach proposed by Berkovits in the year 2000 was based in the observation that there existed a set of ghost variables with $c_g = +22$ and whose contribution to the double pole of the Lorentz currents was -3. So the problems described above would no longer exist if that set of ghosts was added to Siegel's action (2.2). That discovery led to the creation of the pure spinor formalism. Let's now take a look at some of its ingredients in such a way as to motivate the solution found by Berkovits.

2.2.1 Lorentz currents for the ghosts

When trying to construct the Lorentz currents for the fermionic variables in the pure spinor formalism, Berkovits suggested to modify the Lorentz currents (2.10) by the addition of a contribution N^{mn} coming from the ghosts,

$$M^{mn} = \Sigma^{mn} + N^{mn}.$$

The newly defined M^{mn} would satisfy the same OPE (2.12) as in the RNS formalism if

$$N^{mn}(w)N^{pq}(z) \to \frac{\eta^{p[n}N^{m]q} - \eta^{q[n}N^{m]p}}{w-z} - 3\frac{\eta^{m[q}\eta^{p]n}}{(w-z)^2}, \tag{2.13}$$

$$\Sigma^{mn}(w)N^{pq}(z) \to \text{regular}, \tag{2.14}$$

as one can check as follows

$$\begin{aligned} M^{mn}(w)M^{pq}(z) &= (\Sigma^{mn}(w) + N^{mn}(w))(\Sigma^{pq}(z) + N^{pq}(z)) \\ &\to \Sigma^{mn}(w)\Sigma^{pq}(z) + N^{mn}(w)N^{pq}(z) \\ &\to \frac{\eta^{p[n}M^{m]q} - \eta^{q[n}M^{m]p}}{w-z} + \frac{\eta^{m[q}\eta^{p]n}}{(w-z)^2}. \end{aligned}$$

At the same time those ghosts should have the right properties as to contribute $c_g = +22$ to the central charge, otherwise the total central charge would be non-vanishing. Fortunately the right solution to both problems was found when a proposal for the BRST charge was put forward. As we will see, that provided the hint as to what was missing in the long quest for finding a manifestly spacetime supersymmetric and covariant formalism: **pure spinors**.

2.2.2 The BRST operator

The next step in the line of reasoning which led to the pure spinor formalism is the proposal of the BRST operator

$$Q_{BRST} = \oint \lambda^\alpha(z) d_\alpha(z), \tag{2.15}$$

where λ^α are bosonic and $d_\alpha = p_\alpha - \frac{1}{2}(\gamma^m\theta)_\alpha \partial X_m - \frac{1}{8}(\gamma^m\theta)_\alpha(\theta\gamma_m\partial\theta)$. However the BRST charge (2.15) must satisfy the consistency condition $Q^2_{BRST} = 0$, otherwise the BRST charge itself would not be invariant under a variation of the gauge constraint [31]. Using (2.15) we obtain

$$Q^2_{BRST} = \frac{1}{2}\{Q_{BRST}, Q_{BRST}\} = -\frac{1}{2}\oint dz(\lambda\gamma^m\lambda)\Pi_m,$$

therefore the bosonic fields λ^α must satisfy the constraints

$$\lambda\gamma^m\lambda = 0. \tag{2.16}$$

Definition 1 (Pure Spinor)**.** *A ten dimensional Weyl spinor λ^α is said to be a pure spinor if* (2.16) *is satisfied for $m = 0 \ldots 9$.*

The formalism discovered by Berkovits is based on the properties of the pure spinor λ^α, and it is important to study what are the number of degrees of freedom which survives the constraints (2.16). Naively one could think that those ten constraints would imply that a ten dimensional pure spinor would have only $16 - 10 = 6$ degrees of freedom, but that's not the case. To see this it is convenient to perform a Wick rotation and break manifest $SO(10)$ Lorentz symmetry to its $U(5)$ subgroup.

A Weyl spinor of SO(10) decomposes under U(5) as follows

$$16 \to (1_{\frac{5}{2}}, 10_{\frac{1}{2}}, \overline{5}_{-\frac{3}{2}}),$$

where the subscrit denotes the U(1)-charge. Using this decomposition for λ^α we can solve the constraints (2.16) explicitly,

$$\lambda_+ = e^s, \tag{2.17}$$

$$\lambda_{ab} = u_{ab}, \tag{2.18}$$

$$\lambda^a = -\frac{1}{8}e^{-s}\epsilon^{abcde}u_{bc}u_{de}, \tag{2.19}$$

for any s and antisymmetric u_{ab}. To prove this we note that $\lambda^\alpha\gamma^m_{\alpha\beta}\lambda^\beta$ is obtained from the 32-dimensional expression $\lambda^T(C\Gamma^m)\lambda$, where C is the conjugation matrix satisfying $C\Gamma^m = -\Gamma^{m,T}C$ which is given by $C = \prod_{i=1}^5(a_i - a^i)$.

Under the decomposition of $SO(10) \to U(5)$ the constraint (2.16) goes to two independent equations

$$\langle\lambda| Ca^i |\lambda\rangle = 0 \qquad i = 1, 2, 3, 4, 5 \tag{2.20}$$

$$\langle\lambda| Ca_i |\lambda\rangle = 0.. \tag{2.21}$$

In the above expressions the only non-vanishing terms are the ones proportional to[2] $\langle 0| Ca^i a^j a^k a^l a^m |0\rangle = \epsilon^{ijklm}$. Therefore, using the known expansion of a Weyl spinor in terms of creation operators

$$|\lambda\rangle = \lambda_+ |0\rangle + \frac{1}{2}\lambda_{ij} a^j a^i |0\rangle + \frac{1}{24}\lambda^i \epsilon_{ijklm} a^m a^l a^k a^j |0\rangle$$

equation (2.20) becomes

$$\langle\lambda| Ca^p |\lambda\rangle = \lambda_+ \langle 0| Ca^p |\lambda\rangle + \frac{1}{2}\lambda_{ij} \langle 0| a_i a_j Ca^p |\lambda\rangle + \frac{1}{24}\lambda^i \epsilon_{ijklm} \langle 0| a_j a_k a_l a_m Ca^p |\lambda\rangle\,. \quad (2.22)$$

Mas,

$$\begin{aligned}\lambda_+ \langle 0| Ca^p |\lambda\rangle &= \frac{1}{24}\lambda_+\lambda^i \epsilon_{ijklm} \langle 0| Ca^p a^m a^l a^k a^j |0\rangle \\ &= \frac{1}{24}\lambda_+\lambda^i \epsilon_{ijklm}\epsilon^{pmlkj} \\ &= \lambda_+\lambda^p\end{aligned}$$

Analogously, by noting that $a_i C = -Ca^i$ and $a^i C = -Ca_i$ we obtain

$$\begin{aligned}\frac{1}{2}\lambda_{ij} \langle 0| a_i a_j Ca^p |\lambda\rangle &= \frac{1}{4}\epsilon^{pijkl}\lambda_{ij}\lambda_{kl} \\ \frac{1}{24}\lambda^i \epsilon_{ijklm} \langle 0| a_j a_k a_l a_m Ca^p |\lambda\rangle &= \lambda_+\lambda^p.\end{aligned}$$

Plugging the above results into (2.22) we arrive at $2\lambda_+\lambda^a + \frac{1}{4}\epsilon^{abcde}\lambda_{bc}\lambda_{de} = 0$ which is easily solved by

$$\lambda_+ \equiv e^s, \quad \lambda_{ab} \equiv u_{ab}, \quad \lambda^a = -\frac{1}{8}e^{-s}\epsilon^{abcde}u_{bc}u_{de}. \quad (2.23)$$

One can also show that (2.21) is automatically satisfied by the above parameterization, therefore the eleven degrees of freedom of e^s and u_{ab} together with (2.23) correctly describe the ten-dimensional pure spinor λ^α.

Let's now see how the pure spinor λ^α can be used to solve the issues present in the approach of Siegel to the Green-Schwarz formulation.

2.3 The pure spinor formalism

To solve the pure spinor constraint it was convenient to break the manifest $SO(10)$ symmetry to its subgroup $U(5)$, so the solution (2.23) is written in terms of $U(5)$ variables.

[2] To prove this one computes $\langle 0| Ca^1 a^2 a^3 a^4 a^5 |0\rangle = 1$ and notes that the expression is completely antisymmetric in the exchange of its indices.

Therefore using this solution one is able to write down only the $U(5)$-covariant Lorentz currents

$$N^{mn} \longrightarrow (n, n_a^b, n_{ab}, n^{ab}).$$

We will be required to check whether the $U(5)$ Lorentz currents constructed out of the variables $s(z)$, $u_{ab}(z)$ and their conjugate momenta $t(z)$ and $v^{ab}(z)$ satisfy the required condition (2.13). To do this we will first need to know how the OPE (2.13) decomposes under $SO(10) \to U(5)$. This can be summarized by the following statement, which we will prove in the appendix.

Theorem 1. *If the SO(10)-covariant OPE of the Lorentz currents N^{mn} is given by*

$$N^{kl}(y)N^{mn}(z) \to \frac{\delta^{m[l}N^{k]n}(z) - \delta^{n[l}N^{k]m}(z)}{y-z} - 3\frac{\delta^{kn}\delta^{lm} - \delta^{km}\delta^{ln}}{(y-z)^2}, \tag{2.24}$$

then the U(5)-covariant currents $(n, n_a^b, n_{ab}, n^{ab})$ satisfy the following OPE's:

$$n_{ab}(y)n^{cd}(z) \to \frac{-\delta^{[c}_{[a}n^{d]}_{b]}(z) - \frac{2}{\sqrt{5}}\delta^{[c}_a\delta^{d]}_b n(z)}{y-z} + 3\frac{\delta^{[c}_a\delta^{d]}_b}{(y-z)^2} \tag{2.25}$$

$$n_a^b(y)n_c^d(z) \to \frac{\delta_c^b n_a^d(z) - \delta_a^d n_c^d(z)}{y-z} - 3\frac{\delta_a^d\delta_c^b - \frac{1}{5}\delta_a^b\delta_c^d}{(y-z)^2} \tag{2.26}$$

$$n(y)n(z) \to -\frac{3}{(y-z)^2}, \tag{2.27}$$

$$n(y)n_{ab}(z) \to -\frac{2}{\sqrt{5}}\frac{n_{ab}}{y-z} \tag{2.28}$$

$$n(y)n^{ab}(z) \to +\frac{2}{\sqrt{5}}\frac{n^{ab}}{y-z} \tag{2.29}$$

$$n(y)n_b^a(z) \to \textit{regular} \tag{2.30}$$

Furthermore there is one more (consistency) condition to be obeyed when constructing those $U(5)$ Lorentz currents. The pure spinor λ^α must obviously transform as a spinor under the action of N^{mn},

$$\delta\lambda^\alpha = \frac{1}{2}\left[\oint dz\epsilon_{mn}M^{mn}, \lambda^\alpha\right] = \frac{1}{4}\epsilon_{mn}(\gamma^{mn}\lambda)^\alpha$$

As the OPE of λ^α with Σ^{mn} have no poles we conclude that the pure spinor must satisfy,

$$N^{mn}(y)\lambda^\alpha(z) \to \frac{1}{2}\frac{(\gamma^{mn})^\alpha{}_\beta\lambda^\beta(z)}{(y-z)}. \tag{2.31}$$

By the same reasoning, the OPE (2.31) must also be broken to U(5) if we want the check whether the U(5) Lorentz currents to be described below satisfy it. That is

Theorem 2. *If the OPE in SO(10)-covariant language is given by*

$$N^{mn}(y)\lambda^{\alpha}(z) \to \frac{1}{2}\frac{(\gamma^{mn})^{\alpha}{}_{\beta}\lambda^{\beta}(z)}{(y-z)}, \tag{2.32}$$

then the OPE's between $(n, n^a_b, n_{ab}, n^{ab})$ *and* $(\lambda_+, \lambda_{cd}, \lambda^c)$ *are given by*

$$n(y)\lambda_+(z) \to -\frac{\sqrt{5}}{2}\frac{\lambda_+(z)}{y-z} \tag{2.33}$$

$$n(y)\lambda_{cd}(z) \to -\frac{1}{2\sqrt{5}}\frac{\lambda_{cd}(z)}{y-z} \tag{2.34}$$

$$n(y)\lambda^c(z) \to \frac{3}{2\sqrt{5}}\frac{\lambda^c(z)}{y-z} \tag{2.35}$$

$$n^a_b(y)\lambda_+(z) \to \textit{regular} \tag{2.36}$$

$$n^a_b(y)\lambda_{cd}(z) \to \frac{\delta^a_d\lambda_{cb} - \delta^a_c\lambda_{db}}{(y-z)} - \frac{2}{5}\frac{\delta^a_b\lambda_{cd}}{(y-z)} \tag{2.37}$$

$$n^a_b(y)\lambda^c(z) \to \frac{1}{5}\delta^a_b\lambda^c - \delta^c_b\lambda^a \tag{2.38}$$

$$n_{ab}(y)\lambda_+(z) \to \frac{\lambda_{ab}(z)}{y-z} \tag{2.39}$$

$$n_{ab}(y)\lambda_{cd}(z) \to \epsilon_{abcde}\lambda^e \tag{2.40}$$

$$n_{ab}(y)\lambda^c(z) \to \textit{regular} \tag{2.41}$$

$$n^{ab}(y)\lambda_+(z) \to \textit{regular} \tag{2.42}$$

$$n^{ab}(y)\lambda_{cd}(z) \to -\frac{\delta^{[a}_c\delta^{b]}_d\lambda_+(z)}{y-z} \tag{2.43}$$

$$n^{ab}(y)\lambda^c(z) \to -\frac{1}{2}\epsilon^{abcde}\lambda_{de} \tag{2.44}$$

Will it be possible to find an action for $s(z)$, $u_{ab}(z)$, $t(z)$ and $v^{ab}(z)$ and explicitly construct the Lorentz currents $(n, n^b_a, n_{ab}, n^{ab})$ out of those variables in such a way as to reproduce all the above OPE's? If it was impossible to do this then the pure spinor formalism would have never been born. In the following paragraphs we will see the solution found by Berkovits.

2.3.1 The action for the ghosts

The action for the ghosts appearing in the pure spinor constraint is given by

$$S_\lambda = \frac{1}{2\pi}\int d^2z\left(-\partial t\overline{\partial}s + \frac{1}{2}v^{ab}\overline{\partial}u_{ab}\right) \tag{2.45}$$

where $t(z)$ e $v^{ab}(z)$ are the conjugate momenta for $s(z)$ and $u_{ab}(z)$. Furthermore $s(z)$ and $t(z)$ chiral bosons, so that we must impose their equations of motions by hand $\overline{\partial}s = \overline{\partial}t = 0$. The OPE's are given by

$$t(y)s(z) \to \ln(y-z) \tag{2.46}$$

$$v^{ab}(y)u_{cd}(z) \to 2\frac{\delta^{ab}_{cd}}{y-z} = \frac{\delta^{[a}_c\delta^{b]}_d}{y-z}. \tag{2.47}$$

One of the most important results which allowed the birth of the pure spinor formalism is given by the following theorem

Theorem 3. *If the U(5)-symmetric Lorentz currents are built out of the ghosts as follows*

$$n = -\frac{1}{\sqrt{5}}\left(\frac{1}{4}u_{ab}v^{ab} + \frac{5}{2}\partial t - \frac{5}{2}\partial s\right) \tag{2.48}$$

$$n^a_b = u_{bc}v^{ac} - \frac{1}{5}\delta^a_b u_{cd}v^{cd} \tag{2.49}$$

$$n^{ab} = -e^s v^{ab} \tag{2.50}$$

$$n_{ab} = e^{-s}\left(2\partial u_{ab} - u_{ab}\partial t - 2u_{ab}\partial s + u_{ac}u_{bd}v^{cd} - \frac{1}{2}u_{ab}u_{cd}v^{cd}\right) \tag{2.51}$$

then their OPE's among themselves and with λ_+, λ_{ab} e λ^a correctly reproduce the relations (2.25)-(2.30) *and* (2.33)-(2.44)*, if $s(z)$, $t(z)$, $v^{ab}(z)$ e $u_{ab}(z)$ satisfy the OPE's* (2.46) *and* (2.47)*.*

Proof. We will explicitly check a few of those OPE's as the others can be shown along similar lines. For example, one can easily check (B.15) as follows,

$$\begin{aligned} n(y)\lambda_{cd}(z) &= -\frac{1}{\sqrt{5}}\left(\frac{1}{4}u_{ab}v^{ab} + \frac{5}{2}\partial t - \frac{5}{2}\partial s\right)u_{cd}(z) \\ &\to -\frac{1}{4\sqrt{5}}u_{ab}(y)\left(\frac{\delta^{[a}_c\delta^{b]}_d}{y-z}\right) = -\frac{1}{2\sqrt{5}}\frac{\lambda_{cd}(z)}{y-z}. \end{aligned}$$

Similarly, (2.42) is easily seen to be true because $s(z)$ has no poles with itself nor with $v^{ab}(y)$,

$$n^{ab}(y)\lambda_+(z) = -(e^s v^{ab})e^s \to \text{regular}.$$

Using (2.47) we check the validity of (2.43),

$$n^{ab}(y)\lambda_{cd}(z) = -(e^s v^{ab})u_{cd} \to -\frac{\delta^{[a}_c\delta^{b]}_d}{y-z}\lambda_+.$$

The OPE (2.28) requires a bit more work but it also comes out right. Using (2.48) we get

$$n(y)n(z) \to \frac{1}{80}\left(u_{ab}v^{cd} : v^{ab}u_{cd} : + u_{ab} : v^{ab}u_{cd} : v^{cd} + : u_{ab}v^{cd} : v^{ab}u_{cd}\right) - \frac{5}{4}\left(: \partial t \partial s : + : \partial s \partial t :\right),$$

and one can check that the simple pole terms cancel while for the double pole we get

$$\to -\frac{1}{80}\frac{\delta_c^{[a}\delta_d^{b]}\delta_a^{[c}\delta_b^{d]}}{(y-z)^2} - \frac{10}{4}\frac{1}{(y-z)^2} \to -\frac{3}{(y-z)^2},$$

so it correctly reproduces (2.28). Finally we check (2.40),

$$\begin{aligned} n_{ab}(y)\lambda_{cd}(z) &\to e^{-s}\left(u_{ae}u_{bf} : v^{ef}u_{cd} : -\frac{1}{2}u_{ab}u_{ef} : v^{ef}u_{cd} :\right) \\ &\to e^{-s}\left(u_{ae}u_{bf} - \frac{1}{2}u_{ab}u_{ef}\right)\frac{\delta_c^{[e}\delta_d^{f]}}{y-z} \\ &\to e^{-s}\left(u_{ac}u_{bd} - u_{ad}u_{bc} - u_{ab}u_{cd}\right) = \epsilon_{abcde}\lambda^e, \end{aligned}$$

where in the last line we used (2.19). The proof for all the other cases is analogous and will be omited. ■

We will show in the following that the central charge for the ghost action (2.45) is +22, which is indeed the required value for it to annihilate the total central charge when added to Siegel's action.

The energy momentum tensor for the ghosts can be found using Noether's procedure, with the following definition for the variation of the action

$$\delta S_\lambda = \frac{1}{2\pi}\int d^2z[\overline{\partial}\varepsilon T_\lambda(z) + \partial\overline{\varepsilon}\overline{T}_\lambda(\overline{z})],$$

under the conformal transformations of

$$\delta v^{ab} = \partial\varepsilon v^{ab} + \varepsilon\partial v^{ab} + \overline{\varepsilon}\overline{\partial}v^{ab} \tag{2.52}$$

$$\delta u_{ab} = \varepsilon\partial u_{ab} + \overline{\varepsilon}\overline{\partial}u_{ab} \tag{2.53}$$

$$\delta\partial s = \partial\varepsilon\partial s + \varepsilon\partial^2 s + \partial\overline{\varepsilon}\overline{\partial}s + \overline{\partial}\overline{\varepsilon}\partial s \tag{2.54}$$

$$\delta\overline{\partial}t = \varepsilon\partial\overline{\partial}t + \overline{\partial}\varepsilon\partial t + \overline{\partial}\overline{\varepsilon}\overline{\partial}t + \overline{\varepsilon}\overline{\partial}\overline{\partial}t. \tag{2.55}$$

Doing this we obtain

$$T_\lambda(z) = \frac{1}{2}v^{ab}\partial u_{ab} + \partial t\partial s + \partial^2 s.$$

For example,

$$\delta(\overline{\partial}t\partial s) = \left(\varepsilon\partial\overline{\partial}t + \overline{\partial}\varepsilon\partial t + \overline{\partial}\overline{\varepsilon}\overline{\partial}t + \overline{\varepsilon}\overline{\partial}\overline{\partial}t\right)\partial s + \overline{\partial}t\left(\partial\varepsilon\partial s + \varepsilon\partial^2 s + \partial\overline{\varepsilon}\overline{\partial}s + \overline{\partial}\overline{\varepsilon}\partial s\right)$$
$$= \partial(\varepsilon\overline{\partial}t\partial s) + \overline{\partial}(\overline{\varepsilon}\overline{\partial}t\partial s) + \overline{\partial}\varepsilon\partial t\partial s + \partial\overline{\varepsilon}\overline{\partial}s\overline{\partial}t,$$

so up to a surface term, $T(z) = \partial t\partial s$ is the contribution from the variables s, t. The contribution from the variables v^{ab} e u_{ab} can be easily obtained by noticing that it is a $\beta\gamma$ system with $\lambda = 1$, if the following identification is made $\beta \to -1/2v^{ab}$ and $\gamma \to u_{ab}$. As the energy momentum tensor for $\beta\gamma$ system is given by [32] $T(z) = \partial\beta\gamma - \lambda\partial(\beta\gamma) = \frac{1}{2}v^{ab}\partial u_{ab}$, it follows that

$$T(z) = \frac{1}{2}v^{ab}u_{ab} + \partial t\partial s.$$

To justify the addition of the term $\partial^2 s$ in $T(z)$ we compute the OPE of $T(y)$ with the Lorentz current $n(z)$ from Theorem 3. We get

$$T(y)n(z) \to \frac{\sqrt{5}}{(y-z)^3} + \frac{n(z)}{(y-z)^2} + \frac{\partial n(z)}{(y-z)}$$

where the triple pole comes from

$$\left(\frac{1}{2}v^{ab}\partial u_{ab}\right)\left(\frac{1}{4\sqrt{5}}u_{cd}v^{cd}\right) \to \frac{1}{8\sqrt{5}}\frac{\delta_c^{[a}\delta_d^{b]}\delta_a^{[c}\delta_b^{d]}}{(y-z)^3} = \frac{\sqrt{5}}{(y-z)^3}.$$

Therefore the Lorentz current would fail to be a primary field, but that can be fixed by the addition of $\partial^2 s$, because

$$\partial^2 s(y)n(z) \to \frac{\sqrt{5}}{2} : \partial^2 s(y)\partial t(z) := -\frac{\sqrt{5}}{(y-z)^3}.$$

So we have shown that the energy momentum tensor for the ghost variables is given by

$$T_\lambda(z) = \frac{1}{2}v^{ab}\partial u_{ab} + \partial t\partial s + \partial^2 s. \tag{2.56}$$

The central charge can be easily computed by considering the fourth order pole in $T_\lambda(y)T_\lambda(z)$. There are two contributions

$$\frac{1}{4} : v^{ab}(y)\partial u_{cd}(z) :: \partial u_{ab}(z)v^{cd}(y) : = \frac{1}{4}\frac{\delta_c^{[a}\delta_d^{b]}\delta_a^{[c}\delta_b^{d]}}{(y-z)^4} = \frac{10}{(y-z)^4},$$

and

$$: \partial t(y)\partial s(z) :: \partial s(z)\partial t(y) : = \frac{1}{(y-z)^4},$$

whose sum imply that $c_g = +22$. Therefore, as there are no poles between the ghosts and matter variables, the total central charge of the energy momentum tensor in the pure spinor formalism

$$T(z) = -\frac{1}{2}\partial X^m \partial X_m - p_\alpha \partial\theta^\alpha + \frac{1}{2} v^{ab}\partial u_{ab} + \partial t \partial s + \partial^2 s, \tag{2.57}$$

is zero.

The conclusion from the previous discussion is that the addition of the pure spinor ghost action of (2.45) to the Siegel action (2.2) makes the central charge of the theory to vanish and implies that the Lorentz currents have the same OPE as in the RNS formalism. So the pure spinor formalism action for the left-moving fields is given by

$$S = \frac{1}{2\pi}\int dz\big[\frac{1}{2}\partial X^m \overline{\partial} X_m + p_\alpha \overline{\partial}\theta^\alpha - \partial t \overline{\partial} s + \frac{1}{2} v^{ab}\overline{\partial} u_{ab}\big]. \tag{2.58}$$

The variables in the pure spinor formalism have the following supersymmetry transformations

$$\delta X^m = \frac{1}{2}\left(\varepsilon\gamma^m\theta\right), \quad \delta\theta^\alpha = \varepsilon^\alpha, \quad \delta(\text{ghosts}) = 0, \tag{2.59}$$

$$\delta p_\beta = -\frac{1}{2}\varepsilon^\alpha \gamma^m_{\alpha\beta}\partial X_m + \frac{1}{8}\varepsilon^\alpha\theta^\gamma\partial\theta^\delta\gamma^m_{\beta\delta}\gamma_{m\,\gamma\alpha} \tag{2.60}$$

and one can check that they are generated by

$$Q_\alpha = \oint (p_\alpha + \frac{1}{2}\gamma^m_{\alpha\beta}\theta^\beta\partial X_m + \frac{1}{24}\gamma^m_{\alpha\beta}\gamma_{m\,\gamma\delta}\theta^\beta\theta^\gamma\partial\theta^\delta),$$

which satisfy the supersymmetry algebra

$$\{Q_\alpha, Q_\beta\} = \gamma^m_{\alpha\beta}\oint \partial X_m.$$

For example, the variation of (2.58) under (2.59)-(2.60) can be checked to be

$$\delta S = \int dz\Big[\frac{1}{4}(\varepsilon\gamma^m\partial\theta)\overline{\partial} X_m + \frac{1}{4}(\varepsilon\gamma^m\overline{\partial}\theta)\partial X_m - \frac{1}{2}(\varepsilon\gamma^m\overline{\partial}\theta)\partial X_m - \frac{1}{8}(\overline{\partial}\theta\gamma^m\partial\theta)(\varepsilon\gamma_m\theta)\Big]. \tag{2.61}$$

Integrating the first term by parts we get $-1/4\int(\varepsilon\gamma^m\overline{\partial}\partial\theta)X_m$, which can be integrated by parts again to result in $+1/4\int(\varepsilon\gamma^m\overline{\partial}\theta)\partial X_m$. Therefore the sum of the first two terms of (2.61) cancels the third. So the supersymmetry variation of the pure spinor action (2.58) will be zero if $\int(\overline{\partial}\theta\gamma^m\partial\theta)(\varepsilon\gamma_m\theta)$ vanishes. To see that this we integrate it by parts to obtain

$$\begin{aligned}
\int(\overline{\partial}\theta\gamma^m\partial\theta)(\varepsilon\gamma_m\theta) &= -\int(\theta\gamma^m\partial\theta)(\varepsilon\gamma_m\overline{\partial}\theta) - \int(\theta\gamma^m\overline{\partial}\partial\theta)(\varepsilon\gamma_m\theta) \\
&= -\int(\theta\gamma^m\partial\theta)(\varepsilon\gamma_m\overline{\partial}\theta) + \int(\partial\theta\gamma^m\overline{\partial}\theta)(\varepsilon\gamma_m\theta) + \int(\theta\gamma^m\overline{\partial}\theta)(\varepsilon\gamma_m\partial\theta) \\
&= -\int\theta^\alpha\partial\theta^\beta\varepsilon^\gamma\overline{\partial}\theta^\sigma\left(\gamma^m_{\alpha\beta}\gamma_{m\,\gamma\sigma} - \gamma^m_{\beta\sigma}\gamma_{m\,\gamma\alpha} + \gamma^m_{\alpha\sigma}\gamma_{m\,\gamma\beta}\right) \\
&= +2\int\theta^\alpha\partial\theta^\beta\varepsilon^\gamma\overline{\partial}\theta^\sigma\left(\gamma^m_{\beta\sigma}\gamma_{m\,\gamma\alpha}\right) \\
&= -2\int\left(\overline{\partial}\theta\gamma^m\partial\theta\right)\left(\varepsilon\gamma_m\theta\right),
\end{aligned}$$

where we used $\gamma^m_{\alpha(\beta}(\gamma_m)_{\gamma\delta)} = 0$. We therefore conclude that $\int(\overline{\partial}\theta\gamma^m\partial\theta)(\varepsilon\gamma_m\theta) = 0$, which finishes the proof that (2.58) is supersymmetric.

We can define the ghost number of any state $\Psi(y)$ by

$$[\oint dzJ(z), \Psi(y)] = n_g\Psi(y),$$

where the ghost current $J(z)$ is given by [36]

$$J(z) = \frac{1}{2}u_{ab}v^{ab} + \partial t + 3\partial s. \tag{2.62}$$

One can check that the ghost current defined above satisfies the following OPE's [7],

$$J(y)\lambda^\alpha(z) \to \frac{\lambda^\alpha}{y-z} \tag{2.63}$$

$$J(y)J(z) \to -\frac{4}{(y-z)^2} \tag{2.64}$$

$$J(y)T(z) \to -\frac{8}{(y-z)^3} + \frac{J(z)}{(y-z)^2} \tag{2.65}$$

$$J(y)N^{mn}(z) \to \text{regular} \tag{2.66}$$

$$T(y)J(z) \to \frac{8}{(y-z)^3} + \frac{J(z)}{(y-z)^2} + \frac{\partial J(z)}{y-z}. \tag{2.67}$$

For example, to show that (2.63) is true we must compute the OPE's of $J(y)$ with the U(5) components of λ^α to check that the results are compatible. So

$$J(y)\lambda_+(z) \to \quad :\partial t(y)e^s(z):=\to \frac{\lambda_+}{y-z}$$

$$J(y)\lambda_{cd} \to \frac{1}{2}u_{ab}(y):v^{ab}(y)u_{cd}(z): \quad = \frac{1}{2}u_{ab}\frac{\delta^{[a}_c\delta^{b]}_d}{y-z} = \frac{\lambda_{cd}}{y-z}$$

The triple pole of (2.65), for example, comes from the following contractions

$$\frac{1}{4}:u_{cd}(y)v^{ab}(z)::v^{cd}(y)\partial u_{ab}(z):\to -\frac{1}{4}\frac{\delta_c^{[a}\delta_d^{b]}\delta_a^{[c}\delta_b^{d]}}{(y-z)^3}=-\frac{10}{(y-z)^3}$$
$$:\partial t(y)\partial^2 s(z):\to \frac{2}{(y-z)^3}$$

whose sum results in the coefficient -8. The proof for the other OPE's is similar and therefore will be omited.

From (2.63) we can see that the ghost number of the pure spinor λ^α is $+1$. Moreover from (2.66) we see that $J(z)$ is a Lorentz scalar (as it should be) and from (2.67) that there is an anomaly of $+8$ in the ghost current, which has conformal weight $h=1$.

2.4 Massless vertex operators

The physical states in the pure spinor formalism are defined to be in the cohomology of the BRST operator

$$Q=\frac{1}{2\pi i}\oint \lambda^\alpha d_\alpha$$

which satisfy $Q^2=0$ due to the pure spinor condition (2.16) and the OPE (2.5). Therefore we can define the unintegrated and integrated massless vertex operators for the super-Yang-Mills states as follows

$$V=\lambda^\alpha A_\alpha(x,\theta) \tag{2.68}$$

$$U(z)=\partial\theta^\alpha A_\alpha(x,\theta)+A_m(x,\theta)\Pi^m+d_\alpha W^\alpha(x,\theta)+\frac{1}{2}N_{mn}\mathcal{F}^{mn}(x,\theta), \tag{2.69}$$

where the superfields A_α, A_m, W^α and $\mathcal{F}_{mn}$ describe the super-Yang-Mills theory in D=10, which is briefly reviewed in appendix B.

In the RNS formalism the unintegrated vertex operator satisfies $QU=\partial V$, as one can check by recalling that $U=\{\oint b,V\}$ and $T=\{Q,b\}$. The proof then follows from the Jacobi identity

$$QU=[Q,\{\oint b,V\}]=-[V,\{Q,\oint b\}]-[\oint b,\{V,Q\}]=\partial V \tag{2.70}$$

because the cohomology condition requires $\{V,Q\}=0$ and the conformal weight zero of V implies $[\oint T,V]=\partial V$.

In the pure spinor formalism the integrated vertex (2.69) also satisfies (2.70). To see this we use the OPE's (2.4), (2.5) and (2.6) and the equations of motion for the SYM superfields listed in Appendix B to get

$$Q(\partial\theta^\alpha A_\alpha) = (\partial\lambda^\alpha)A_\alpha - \partial\theta^\alpha\lambda^\beta D_\beta A_\alpha$$

$$Q(\Pi^m A_m) = (\lambda\gamma^m\partial\theta)A_m + \Pi^m\lambda^\alpha(D_\alpha A_m)$$

$$Q(d_\alpha W^\alpha) = -(\lambda\gamma^m W)\Pi_m - d_\beta\lambda^\alpha D_\alpha W^\beta$$

$$Q(\frac{1}{2}N_{mn}\mathcal{F}^{mn}) = \frac{1}{4}(\gamma^{mn}\lambda)^\alpha d_\alpha\mathcal{F}_{mn} + \frac{1}{2}N_{mn}\lambda^\alpha D_\alpha\mathcal{F}^{mn}$$

Therefore

$$QU = (\partial\lambda^\alpha)A_\alpha - \partial\theta^\beta\lambda^\alpha(D_\alpha A_\beta - \gamma^m_{\alpha\beta}A_m) + \lambda^\alpha\Pi^m(D_\alpha A_m - (\gamma_m W)_\alpha)$$

$$-\lambda^\alpha d_\beta(D_\alpha W^\beta + \frac{1}{4}(\gamma^{mn})_\alpha{}^\beta\mathcal{F}_{mn}) + N_{mn}(\lambda\gamma^n\partial^m W). \qquad (2.71)$$

Using the equations of motion listed in Appendix B we get

$$QU = (\partial\lambda^\alpha)A_\alpha + \lambda^\alpha\partial\theta^\beta D_\beta A_\alpha + \lambda^\alpha\Pi^m\partial_m A_\alpha \qquad (2.72)$$

where the last term in (2.71) vanished by the pure spinor condition $(\lambda\gamma^n)_\alpha(\lambda\gamma_n)_\beta = 0$ and the equation of motion $\gamma^m_{\alpha\beta}\partial_m W^\beta = 0$,

$$N_{mn}(\lambda\gamma^n\partial^m W) = \frac{1}{2}(w\gamma^m\gamma^n\lambda)(\lambda\gamma^n\partial^m W) - (w\lambda)(\lambda\gamma^m\partial_m W) = 0.$$

Using the definitions (2.3) and (2.7) one easily checks that (2.72) becomes

$$QU = (\partial\lambda^\alpha)A_\alpha + \lambda^\alpha(\partial\theta^\beta\partial_\beta A_\alpha + \partial X^m\partial_m A_\alpha)$$

$$= (\partial\lambda^\alpha)A_\alpha + \lambda^\alpha\partial A_\alpha = \partial(\lambda A) = \partial V,$$

as we wanted to show.

The unintegrated vertex operator satisfies $QV = 0$ if the superfield A_α is on-shell, *i.e.*, if equation (B.8) is obeyed,

$$QV = \oint \lambda^\alpha(z)d_\alpha(z)\lambda^\beta(w)A_\beta(x,\theta) = \lambda^\alpha\lambda^\beta D_\alpha A_\beta = 0,$$

where we used that $\lambda^\alpha\lambda^\beta = (1/3840)(\lambda\gamma^{mnpqr}\lambda)\gamma^{\alpha\beta}_{mnpqr}$ for pure spinors λ^α.

2.5 Tree-level prescription

The prescription to compute N-point superstring amplitudes at tree-level is given by

$$\mathcal{A} = \langle V^1 V^2 V^3 \int U^4 \dots \int U^N \rangle \tag{2.73}$$

where the angle brackets is defined in such a way as to be non-vanishing only when there are three pure spinor λ's and five θ's in a combination proportional to

$$\langle (\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta) \rangle = 1. \tag{2.74}$$

One can check that the measure (2.74) is in the cohomology of the pure spinor BRST operator (2.15). It is BRST-closed due to the pure spinor constraint (2.16). And it is not BRST-trivial because there is no Lorentz scalar built out of two λ's and six θ's. To check this one uses the theory of group representations as follows.

The representation of two pure spinors λ^α is given by $[0,0,0,0,2]$ while six antisymmetric thetas are represented by $[0,1,0,2,0]+[2,0,1,0,0]$. Therefore

$$[0,0,0,0,2] \otimes \Big[[0,1,0,2,0] + [2,0,1,0,0] \Big] = 1[0,0,0,1,1] + 1[0,0,0,2,2] + 2[0,0,1,2,0] + \dots$$

has no scalar component.

The pure spinor measure (2.74) together with BRST-closedness of the vertex operators imply that the amplitude prescription is supersymmetric. To see this one notes that the only possibility of getting a non-vanishing result after a supersymmetry transformation $\delta\theta^\alpha = \epsilon^\alpha$ is if the amplitude of (2.73) contains the term

$$\mathcal{A} = \langle (\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)(\theta^\alpha\Phi_\alpha + \dots) \rangle \tag{2.75}$$

for some Φ_α. If that were true then the supersymmetry variation $\delta_S\mathcal{A}$ would be

$$\delta_S\mathcal{A} = \int dz_4 \cdots \int dz_N \epsilon^\alpha \Phi_\alpha. \tag{2.76}$$

But note that the result of the amplitude calculation of (2.73)

$$\int dz_4 \cdots \int dz_N \lambda^\alpha\lambda^\beta\lambda^\gamma f_{\alpha\beta\gamma}(\theta)$$

must satisfy the BRST-closedness property of

$$\int dz_4 \cdots \int dz_N \lambda^\alpha\lambda^\beta\lambda^\gamma\lambda^\delta D_\delta f_{\alpha\beta\gamma}(\theta) = 0. \tag{2.77}$$

Plugging (2.75) into (2.77) we conclude that

$$\int dz_4 \cdots \int dz_N \lambda^\alpha \lambda^\beta \lambda^\gamma \lambda^\delta \Phi_\delta = 0,$$

which is only possible if Φ_δ is a total derivative, implying that the supersymmetry variation of (2.76) vanishes, $\delta_S \mathcal{A} = 0$.

2.6 Multiloop prescription

The prescription to compute multiloop amplitudes in the minimal pure spinor formalism was spelled out in [7], which we now briefly review.

The multiloop prescription in the pure spinor formalism was made possible by the construction of the analogous operators of the picture changing operators in the RNS formalism, which can be understood as being necessary to absorb the zero-modes of the various variables. As it is well-known [41], the zero-modes of bosonic variables are to be dealt with with the introduction of delta functions depending over the variable which has the zero mode. The fermionic zero modes require the insertion of as much fermionic variables as is the number of zero modes, otherwise the Berezin integration will produce a vanishing result.

So the analysis of zero modes will play a crucial rôle in the multiloop prescription. But for our purposes in this thesis it will be sufficient to know that a conformal weight one variable Φ_1 has g zero-modes in a genus g Riemann surface, while a conformal weight zero variable Φ_0 always has one zero mode in every genus.

In the pure spinor formalism the zero modes of λ^α, N^{mn} and J will require insertions of delta functions involving these variables. They are given as follows

$$Y_C = C_\alpha \theta^\alpha \delta(C_\beta \lambda^\beta), \quad Z_B = \frac{1}{2} B_{mn} (\lambda \gamma^{mn} d) \delta(B^{pq} N_{pq}), \quad Z_J = (\lambda^\alpha d_\alpha) \delta(J), \qquad (2.78)$$

where C_α and B_{mn} are constant tensors. They will be responsable for killing the eleven zero modes of λ^α and $11g$ zero modes of w_α. Therefore after eliminating the conformal weight one variables through their OPE's one will be left with an expression containing only the zero modes of all the variables which are part of the pure spinor formalism. Those zero modes will be absorbed by the insertions of the operators (2.78), but one will need explicit measures to integrate what is left.

The measure for integration over the eleven λ zero-modes is given by

$$[d\lambda]\lambda^\alpha\lambda^\beta\lambda^\gamma = \epsilon_{\rho_1\ldots\rho_{11}\kappa_1\ldots\kappa_5} T^{((\alpha\beta\gamma))[\kappa_1\kappa_2\kappa_3\kappa_4\kappa_5]} d\lambda^{\rho_1}\ldots d\lambda^{\rho_1 1}$$

while for the w_α zero modes it reads

$$(d^{11}N)^{[[m_1n_1][m_2n_2]\ldots[m_{10}n_{10}]]} = [dN]$$

$$\Big[(\lambda\gamma^{m_1n_1m_2m_3m_4}\lambda)(\lambda\gamma^{m_5n_5n_2m_6m_7}\lambda)(\lambda\gamma^{m_8n_8n_3n_6m_9}\lambda)(\lambda\gamma^{m_{10}n_{10}n_4n_7n_9}\lambda) + \text{permutations}\Big]$$

where

$$(d^{11}N)^{[[m_1n_1][m_2n_2]\ldots[m_{10}n_{10}]]} \equiv dN^{[m_1n_1]} \wedge dN^{[m_2n_2]} \wedge \ldots \wedge dN^{[m_{10}n_{10}]} \wedge dJ.$$

and

$$T^{((\alpha\beta\gamma))[\kappa_1\kappa_2\kappa_3\kappa_4\kappa_5]} = (\gamma_m)^{\kappa[\kappa_1}(\gamma_n)^{\sigma|\kappa_2}(\gamma_p)^{\tau\kappa_3}(\gamma^{mnp})^{\kappa_4\kappa_5]}(\delta_\kappa^{(\alpha}\delta_\sigma^\beta\delta_\tau^{\gamma)} - \frac{1}{40}\gamma_q^{(\alpha\beta}\delta_\kappa^{\gamma)}\gamma^q_{\sigma\tau}).$$

To compute multiloop amplitudes over a g-genus Riemann surface one needs to have a measure for the integration over the moduli space of Riemann surfaces. The standard way to achieve this is through the insertion of $3g-3$ factors containing the b-ghost and the Beltrami differential, which is a conformal weight $(-1,1)$ differential defined by

$$\mu_z{}^{\bar{z}} = g^{z\bar{z}}\frac{\partial g_{zz}}{\partial\tau}.$$

That insertion has the property of being a density for the moduli integration, because the Beltrami differential transforms as

$$\mu_z{}^{\bar{z}} = \tilde{\mu}_z{}^{\bar{z}}\frac{\partial\tilde{\tau}}{\partial\tau}.$$

Explicitly the b-ghost insertion reads

$$\langle b\cdot\mu\rangle = \int d^2z b_{zz}\mu^z{}_{\bar{z}}$$

However the b-ghost must satisfy the property of $\{Q, b(z)\} = T(z)$ because $\langle b\cdot\mu\rangle$ must be BRST-invariant after the integration over moduli space. But in the pure spinor formalism there is no such object, because there is no gauge invariant operator with ghost number -1 (with respect to $J = (\lambda w)$).

The idea to overcome this difficulty was to construct an operator $b(u,z)$ such that

$$\{Q, b_B(u,z)\} = T(u)Z_B(z)$$

because whenever one needs to insert the $3g-3$ b-ghosts in the scattering amplitude prescription one also needs to insert $10g$ of Z_B and $1g$ of Z_J to deal with the zero modes of w_α. Then the idea was to borrow $3g-3$ Z_B's into the factor containing the measure for the moduli space. Therefore the insertion of $\langle b_B \cdot \mu\rangle$ in the pure spinor amplitude prescription will respect its BRST-closedness property up to a total derivative in moduli space.

The multiloop amplitude prescription for genus higher than one is given by

$$A = \int d^2\tau_1 \ldots d^2\tau_{3g-3} \langle \prod_{P=1}^{3g-3} \int d^2u_P \mu_P(u_P) \tilde{b}_{B_P}(u_P, z_P)$$

$$\prod_{P=3g-2}^{10g} Z_{B_P}(z_P) \prod_{R=1}^{g} Z_J(v_R) \prod_{I=1}^{11} Y_{C_I}(y_I)\ |^2 \prod_{T=1}^{N} \int d^2t_T U_T(t_T) \rangle,$$

where the b_B-ghost is a complicated operator whose expression can be looked in [7]. For the genus one surface the prescription is given by

$$A_{\text{one-loop}} \int d^2\tau \langle\ |\ \int d^2u \mu(u) \widetilde{b}_{B_1}(u, z_1)$$

$$\prod_{P=2}^{10} Z_{B_P}(z_P) Z_J(v) \prod_{I=1}^{11} Y_{C_I}(y_I)\ |^2\ V_1(t_1) \prod_{T=2}^{N} \int d^2t_T U_T(t_T)\ \rangle,$$

where due to translational invariance of the torus one can fix the position of one unintegrated vertex operator V_1.

The $\langle\quad\rangle$ brackets means the integration over the zero modes of the various variables using the measures described above together with the Berezin integrals over $\int d^{16}\theta$ and $\int d^{16}d$.

2.7 The non-minimal pure spinor formalism

In the year 2005 a modification of the pure spinor formalism was proposed in [26] which features the addition of the left-moving non-minimal variables (r_α, s^β) and $(\overline{\lambda}_\alpha, \overline{w}^\alpha)$. The action is given by

$$S_{\text{NMPS}} = \frac{1}{2\pi} \int d^2z (\frac{1}{2}\partial x^m \overline{\partial} x_m + p_\alpha \overline{\partial}\theta^\alpha - w_\alpha \overline{\partial}\lambda^\alpha - \overline{w}^\alpha \overline{\partial\lambda}_\alpha + s^\alpha \overline{\partial} r_\alpha) \tag{2.79}$$

where $\overline{w}^\alpha$ and s^α are the conformal weight one conjugate momenta of the bosonic pure spinor $\overline{\lambda}_\alpha$ and the fermionic spinor r_α which satisfies

$$(\overline{\lambda}\gamma^m r) = 0.$$

Their OPE's are given by

$$\overline{\lambda}_\alpha(z)\overline{w}^\beta(y) \to \frac{\delta^\beta_\alpha}{z-y}, \quad s^\alpha(z)r_\beta(w) \to \frac{\delta^\alpha_\beta}{z-w}.$$

Analogously to the minimal pure spinor formalism variables where

$$N_{mn} = \frac{1}{2}(w\gamma_{mn}\lambda), \quad J_\lambda = w_\alpha\lambda^\alpha, \quad T_\lambda = w_\alpha\partial\lambda^\alpha,$$

the new variables also have their associated Lorentz and ghost currents,

$$\overline{N}_{mn} = \frac{1}{2}(\overline{w}\gamma_{mn}\overline{\lambda} - s\gamma_{mn}r), \quad \overline{J}_{\overline{\lambda}} = \overline{w}^\alpha\overline{\lambda}_\alpha - s^\alpha r_\alpha, \quad T_{\overline{\lambda}} = \overline{w}^\alpha\partial\overline{\lambda}_\alpha - s^\alpha\partial r_\alpha,$$

Furthermore one also defines

$$S_{mn} = \frac{1}{2}(s\gamma_{mn}\overline{\lambda}), \quad S = s^\alpha\overline{\lambda}_\alpha, \quad J_r = (rs).$$

and the total ghost current to be[3]

$$J = w_\alpha\lambda^\alpha - s^\alpha r_\alpha - \frac{2}{(\lambda\overline{\lambda})}\left[(\overline{\lambda}\partial\lambda) + (r\partial\theta)\right] + \frac{2}{(\lambda\overline{\lambda})^2}(\lambda r)(\overline{\lambda}\partial\theta), \tag{2.80}$$

which is BRST equivalent to

$$J_b = J_\lambda - \overline{J}_{\overline{\lambda}} + J_r = w_\alpha\lambda^\alpha - \overline{w}^\alpha\overline{\lambda}_\alpha.$$

The non-minimal BRST operator is defined by

$$Q = \int dz(\lambda^\alpha d_\alpha + \overline{w}^\alpha r_\alpha). \tag{2.81}$$

Using the Kugo-Ojima (KO) quartet mechanism [17][18] one can show that the cohomology of the non-minimal BRST operator (2.81) doesn't depend on the "quartet" of non-minimal variables $(r_\alpha, s^\alpha), (\overline{\lambda}_\alpha, \overline{w}^\alpha)$. That will allow us to choose a gauge were the external vertex operators are independent of the non-minimal variables, so that the same vertices as in the minimal pure spinor formalism can be used.

Furthermore, due to the existence of the pure spinor field $\overline{\lambda}_\alpha$ it is possible to construct a "simple" b-ghost satisfying $\{Q, b(z)\} = T(z)$, where

$$b = s^\alpha\partial\overline{\lambda}_\alpha + \frac{1}{4(\overline{\lambda}\lambda)}\Big[(2\Pi^m(\overline{\lambda}\gamma_m d) - N_{mn}(\overline{\lambda}\gamma^{mn}\partial\theta) - J_\lambda(\overline{\lambda}\partial\theta) - (\overline{\lambda}\partial^2\theta)\Big]$$

[3]There is a typo in equations (3.14) and (3.15) of [26], where $\lambda^\alpha r_\alpha$ was written as $\overline{\lambda}_\alpha r^\alpha$.

$$+\frac{(\overline{\lambda}\gamma^{mnp}r)(d\gamma_{mnp}d+24N_{mn}\Pi_p)}{192(\overline{\lambda}\lambda)^2}-\frac{(r\gamma_{mnp}r)(\overline{\lambda}\gamma^m d)N^{np}}{16(\overline{\lambda}\lambda)^3}+\frac{(r\gamma_{mnp}r)(\overline{\lambda}\gamma^{pqr}r)N^{mn}N_{qr}}{128(\overline{\lambda}\lambda)^4} \tag{2.82}$$

and the total energy momentum tensor is given by

$$T(z)=-\frac{1}{2}\partial x^m\partial x_m-(p\partial\theta)+(w\partial\lambda)+(\overline{w}\partial\overline{\lambda})-(s\partial r). \tag{2.83}$$

Now the key aspect of this non-minimal construction follows from the observation that the operators $T(z), G^+(z)=2j_{\mathrm{BRST}}, G^-(z)=b$ and $J(z)$ satisfy the twisted $\hat{c}=3$ $N=2$ algebra

$$T(z)T(w)\to\frac{c/2}{(z-w)^4}+\frac{2T}{(z-w)^2}+\frac{\partial T}{(z-w)}$$

$$T(z)G^{\pm}\to\frac{3}{2}\frac{G^{\pm}}{(z-w)^2}+\frac{\partial G^{\pm}}{(z-w)}$$

$$G^+(z)G^-(w)\to\frac{2c/3}{(z-w)^3}+\frac{2J}{(z-w)^2}+\frac{T}{(z-w)}$$

$$T(z)J(w)\to\frac{\hat{c}}{(z-w)^3}+\frac{J}{(z-w)^2}+\frac{\partial J}{(z-w)}$$

$$J(z)G^{\pm}(w)\to\pm\frac{G^{\pm}}{(z-w)}$$

$$J(z)J(w)\to\frac{c/3}{(z-w)^2}.$$

In particular we note that the anomaly of $\hat{c}=+3$ in the ghost current of (2.80) is the same as the anomaly of $J=-bc$ in bosonic string theory. The anomaly of $+3$ in the ghost current implies the non-conservation of $3g-3$ units of charge in a genus g Riemann surface, via the Riemann-Roch theorem. That is the same as the number of moduli parameters of the surface. It is this equality that allows one to use topological string methods in the computation of superstring scattering amplitudes in the non-minimal pure spinor formalism (see for example [19]).

2.8 The scattering amplitude prescription

We will now briefly review how scattering amplitudes are to be computed using the non-minimal pure spinor formalism.

2.8.1 Tree-level prescription

N-point tree-level scattering amplitudes are computed by a correlation function with three unintegrated vertices (2.68) and $N-3$ integrated vertices (2.69),

$$\mathcal{A} = \langle \mathcal{N} V^1 V^2 V^3 \int U^4 \cdots \int U^N \rangle. \tag{2.84}$$

The computation of (2.84) proceeds as usual in a CFT. First one integrates out the conformal weight one variables through their OPE's to get an expression containing only zero modes for λ's and θ's,

$$\mathcal{A} = \int [d\lambda][d\overline{\lambda}][dr] d^{16}\theta \mathcal{N} \lambda^\alpha \lambda^\beta \lambda^\gamma f_{\alpha\beta\gamma}(\theta).$$

The measures $[d\lambda], [d\overline{\lambda}]$ and $[dr]$ are given by

$$[d\lambda]\lambda^\alpha\lambda^\beta\lambda^\gamma = \epsilon_{\rho_1\ldots\rho_{11}\kappa_1\ldots\kappa_5} T^{((\alpha\beta\gamma))[\kappa_1\kappa_2\kappa_3\kappa_4\kappa_5]} d\lambda^{\rho_1}\ldots d\lambda^{\rho_1 1} \tag{2.85}$$

$$[d\overline{\lambda}]\overline{\lambda}_\alpha\overline{\lambda}_b\overline{\lambda}_\gamma = \epsilon^{\alpha_1\ldots\alpha_{11}\kappa_1\ldots\kappa_5} T_{((\alpha\beta\gamma))[\kappa_1\kappa_2\kappa_3\kappa_4\kappa_5]}\, d\overline{\lambda}_{\alpha_1}\cdots d\overline{\lambda}_{\alpha_{11}} \tag{2.86}$$

$$[dr] = \epsilon_{\alpha_1\ldots\alpha_{11}\kappa_1\ldots\kappa_5} T^{((\alpha\beta\gamma))[\kappa_1\kappa_2\kappa_3\kappa_4\kappa_5]} \overline{\lambda}_\alpha\overline{\lambda}_\beta\overline{\lambda}_\gamma\, \partial_r^{\alpha_1}\cdots\partial_r^{\alpha_{11}} \tag{2.87}$$

This is almost the same recipe as in the minimal formalism, the difference is the insertion of a regularization factor $\mathcal{N}$, where

$$\mathcal{N} = \exp(\{Q,\chi\}) = \mathrm{e}^{-(\lambda\overline{\lambda})-(r\theta)} \quad \text{for} \quad \chi = -(\overline{\lambda}\theta).$$

The purpose of the regularization factor is due to the fact that the integration over λ and $\overline{\lambda}$ may diverge because they are non-compact. However, as $\mathcal{N} = 1 + Q\Omega$ the integral will be independent of the choice for the regularization.

Using the measures (2.85) – (2.87) one can show that

$$\mathcal{A} = \int [d\lambda][d\overline{\lambda}][dr] d^{16}\theta \mathcal{N} \lambda^\alpha \lambda^\beta \lambda^\gamma f_{\alpha\beta\gamma}(\theta) = \langle \lambda^\alpha \lambda^\beta \lambda^\gamma f_{\alpha\beta\gamma}(\theta) \rangle$$

and therefore the non-minimal prescription for tree-level amplitudes is equivalent to the minimal pure spinor formalism.

2.8.2 Multiloop prescription

The prescription to compute $g-$loop amplitudes is given by

$$\mathcal{A} = \int d^{3g-3}\tau \langle \mathcal{N}(y) \prod_{i=1}^{3g-3} (\int dw_i \mu_i(w_j) b(w_j)) \prod_{j=1}^{N} \int dz_j U(z_j) \rangle \tag{2.88}$$

where $U(z)$ is the same integrated vertex operator of (2.69) and the b-ghost is given by (2.82). After the integration of non-zero modes appearing in the correlator (2.88) one is left with the problem of how to integrate over the $g-$zero modes of the conformal weight one variables

$$N_{mn}(z),\ \overline{N}_{mn}(z),\ J_{\lambda}(z),\ J_{\overline{\lambda}}(z),\ d_{\alpha}(z),\ S_{mn}(z) \text{ and } S(z).$$

and also the zero modes of the conformal weight zero variables $\lambda^{\alpha}, \overline{\lambda}_{\alpha}$ and r_{α}. In general, a conformal weight +1 field Φ_1 is written in a genus g Riemann surface as follows

$$\Phi_1(z) = \hat{\Phi}_1(z) + \sum_{I=1}^{g} \Phi_1^I w_I(z)$$

where $w_I(z)$ are the holomorphic one-forms and $\hat{\Phi}_1(z)$ has no zero-mode. They satisfy

$$\int_{a_I} w_J = \delta_{IJ}, \quad \int_{a_I} dz \hat{\Phi}_1(z) = 0 \quad \forall I = 1, \ldots, g.$$

Therefore one can show that, for example

$$w_{\alpha}^I = \int_{a_I} dz w(z)_{\alpha},$$

and this notation will be used in the following discussion. The integration over the zero modes of the pure spinor fields and of r_{α} is performed with the measures (2.85) – (2.87) described above, while the other zero modes are integrated with the measures defined by

$$[dw] = (\lambda\gamma^m)_{\kappa_1}(\lambda\gamma^n)_{\kappa_2}(\lambda\gamma^p)_{\kappa_3}(\gamma_{mnp})_{\kappa_4\kappa_5}\epsilon^{\kappa_1\ldots\kappa_5\rho_1\ldots\rho_{11}}dw_{\rho_1}\cdots dw_{\rho_{11}}, \tag{2.89}$$

$$[d\overline{w}] = (\overline{\lambda}\gamma^m)^{\kappa_1}(\overline{\lambda}\gamma^n)^{\kappa_2}(\overline{\lambda}\gamma^p)^{\kappa_3}(\gamma_{mnp})^{\kappa_4\kappa_5}\ \epsilon_{\kappa_1\ldots\kappa_5\alpha_1\ldots\alpha_{11}}d\overline{w}^{\alpha_1}\cdots d\overline{w}^{\alpha_{11}} \tag{2.90}$$

$$[ds] = (\lambda\overline{\lambda})^{-3}(\lambda\gamma^m)_{\kappa_1}(\lambda\gamma^n)_{\kappa_2}(\lambda\gamma^p)_{\kappa_3}(\gamma_{mnp})_{\kappa_4\kappa_5}\epsilon^{\kappa_1\ldots\kappa_5\rho_1\ldots\rho_{11}}\partial^s_{\rho_1}\cdots\partial^s_{\rho_{11}}, \tag{2.91}$$

Note that the measure (2.89) is gauge invariant under $\delta w_{\alpha} = (\lambda\gamma^m)_{\alpha}\Omega_m$ because

$$(d\lambda\gamma^q)_{[\delta_1}(\lambda\gamma^m)_{\kappa_1}(\lambda\gamma^n)_{\kappa_2}(\lambda\gamma^p)_{\kappa_3}(\gamma_{mnp})_{\kappa_4\kappa_5]} = 0,$$

which comes from the fact that there is no vector representation in the decomposition of $\lambda^4\theta^6$ (here the θ^6 factor is to emulate the antisymmetry over the spinor indices). To define the regularization factor we use $\chi = -(\overline{\lambda}\theta) - (w^I s^I)$ to obtain

$$\mathcal{N}(y) = \exp\big[-(\lambda\overline{\lambda}) - (r\theta) - (w^I\overline{w}^I) + (s^I d^I)\big]. \tag{2.92}$$

Note that here we are using a different (non gauge invariant) χ from what was originally defined in [26]. However the non gauge invariance of (2.92) should not affect the amplitudes because $\mathcal{N} - 1$ continues to be BRST-trivial even if it is not gauge invariant.

Therefore the evaluation of (2.88) will give rise to an expression of the form

$$\mathcal{A} = \int [d\lambda][d\overline{\lambda}][dr] \prod_{I=1}^{g} [dw^I][d\overline{w}^I][ds^I](d^{16}d^I)d^{16}\theta\,\mathcal{N} f(\theta).$$

From the measures (2.85) – (2.87) and (2.89) – (2.91) one can deduce the following behaviour as $(\lambda\overline{\lambda}) \to 0$

$$\int [d\lambda][d\overline{\lambda}][dr] \prod_{I=1}^{g} [dw^I][d\overline{w}^I][ds^I](d^{16}d^I)d^{16}\theta\,\mathcal{N} \to \lambda^{8+3g}\overline{\lambda}^{11}, \tag{2.93}$$

therefore $f(\lambda, \overline{\lambda}, r, \theta)$ must diverge slower than $\lambda^{-8-3g}\overline{\lambda}^{-11}$ as $(\lambda\overline{\lambda}) \to 0$ so that (2.93) is guaranteed not to diverge. Since each b-ghost diverges as $\lambda^{-4}\overline{\lambda}^{-3}$ the maximum number of loops in which this regularization can be safely used is $g = 2$, where f could diverge as $\lambda^{-14}\overline{\lambda}^{-11}$ but whose $3g - 3$ b-ghosts makes it diverge at most like $\lambda^{-12}\overline{\lambda}^{-9}$.

We will see in the next chapter that in fact this multiloop prescription was successfully used to compute massless four-point amplitudes up to two-loops [4].

Due to the fact that the external vertices don't depend on the non-minimal variables and that the r_α's appearing in the b-ghost can be substituted by D_α, we can easily guess the result of the integrations over $[dw]$, $[d\overline{w}]$ and $[ds]$. That will enable us to easily obtain the kinematic factors at one-loop, for example.

Note that at one-loop there are eleven zero-modes of s^α, which can only[4] come from the term (sd) in the regularizator $\mathcal{N}$. Therefore the remaining five d_α zero modes must come from the b-ghost and the external vertices. Therefore by ghost number conservation we obtain

$$\int d^{16}d[dw][d\overline{w}][ds] \exp\big[-(w\overline{w}) + (sd) - (\lambda\overline{\lambda}) - (r\theta)\big] d_{\kappa_1}\cdots d_{\kappa_5} f^{\kappa_1\ldots\kappa_5}(r_\alpha, \theta) =$$

$$= (\lambda^3)_{[\kappa_1\kappa_2\kappa_3\kappa_4\kappa_5]} f^{\kappa_1\ldots\kappa_5}(D_\alpha, \theta)$$

where $(\lambda^3)_{[\kappa_1\kappa_2\kappa_3\kappa_4\kappa_5]}$ is some tensor with five antisymmetric free indices containing three pure spinors. The unique such tensor is given by

$$(\lambda\gamma_m)_{\kappa_1}(\lambda\gamma_n)_{\kappa_2}(\lambda\gamma_p)_{\kappa_3}(\gamma^{mnp})_{\kappa_4\kappa_5}. \tag{2.94}$$

[4]The term $s^\alpha\partial\overline{\lambda}_\alpha$ of the b-ghost does not contribute because there is no $\overline{w}^\alpha$ in the external vertices to kill the non zero-modes of $\partial\overline{\lambda}_\alpha$.

Thus we can see that the effect of evaluating the pure spinor measures is to substitute five d_α's from the b-ghost and the external vertices by (2.94). Explicitly,

$$d_{\kappa_1} d_{\kappa_2} d_{\kappa_3} d_{\kappa_4} d_{\kappa_5} \to (\lambda\gamma_m)_{\kappa_1} (\lambda\gamma_n)_{\kappa_2} (\lambda\gamma_p)_{\kappa_3} (\gamma^{mnp})_{\kappa_4\kappa_5}. \tag{2.95}$$

It is interesting to note that the right hand side of (2.95) already is completely antisymmetric in $[\kappa_1 \ldots \kappa_5]$ because of the pure spinor condition. To see this one notices that the only non-obvious antisymmetry to check is over the exchange of the indices κ_1 and κ_4, for example. However, as $(\lambda\gamma^p)_\alpha(\lambda\gamma_p)_\beta = 0$ we can write $(\gamma^{mnp})_{\kappa_4\kappa_5} = \gamma^m_{\kappa_4\sigma}(\gamma^n\gamma^p)^\sigma{}_{\kappa_5}$ and use the gamma matrix identity $\eta_{rs}\gamma^r_{\alpha(\beta}\gamma^s_{\gamma\delta)} = 0$ to obtain

$$\lambda^\alpha(\gamma_m)_{\alpha\kappa_1}(\lambda\gamma_n)_{\kappa_2}(\lambda\gamma_p)_{\kappa_3}\gamma^m_{\kappa_4\sigma}(\gamma^n\gamma^p)^\sigma{}_{\kappa_5} = -(\lambda\gamma^m\gamma^n\gamma^p)_{\kappa_5}(\lambda\gamma_n)_{\kappa_2}(\lambda\gamma_p)_{\kappa_3}(\gamma_m)_{\kappa_4\kappa_5}$$

$$-(\lambda\gamma_m)_{\kappa_4}(\lambda\gamma_n)_{\kappa_2}(\lambda\gamma_p)_{\kappa_3}(\gamma^{mnp})_{\kappa_1\kappa_5}.$$

The proof follows from the vanishing of the first term of the right hand side due to the pure spinor condition.

Chapter 3

Computing Pure Spinor Scattering Amplitudes

3.1 Massless three-point amplitude at tree-level

As a brief example we will compute the scattering of three massless particles at tree-level. This is the simplest example possible because the prescription (2.73) implies that there are no integrated vertices and therefore there is no need to compute OPE's. Only the zero modes contribute to the amplitude and their contribution is completely determined by the measure (2.74).

The amplitude to compute is given by

$$\mathcal{A} = \langle V_1(z_1)V_2(z_2)V_3(z_3)\rangle + (2 \leftrightarrow 3), \tag{3.1}$$

where $V = \lambda^\alpha A_\alpha(\theta)\mathrm{e}^{ik\cdot X}$ and the theta expansion of $A_\alpha(\theta)$ is given in Appendix B. The sum over the permutation of labels 2 and 3 has to be done because a general Möbius transformation does not change the cyclic ordering of the vertex operators[1], so both orderings must be summed over.

The contribution from the exponentials is proportional to a constant because the particles are massless. Therefore $k_j^2 = 0$ implies $k_i \cdot k_j = 0$ due to momentum conservation.

[1]That is because a non-cyclic transformation always has a fixed point. For example, it is impossible to map $y_1y_2y_3$ into $y_1y_3y_2$ because the fixed point y_1 implies that the Möbius transformation is the identity,

$$y_1 = \frac{1y_1 + 0}{0 + 1}.$$

The non trivial part of the computation comes from the θ zero modes.

To compute the scattering of three gluons we use the $A_\alpha(\theta)$ expansion of appendix B to get three different possibilities to obtain five thetas, given by

$A^1_\alpha(\theta)$	$A^2_\alpha(\theta)$	$A^3_\alpha(\theta)$
1	1	3
1	3	1
3	1	1

Explicitly we get, for one of the permutations of (3.1),

$$\mathcal{A}_{\rm BBB} = -\frac{1}{64}\left(k^3_m e^1_r e^2_s e^3_n - k^2_m e^1_r e^2_n e^3_s + k^1_m e^1_n e^2_r e^3_s\right)\langle(\lambda\gamma^r\theta)\,(\lambda\gamma^s\theta)\,(\lambda\gamma_p\theta)\,(\theta\gamma^{pmn}\theta)\rangle\,. \quad (3.2)$$

As we will see in Appendix A, the above correlator is given by

$$\langle(\lambda\gamma^r\theta)(\lambda\gamma^s\theta)(\lambda\gamma_p\theta)(\theta\gamma^{pmn}\theta)\rangle = \frac{1}{120}\delta^{rsp}_{pmn} = \frac{1}{45}\delta^{rs}_{mn}.$$

Then (3.2) evaluates to

$$\mathcal{A}_{\rm BBB} = -\frac{1}{2880}\Big[(e_1\cdot e_2)(e_3\cdot k_2) + (e_1\cdot e_3)(e_2\cdot k_1) + (e_2\cdot e_3)(e_1\cdot k_3)\Big] \quad (3.3)$$

where we used momentum conservation and $e_i\cdot k_i = 0$. Note that (3.3) is antisymmetric in $(2\leftrightarrow 3)$ and therefore the whole amplitude vanishes for photons, whereas for gluons it is non-vanishing due to the Chan-Patton factors. Up to an overall constant, this is the same result as in the RNS formalism (see equation (7.4.30) of [33]).

As (3.1) is supersymmetric, the contribution of fermionic states is as easy to compute as the bosonic case considered above. For example, the $B^1F^2F^3$ scattering amplitude is given by the following theta distribution

$A^1_\alpha(\theta)$	$A^2_\alpha(\theta)$	$A^3_\alpha(\theta)$
1	2	2

which is computed to be

$$\mathcal{A}_{\rm BFF} = -\frac{1}{288}e^1_{n_1}(\chi^2\gamma^r\chi^3)\langle(\lambda\gamma^n 1\theta)(\lambda\gamma^m\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mrp}\theta)\rangle = \frac{1}{2880}e^1_m(\chi^2\gamma^m\chi^3),$$

which again is non-vanishing after summing $(2\leftrightarrow 3)$ only for a non-abelian group.

3.2 Massless four-point amplitude at tree-level

It has been known for over eight years now that tree-level amplitudes computed with the pure spinor formalism are equivalent to their RNS counterparts [6]. Nevertheless, apart from the trivial massless tree-point amplitude, no other tree-level amplitude has been explicitly computed with the pure spinor formalism. When one's attention is directed towards pure spinor superspace expressions for kinematic factors, the natural amplitude to study is the scattering of four massless strings. In [5] this task has been completed and the following pure spinor superspace expression for the kinematic factor was obtained,

$$K_0 = 2\langle(\partial_m A_n)(\lambda A)\partial^m(\lambda A)(\lambda\gamma^n W)\rangle - \langle(\lambda A)\partial^m(\lambda A)\partial^n(\lambda A)\mathcal{F}_{mn}\rangle.$$

With this superspace representation for the kinematic factor one can show through pure spinor manipulations that this is in fact proportional to the kinematic factor for this same amplitude, but at the one-loop level. That this could be shown in a few pages is a remarkable display of the uselfulness in having kinematic factors written in pure spinor superspace. We will now review the computation of [5].

Following the tree-level prescription of (2.73), the amplitude to compute is

$$\mathcal{A} = \langle V^1(z_1,\overline{z}_1)V^2(z_2,\overline{z}_2)V^3(z_3,\overline{z}_3)\int_C d^2z_4 U(z_4,\overline{z}_4)\rangle. \tag{3.4}$$

The closed string vertices are given by the holomorphic square of the open string vertices, $V(z,\overline{z}) = \mathrm{e}^{ik\cdot X}\lambda^\alpha\overline{\lambda}^\beta A_\alpha(\theta)\overline{A}_\beta(\theta)$ and $U(z,\overline{z}) = \mathrm{e}^{ik\cdot X}U(z)\overline{U}(\overline{z})$, where the integrated vertex operator is given by (2.69).

In the computation of (3.4) we note that standard SL(2,C) invariance allows us to fix $z_1 = 0, z_2 = 1$ and $z_3 = \infty$, so the expectation value for the exponentials simplify,

$$\langle\prod_{i=1}^4 :\mathrm{e}^{ik^i\cdot X(z_i,\overline{z}_i)}:\rangle = |z_4|^{-\frac{1}{2}\alpha' t}|1-z_4|^{-\frac{1}{2}\alpha' u} \equiv M(z_4,\overline{z}_4).$$

Now we remove the conformal weight one operators of the integrated vertex (2.69) in (3.4) by using their OPE's. The first term of (2.69) does not contribute because there is no p_α's in the unintegrated vertices, while the second gives

$$\langle A^4_m\Pi^m(z_4)\prod_{j=1}^4 :\mathrm{e}^{ik^j\cdot X(z_j,\overline{z}_j)}:\rangle = \sum_{j=1}^3 \frac{\alpha'}{2}\frac{ik^m_j}{z_j - z_4}\langle(\lambda A^1)(\lambda A^2)(\lambda A^3)A^4_m\rangle M(z_4,\overline{z}_4). \tag{3.5}$$

Using the standard OPE's

$$N^{mn}(z_4)\lambda^\alpha(z_j) = \frac{\alpha'}{4}\frac{(\lambda\gamma^{mn})^\alpha}{z_j - z_4}, \quad d_\alpha(z_4)V(z_j) = -\frac{\alpha'}{2}\frac{D_\alpha V}{z_j - z_4}, \tag{3.6}$$

we obtain the following OPE identity:

$$\langle(\lambda A^1)(\lambda A^2)(\lambda A^3)\left(d_\alpha(z_4)W_4^\alpha + \frac{1}{2}N^{mn}(z_4)\mathcal{F}^4_{mn}\right)\rangle =$$

$$= \frac{\alpha'}{2(z_1 - z_4)}\langle A^1_m(\lambda A^2)(\lambda A^3)(\lambda\gamma^m W^4)\rangle - (1 \leftrightarrow 2) + (1 \leftrightarrow 3). \tag{3.7}$$

To show this, one uses (3.6) to get

$$\langle(\lambda A^1)(z_1)(\lambda A^2)(z_2)(\lambda A^3)(z_3)d_\alpha(z_4)W_4^\alpha\rangle =$$

$$\frac{\alpha'}{2(z_1 - z_4)}\langle D_\alpha(\lambda A^1)(\lambda A^2)(\lambda A^3)W_4^\alpha\rangle - (1 \leftrightarrow 2) + (1 \leftrightarrow 3).$$

Concentrating for simplicity on the first term, the use of the super-Yang-Mills identity $D_\alpha(\lambda A) = -(\lambda D)A_\alpha + (\lambda\gamma^m)_\alpha A_m$ allows the numerator to be rewritten as

$$\langle D_\alpha(\lambda A^1)(\lambda A^2)(\lambda A^3)W_4^\alpha\rangle = -\langle(\lambda D A^1_\alpha)(\lambda A^2)(\lambda A^3)W_4^\alpha\rangle + \langle A^1_m(\lambda A^2)(\lambda A^3)(\lambda\gamma^m W^4)\rangle. \tag{3.8}$$

As BRST-exact terms decouple, the first term in the right hand side of (3.8) becomes

$$-\frac{\alpha'}{2(z_1 - z_4)}\langle(\lambda D A^1_\alpha)(\lambda A^2)(\lambda A^3)W_4^\alpha\rangle = -\frac{\alpha'}{2(z_1 - z_4)}\langle A^1_\alpha(\lambda A^2)(\lambda A^3)(\lambda D)W_4^\alpha\rangle$$

$$= -\frac{\alpha'}{8(z_1 - z_4)}\langle(\lambda\gamma^{mn}A^1)(\lambda A^2)(\lambda A^3)\mathcal{F}^4_{mn}\rangle.$$

However, this term is exactly canceled by the $(z_1 - z_4)^{-1}$ contribution from the OPE

$$\frac{1}{2}\langle(\lambda A^1)(\lambda A^2)(\lambda A^3)(N^{mn}\mathcal{F}^4_{mn})\rangle = \frac{\alpha'}{8(z_1 - z_4)}\langle(\lambda\gamma^{mn}A^1)(\lambda A^2)(\lambda A^3)\mathcal{F}^4_{mn}\rangle + \ldots,$$

which finishes the proof of (3.7).

With the results (3.5) and (3.7), the correlation in the amplitude (3.4) reduces to

$$\mathcal{A} = \left(\frac{\alpha'}{2}\right)^2 \int_C d^2 z_4 \left(\frac{F_{12}}{z_4} + \frac{F_{21}}{1 - z_4}\right)\left(\frac{\overline{F}_{12}}{\overline{z}_4} + \frac{\overline{F}_{21}}{1 - \overline{z}_4}\right)|z_4|^{-\frac{1}{2}\alpha' t}|1 - z_4|^{-\frac{1}{2}\alpha' u},$$

where $F_{12} = ik_1^m\langle(\lambda A^1)(\lambda A^2)(\lambda A^3)A^4_m\rangle + \langle A^1_m(\lambda A^2)(\lambda A^3)(\lambda\gamma^m W^4)\rangle$ and F_{21} is obtained by exchanging $1 \leftrightarrow 2$. The integral can be evaluated using the following formula

$$\int_C d^2 z z^N(1 - z)^M \overline{z}^{\overline{N}}(1 - \overline{z})^{\overline{M}} = 2\pi\frac{\Gamma(1 + N)\Gamma(1 + M)}{\Gamma(2 + N + M)}\frac{\Gamma(-1 - \overline{N} - \overline{M})}{\Gamma(-\overline{N})\Gamma(-\overline{M})}.$$

After a few manipulations one finally gets

$$\mathcal{A} = -2\pi(\frac{\alpha'}{2})^4 K_0 \overline{K}_0 \frac{\Gamma(-\frac{\alpha' t}{4})\Gamma(-\frac{\alpha' u}{4})\Gamma(-\frac{\alpha' s}{4})}{\Gamma(1+\frac{\alpha' t}{4})\Gamma(1+\frac{\alpha' u}{4})\Gamma(1+\frac{\alpha' s}{4})},$$

where $K_0 = \frac{1}{2}(uF_{12} + tF_{21})$ is given by

$$K_0 = \langle(\partial_m A^1_n)(\lambda A^2)\partial^m(\lambda A^3)(\lambda\gamma^n W^4)\rangle - \frac{1}{2}\langle\partial^m(\lambda A^1)\partial^n(\lambda A^2)(\lambda A^3)\mathcal{F}^4_{mn}\rangle + (1 \leftrightarrow 2), \quad (3.9)$$

which is the sought-for kinematic factor in pure spinor superspace.

Note that K_0 is BRST-closed because

$$QF_{12} = -\frac{t}{2}\langle(\lambda A^1)(\lambda A^2)(\lambda A^3)(\lambda A^4)\rangle, \quad QF_{21} = +\frac{u}{2}\langle(\lambda A^1)(\lambda A^2)(\lambda A^3)(\lambda A^4)\rangle.$$

When trying to relate the above tree-level kinematic factor with its one-loop cousin, it is convenient to rewrite (3.9) without explicit labels,

$$K_0 = 2\langle(\partial_m A_n)(\lambda A)\partial^m(\lambda A)(\lambda\gamma^n W)\rangle - \langle(\lambda A)\partial^m(\lambda A)\partial^n(\lambda A)\mathcal{F}_{mn}\rangle. \quad (3.10)$$

We postpone the explicit evaluation in components of (3.9) to section 3.6. Before that we will show how (3.9) relates to amplitudes at higher-loop orders.

3.3 Massless four-point amplitude at one-loop

We can compute the massless four-point amplitude at the one-loop order with the two different pure spinor formalism prescriptions described in sections 2.6 an 2.8.2. It will be shown that they are equivalent up to a constant factor. Note that it has been recently proved that these two prescriptions are equivalent in general [16], and the proof presented here can be regarded as an example of that.

If one is not interested in the overall coefficient, it also happens that the kinematic factor is readily obtained by a zero-mode saturation argument, avoiding the long procedure of functionaly integrating using the measures $[dr]$, $[ds]$, $[d\lambda]$ etc. This is the route taken in the papers [7][4] and which will be described here.

3.3.1 Minimal pure spinor computation

Using the minimal pure spinor prescription the open superstring amplitude is given by

$$\mathcal{A} = \int d\tau \langle \int dw \mu(w) \widetilde{b}_{B_1}(w, z_1) \prod_{P=2}^{10} Z_{B_P}(z_P) Z_J(v) \prod_{I=1}^{11} Y_{C_I}(y_I)\ V_1(t_1) \prod_{T=2}^{N} \int d^2 t_T U_T(t_T)\ \rangle,$$

and the kinematic factor in pure spinor superspace is obtained by considering how the sixteen zero modes for d_α can be saturated.

From (2.78) we see that the nine Z_B and one Z_J will provide ten d_α zero modes. Since there is no term in the b-ghost which contains three or five d_α's, the amplitude will be non-vanishing if the b-ghost contributes with four d's and the three integrated vertices provide two d's through the term $(dW)(dW)$. Furthermore, as there is a delta function derivative of N^{mn} coming from the b-ghost, the amplitude will be non-vanishing if one of the external vertices provide an explicit N^{mn}, so that the analogous delta function property of $\int dx \delta'(x) x = -1$ can be used.

Looking at the integrated vertex (2.69) we see that the term containing N^{mn} has the superfield $\mathcal{F}_{mn}$, so we have shown the kinematic factor to be composed out of the following superfields

$$(\lambda)^2(\lambda A)W^2\mathcal{F}, \tag{3.11}$$

where we already used the fact (which can be shown by integrating the measures) pure spinor superspace expressions contain three pure spinors λ^α. We are now required to check how many different Lorentz invariant contractions can be constructed out of these fields in (3.11). If there is a unique contraction then we can shortcut the functional integration procedure and immediately write down the answer in pure spinor superspace. It is a happy fortuitous fact that this is the case here, in deep contrast to the massless five-point amplitude of section 3.9.2.

Fortunately, it is easy to show there is a unique Lorentz-invariant way to contract the indices in (3.11). To show this, first choose a Lorentz frame in which the only non-zero component of λ^α is in the λ^+ direction. This choice preserves a $U(1) \times SU(5)$ subgroup of $SO(10)$, under which a Weyl spinor U^α and an anti-Weyl spinor V_α decompose as

$$U^\alpha \longrightarrow \left(U^+_{\frac{5}{2}}, U_{\frac{1}{2}[ab]}, U^a_{-\frac{3}{2}}\right), \quad V_\alpha \longrightarrow \left(V_{-\frac{5}{2}+}, V^{[ab]}_{-\frac{1}{2}}, V_{+\frac{3}{2}a}\right),$$

where the subscript denotes the $U(1)$ charge. So the unique way to cancel the $+15/2$ U(1)-charge of the three λ^α's is when the superfields contribution is

$$K = \langle (\lambda^+)^3 A_+ W^a W^b F_{ab} \rangle$$

which can be written in covariant SO(10) language as

$$K = \langle (\lambda A)(\lambda \gamma^m W)(\lambda \gamma^n W)\mathcal{F}_{mn} \rangle, \tag{3.12}$$

which is the final pure spinor superspace expression for this important amplitude. Now let's analyse it a bit.

Gauge invariance of the kinematic factor

The appearance of the explicit superfield A_α in the kinematic factor of (3.12) might spoil the gauge invariance of the amplitude, as it transforms as (B.7)

$$\delta A_\alpha = D_\alpha \Omega. \tag{3.13}$$

However it is easy to check that using the properties of pure spinor superspace and the equations of motion of the SYM superfields (B.10) and (B.11), the kinematic factor (3.12) is indeed gauge invariant. This is because the variation (3.13) implies that the gauge transformation of the unintegrated vertex operator is BRST-exact $\delta(\lambda A) = \lambda^\alpha D_\alpha \Omega = Q_{\text{BRST}}\Omega$, which allows the BRST-charge to be "integrated by parts" using the property that pure spinor superspace expressions of BRST-exact terms are zero. So the gauge variation of (3.12) is given by

$$\delta K = \langle Q(\Omega)(\lambda\gamma^m W)(\lambda\gamma^n W)\mathcal{F}_{mn}\rangle = -\langle \Omega Q[(\lambda\gamma^m W)(\lambda\gamma^n W)\mathcal{F}_{mn}]\rangle = 0$$

where we used

$$Q(\lambda\gamma^m W) = -\frac{1}{4}(\lambda\gamma^m\gamma^{rs}\lambda)\mathcal{F}_{rs} = 0$$

and

$$(\lambda\gamma^m W)(\lambda\gamma^n W)Q\mathcal{F}_{mn} = 2(\lambda\gamma^m W)(\lambda\gamma^n W)\partial_{[m}(\lambda\gamma_{n]}W) = 0,$$

which can be shown using the equations of motion and the defining pure spinor property of $(\lambda\gamma^m\lambda) = 0$. We have then shown that the massless four-point amplitude at one-loop level is indeed gauge invariant.

An equivalent pure spinor superspace expression

If we use the SYM identity (B.16)

$$QA_m = (\lambda\gamma^m W) + \partial^m(\lambda A)$$

and the vanishing of BRST-exact terms in pure spinor superspace, we can rewrite (3.12) in a manifestly gauge invariant way. To see this substitute (λA) in $\langle(\lambda A)(\lambda\gamma^m W)(\lambda\gamma^n W)\mathcal{F}_{mn}\rangle$ by

$$(\lambda A) = \frac{1}{(H\cdot k)}H^p QA_p - \frac{1}{(H\cdot k)}H_p(\lambda\gamma^p W) \tag{3.14}$$

where H^p is an arbitrary vector such that $(H \cdot k) \neq 0$. Note that the first term in the right hand side of (3.14) will not contribute because

$$\langle (QA_p)(\lambda\gamma^m W)(\lambda\gamma^n W)\mathcal{F}_{mn}\rangle$$

is BRST-exact. So we get

$$\langle (\lambda A^1)(\lambda\gamma^m W^2)(\lambda\gamma^n W^3)\mathcal{F}^4_{mn}\rangle = -\frac{1}{(H\cdot k^1)} H_p \langle (\lambda\gamma^p W^1)(\lambda\gamma^m W^2)(\lambda\gamma^n W^3)\mathcal{F}^4_{mn}\rangle. \quad (3.15)$$

We can easily use the method of appendix A to evaluate the purely bosonic part of the right hand side of (3.15). We obtain the following table for the distribution of thetas,

$W_1^\alpha(\theta)$	$W_2^\alpha(\theta)$	$W_3^\alpha(\theta)$	$F^4_{mn}(\theta)$
1	1	1	2
1	1	3	0
1	3	1	0
3	1	1	0

which, using the superfields of appendix B, expands to

$$\frac{1}{256(H\cdot k^1)} F^1_{m_1n_1}F^2_{m_2n_2}F^3_{m_3n_3}F^4_{m_4n_4}\Big[$$
$$+\langle (\lambda\gamma^p\gamma^{m_1n_1}\theta)(\lambda\gamma^m\gamma^{m_2n_2}\theta)(\lambda\gamma^{k_4}\gamma^{m_3n_3}\theta)(\theta\gamma_{[m}\gamma^{m_4n_4}\theta)\rangle k^4_{n]}$$
$$+\frac{1}{3}\langle (\lambda\gamma^p\gamma^{m_1n_1}\theta)(\lambda\gamma^{[m_4|}\gamma^{m_2n_2}\theta)(\lambda\gamma^{|n_4]}\gamma^{k_3a}\theta)(\theta\gamma_a\gamma^{m_3n_3}\theta)\rangle k^3_{k_3}$$
$$+\frac{1}{3}\langle (\lambda\gamma^p\gamma^{m_1n_1}\theta)(\lambda\gamma^{[m_4|}\gamma^{k_2a}\theta)(\lambda\gamma^{|n_4]}\gamma^{m_3n_3}\theta)(\theta\gamma_a\gamma^{m_2n_2}\theta)\rangle k^2_{k_2}$$
$$+\frac{1}{3}\langle (\lambda\gamma^p\gamma^{k_1a}\theta)(\lambda\gamma^{[m_4|}\gamma^{m_2n_2}\theta)(\lambda\gamma^{|n_4]}\gamma^{m_3n_3}\theta)(\theta\gamma_a\gamma^{m_1n_1}\theta)\rangle k^1_{k_1}\Big].$$

After a long but straightforward calculation we obtain,

$$= -\frac{1}{5760}\Big[+\frac{1}{2}t^2(e^1\cdot e^3)(e^2\cdot e^4) + \frac{1}{2}tu(e^1\cdot e4)(e^2\cdot e^3) + \frac{1}{2}tu(e^1\cdot e3)(e^2\cdot e^4)$$
$$-\frac{1}{2}tu(e^1\cdot e2)(e^3\cdot e^4) + \frac{1}{2}u^2(e^1\cdot e^4)(e^2\cdot e^3)$$
$$+t(k^4\cdot e^2)(k^4\cdot e^3)(e^1\cdot e^4) - t(k^4\cdot e^1)(k^4\cdot e^3)(e^2\cdot e^4) - t(k^3\cdot e^4)(k^4\cdot e^2)(e^1\cdot e^3)$$
$$-t(k^3\cdot e^1)(k^4\cdot e^3)(e^2\cdot e^4) + t(k^3\cdot e^1)(k^4\cdot e^2)(e^3\cdot e^4) - t(k^2\cdot e^4)(k^4\cdot e^3)(e^1\cdot e^2)$$
$$+t(k^2\cdot e^4)(k^3\cdot e^2)(e^1\cdot e^3) - t(k^2\cdot e^4)(k^3\cdot e^1)(e^2\cdot e^3) + t(k^2\cdot e^3)(k^4\cdot e^2)(e^1\cdot e^4)$$

$$
\begin{aligned}
&-t(k^2\cdot e^3)(k^4\cdot e^1)(e^2\cdot e^4)-t(k^2\cdot e^3)(k^2\cdot e^4)(e^1\cdot e^2)-u(k^3\cdot e^4)(k^4\cdot e^1)(e^2\cdot e^3)\\
&-u(k^3\cdot e^2)(k^4\cdot e^3)(e^1\cdot e^4)+u(k^3\cdot e^2)(k^4\cdot e^1)(e^3\cdot e^4)+u(k^3\cdot e^2)(k^3\cdot e^4)(e^1\cdot e^3)\\
&-u(k^3\cdot e^1)(k^3\cdot e^4)(e^2\cdot e^3)+u(k^2\cdot e^4)(k^3\cdot e^2)(e^1\cdot e^3)-u(k^2\cdot e^4)(k^3\cdot e^1)(e^2\cdot e^3)\\
&+u(k^2\cdot e^3)(k^4\cdot e^2)(e^1\cdot e^4)-u(k^2\cdot e^3)(k^4\cdot e^1)(e^2\cdot e^4)-u(k^2\cdot e^3)(k^3\cdot e^4)(e^1\cdot e^2)\\
&\qquad -u(k^2\cdot e^3)(k^2\cdot e^4)(e^1\cdot e^2)\Big],
\end{aligned}
$$

where we used the Mandelstam variables and momentum conservation as $s=-t-u$. Therefore the answer does not depend on H_p and we will see in (3.80) that it matches with the computation of the left hand side of (3.15), by considering the identity (3.68).

3.3.2 Non-minimal pure spinor computation

In this section we will compute the same one-loop amplitude of section 3.3.1, but now using the NMPS prescription of 2.8.2.

At the genus one surface the variables s^α and d_α have eleven and sixteen zero-modes, respectively. Using the one-loop prescription of the non-minimal formalism, the only place which can provide the 11 zero modes of s^α is the regulator $\mathcal{N}$ of (2.92). But they are multiplied by the eleven d_α zero modes, and so the remaining five zero modes of d_α must come either from the vertex operators or from the single b ghost.

Since the three integrated vertex operators can provide at most three d_α zero modes through the terms $(W^\alpha d_\alpha)$, the single b ghost of (2.82) must provide two d_α zero modes through the term

$$\frac{(\overline{\lambda}\gamma^{mnp}r)(d\gamma_{mnp}d)}{192(\lambda\overline{\lambda})^2}. \tag{3.16}$$

After integrating over the zero modes of the dimension one fields $(w_\alpha,\overline{w}^\alpha,d_\alpha,s^\alpha)$ using the measure factors described in 2.8.2, one is left with an expression proportional to

$$\int d^{16}\theta\int[d\lambda][d\overline{\lambda}][dr](\lambda\overline{\lambda})^{-2}(\lambda)^4(\overline{\lambda}\gamma^{mnp}r)AWWW\exp(-\lambda\overline{\lambda}-r\theta) \tag{3.17}$$

$$=\int d^{16}\theta\int[d\lambda][d\overline{\lambda}][dr]\exp(-\lambda\overline{\lambda}-r\theta)(\lambda\overline{\lambda})^{-2}(\lambda)^4(\overline{\lambda}\gamma^{mnp}D)AWWW \tag{3.18}$$

where $D_\alpha=\frac{\partial}{\partial\theta^\alpha}+(\gamma^m\theta)_\alpha\partial_m$ is the usual superspace derivative and the index contractions on

$$(\lambda)^4(\overline{\lambda}\gamma^{mnp}D)AWWW \tag{3.19}$$

have not been worked out. Note that (3.18) is obtained from (3.17) by writing $r_\alpha \exp(-r\theta) = \frac{\partial}{\partial\theta^\alpha}\exp(-r\theta)$, integrating by parts with respect to θ, and using conservation of momentum to ignore total derivatives with respect to x. Furthermore, the factor of $(\lambda)^4$ in (3.17) comes from the λ in the unintegrated vertex operator, the 11 factors of λ and $\overline{\lambda}$ which multiply the zero modes of d_α and s_α in $\mathcal{N}$, the factor of $(\lambda)^{-8}(\overline{\lambda})^{-8}$ in the measure factor of w_α and $\overline{w}^\alpha$, and the factor of $(\overline{\lambda})^{-3}$ in the measure factor of s^α.

Fortunately, it is easy to show there is a unique Lorentz-invariant way to contract the indices in (3.19). Since $(\lambda^+)^4$ carries $+10$ $U(1)$ charge, $(\overline{\lambda}\gamma^{mnp}D)AWWW$ must carry -10 $U(1)$ charge which is only possible if $(\overline{\lambda}\gamma^{mnp}D)$ carries -3 charge, A_α carries $-\frac{5}{2}$ charge, and each W^α carries $-\frac{3}{2}$ charge. Contracting the $SU(5)$ indices, one finds that the unique $U(1)\times SU(5)$ invariant contraction of the indices is

$$(\lambda^+)^4(\overline{\lambda}\gamma_{abc}D)A_+W^aW^bW^c. \tag{3.20}$$

Returing to covariant notation, one can easily see that (3.19) must be proportional to the Lorentz-invariant expression

$$(\overline{\lambda}\gamma_{mnp}D)(\lambda A)(\lambda\gamma^m W)(\lambda\gamma^n W)(\lambda\gamma^p W), \tag{3.21}$$

which reduces to (3.20) in the frame where λ^+ is the only non-zero component of λ^α.

Note that this same conclusion can be obtained covariantly from the prescription if we use the trick given by (2.95). That is because from the zero mode counting the kinematic factor is proportional to

$$\langle(\overline{\lambda}\gamma^{rst}D)(d\gamma_{rst}d)(\lambda A)(dW)(dW)(dW)\rangle,$$

which, upon use of (2.95), immediately becomes (3.21).

However, to express the kinematic factor as an integral over pure spinor superspace as in (3.12), it is convenient to have an expression in which all $\overline{\lambda}_\alpha$'s appear in the combination $(\lambda^\alpha\overline{\lambda}_\alpha)$. If all $\overline{\lambda}$'s appear in this combination one can use that, up to a constant,

$$\int d^{16}\theta \int [d\lambda][d\overline{\lambda}][dr]\exp(-\lambda\overline{\lambda} - r\theta)(\lambda\overline{\lambda})^{-n}\lambda^\alpha\lambda^\beta\lambda^\gamma f_{\alpha\beta\gamma} = \langle\lambda^\alpha\lambda^\beta\lambda^\gamma f_{\alpha\beta\gamma}\rangle. \tag{3.22}$$

To convert (3.21) to this form, it is convenient to return to the frame in which λ^+ is the only non-zero component of λ^α and write (3.20) as

$$(\lambda^+)^4\epsilon_{abcde}(\overline{\lambda}^{[de]}D_+ - \overline{\lambda}_+D^{[de]})A_+W^aW^bW^c. \tag{3.23}$$

Using the superspace equations of motion for A_α and W^α, it is easy to show that

$$D_+A_+ = D_+W^a = 0, \quad D^{[de]}A_+ + D_+A^{[de]} = 0, \quad \epsilon_{abcde}D^{[ab]}W^c = \mathcal{F}_{de}. \tag{3.24}$$

So (3.23) is proportional to two terms which are

$$(\lambda^+)^4\overline{\lambda}_+\epsilon_{abcde}(D_+A^{[de]})W^aW^bW^c \quad \text{and} \quad (\lambda^+)^4\overline{\lambda}_+A_+W^aW^b\mathcal{F}_{ab}. \tag{3.25}$$

The second term in (3.25), when written in covariant language, is proportional to

$$(\lambda\overline{\lambda})(\lambda A)(\lambda\gamma^m W)(\lambda\gamma^n W)\mathcal{F}_{mn}, \tag{3.26}$$

which produces the desired pure spinor superspace integral of (3.12). And the first term in (3.25) can be written as

$$(\lambda\overline{\lambda})\big[(\lambda D)(\lambda\gamma^{mn}A)\big](\lambda\gamma^p W)(W\gamma_{mnp}W), \tag{3.27}$$

which produces the pure spinor superspace integral

$$\langle\big[(\lambda D)(\lambda\gamma^{mn}A)\big](\lambda\gamma^p W)(W\gamma_{mnp}W)\rangle. \tag{3.28}$$

But since BRST-trivial operators decouple,

$$\langle(\lambda D)\big[(\lambda\gamma^{mn}A)(\lambda\gamma^p W)(W\gamma_{mnp}W)\big]\rangle = 0,$$

which implies that (3.28) is equal to

$$\begin{aligned}\langle(\lambda\gamma^{mn}A)(\lambda D)\big[(\lambda\gamma^p W)(W\gamma_{mnp}W)\big]\rangle &= -\frac{1}{2}(\lambda\gamma^{mn}A)(\lambda\gamma^p W)(\lambda\gamma^{rs}\gamma_{mnp}W)\mathcal{F}_{rs} \\ &= -24\langle(\lambda A)(\lambda\gamma^r W)(\lambda\gamma^s W)\mathcal{F}_{rs}\end{aligned}$$

where we used the equation of motion for the superfield W^α and several gamma matrix identities in the last step. So we finally have shown that the non-minimal computation of the kinematic factor is proportional to the minimal kinematic factor of (3.12), which was indeed to be expected due to the proof presented in [16].

3.3.3 Covariant proof of equivalence

In the last section we used the U(5) decomposition of the superfields to show the equivalence between the kinematic factor (3.12), obtained with the minimal pure spinor formalism, and the non-minimal expression of (3.21). The proof was rather straightforward but it doesn't teach us how to deal with expressions containing four λ^α's and one $\overline{\lambda}_\alpha$ in general, where it is not always the case that going to the U(5) frame where $\lambda_{[ab]} = \lambda^a = 0$ leads to manageable expressions.

So it is worth devoting some time in trying to find a covariant (and general) method to deal with kinematic factors containing $\lambda^4\overline{\lambda}$ which are obtained in the non-minimal pure spinor formalism. These expressions are of the following form

$$\int d^{16}\theta \int [d\lambda][d\overline{\lambda}][dr]\mathrm{e}^{-(\lambda\overline{\lambda})-(r\theta)}(\lambda\overline{\lambda})^{-n}\lambda^\alpha\lambda^\beta\lambda^\gamma\lambda^\delta\overline{\lambda}_\epsilon f^\epsilon_{\alpha\beta\gamma\delta}(\theta) \tag{3.29}$$

and are obtained after integration over the non-minimal measures $[ds]$,$[dw]$ and $[d\overline{w}]$. The reason to integrate over these particular variables is because the end result contains the same integrations to perform as the tree-level amplitude prescription. This is interesting because for tree-level amplitudes there is the notion of pure spinor superspace, where one uses the fact that the integrations over $[dr]$,$[d\lambda]$,$[d\overline{\lambda}]$ and $d^{16}\theta$ select the terms proportional to $\langle(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)\rangle$.

What we now require is a rule analogous to (3.22) for expressions of the type (3.29) or, in other words, we need to have a covariant prescription to integrate[2] over four λ's and one $\overline{\lambda}$,

$$\langle\lambda^\alpha\lambda^\beta\lambda^\gamma\lambda^\delta\overline{\lambda}_\epsilon f^\epsilon_{\alpha\beta\gamma\delta}\rangle_{(4,1)}. \tag{3.30}$$

To derive such a rule one needs to write down a tensor $T_\epsilon^{\alpha\beta\gamma\delta}$ which is symmetric and gamma-matrix traceless with respect to the four Weyl indices,

$$\begin{aligned} T_\epsilon^{\alpha\beta\gamma\delta} = {} & \delta^\alpha_\epsilon T^{\beta\gamma\delta} + \delta^\beta_\epsilon T^{\alpha\gamma\delta} + \delta^\gamma_\epsilon T^{\alpha\beta\delta} + \delta^\delta_\epsilon T^{\alpha\beta\gamma} - \frac{1}{12}\Big[\gamma^m_{\epsilon\kappa}\gamma_m^{\alpha\beta}T^{\gamma\delta\kappa} + \gamma^m_{\epsilon\kappa}\gamma_m^{\alpha\delta}T^{\beta\gamma\kappa} \\ & + \gamma^m_{\epsilon\kappa}\gamma_m^{\alpha\gamma}T^{\beta\delta\kappa} + \gamma^m_{\epsilon\kappa}\gamma_m^{\beta\gamma}T^{\alpha\delta\kappa} + \gamma^m_{\epsilon\kappa}\gamma_m^{\beta\delta}T^{\alpha\gamma\kappa} + \gamma^m_{\epsilon\kappa}\gamma_m^{\gamma\delta}T^{\alpha\beta\kappa}\Big]. \end{aligned} \tag{3.31}$$

This is valid because there is only one scalar built out of four pure spinors λ^α, one pure spinor $\overline{\lambda}_\alpha$ and five unconstrained θ^α's. Using the theory of group representations one can show the following to be true

$$\overline{\lambda}\lambda^4 = [0,0,0,1,0] \otimes [0,0,0,0,4] = 1X[0,0,0,0,3] + 1X[0,0,0,1,4] + 1X[0,1,0,0,3]$$

$$\theta^5 = 1X[0,0,0,3,0] + 1X[1,1,0,1,0]$$

[2]Note that one could in principle use the explicit form for the measures $[dr]$,$[d\lambda]$,$[d\overline{\lambda}]$ to integrate the pure spinor variables and arrive at the final answer. But in doing so one looses the elegance and the simplifying features of expressions written in pure spinor superspace, so we avoid that route here.

so that

$$\overline{\lambda}\lambda^4\theta^5 = 1X[0,0,0,0,0] + 2X[0,0,0,0,4] + 5X[0,0,0,1,1] + 1X[0,0,0,1,5] + \dots$$

where the $\dots$ are higher rank representations. If λ^α were not a pure spinor then there would be three different scalars in the above decomposition.

Using (3.31) we can translate pure spinor superspace expressions of the type $(4,1)$ into a sum of familiar $(3,0)$ expressions

$$\langle \lambda^\alpha\lambda^\beta\lambda^\gamma\lambda^\delta\overline{\lambda}_\epsilon f^\epsilon_{\alpha\beta\gamma\delta}\rangle_{(4,1)} = \tag{3.32}$$

$$= \langle \lambda^\beta\lambda^\gamma\lambda^\delta f^\alpha_{\alpha\beta\gamma\delta}\rangle + \langle \lambda^\alpha\lambda^\gamma\lambda^\delta f^\beta_{\alpha\beta\gamma\delta}\rangle + \langle \lambda^\alpha\lambda^\beta\lambda^\delta f^\gamma_{\alpha\beta\gamma\delta}\rangle + \langle \lambda^\alpha\lambda^\beta\lambda^\gamma f^\delta_{\alpha\beta\gamma\delta}\rangle$$
$$-\frac{1}{12}\Big[\langle(\lambda\gamma^m)_\epsilon\gamma_m^{\alpha\beta}\lambda^\gamma\lambda^\delta f^\epsilon_{\alpha\beta\gamma\delta}\rangle + \langle(\lambda\gamma^m)_\epsilon\gamma_m^{\alpha\delta}\lambda^\beta\lambda^\gamma f^\epsilon_{\alpha\beta\gamma\delta}\rangle + \langle(\lambda\gamma^m)_\epsilon\gamma_m^{\alpha\gamma}\lambda^\beta\lambda^\delta f^\epsilon_{\alpha\beta\gamma\delta}\rangle$$
$$\langle(\lambda\gamma^m)_\epsilon\gamma_m^{\beta\gamma}\lambda^\alpha\lambda^\delta f^\epsilon_{\alpha\beta\gamma\delta}\rangle + \langle(\lambda\gamma^m)_\epsilon\gamma_m^{\beta\delta}\lambda^\alpha\lambda^\gamma f^\epsilon_{\alpha\beta\gamma\delta}\rangle + \langle(\lambda\gamma^m)_\epsilon\gamma_m^{\gamma\delta}\lambda^\alpha\lambda^\beta f^\epsilon_{\alpha\beta\gamma\delta}\rangle\Big] \tag{3.33}$$

We will now proceed to show that using (3.33) one can derive that

$$\langle(\overline{\lambda}\gamma_{mnp}D)[(\lambda A)(\lambda\gamma^m W)(\lambda\gamma^n W)(\lambda\gamma^p W)]\rangle_{(4,1)} \propto (\lambda\overline{\lambda})\langle(\lambda A)(\lambda\gamma^m W)(\lambda\gamma^n W)\mathcal{F}_{mn}\rangle, \tag{3.34}$$

which is the covariant equivalence proof we are looking for. Acting with the derivative over the superfields in (3.34) one obtains

$$\langle(\overline{\lambda}\gamma_{mnp}D)[(\lambda A)(\lambda\gamma^m W)(\lambda\gamma^n W)(\lambda\gamma^p W)]\rangle =$$
$$\langle\big[(\overline{\lambda}\gamma_{mnp}D)(\lambda A)\big](\lambda\gamma^m W)(\lambda\gamma^n W)(\lambda\gamma^p W)\rangle - \langle(\lambda A)(\overline{\lambda}\gamma_{mnp}D)[(\lambda\gamma^m W)(\lambda\gamma^n W)(\lambda\gamma^p W)]\rangle. \tag{3.35}$$

However using pure spinor properties and the equations of motion of super-Yang-Mills theory we find that the second term in the right hand side of (3.35) is given by

$$-\langle(\lambda A)(\overline{\lambda}\gamma_{mnp}D)(\lambda\gamma^m W)(\lambda\gamma^n W)(\lambda\gamma^p W)\rangle = 36(\lambda\overline{\lambda})\langle(\lambda A)(\lambda\gamma^m W)(\lambda\gamma^n W)\mathcal{F}_{mn}\rangle, \tag{3.36}$$

so that (3.21) is manifestly equivalent to (3.12) when the derivative acts over the W's. To finish the equivalence proof we need to evaluate the expression

$$\langle\big[(\overline{\lambda}\gamma_{mnp}D)(\lambda A)\big](\lambda\gamma^m W)(\lambda\gamma^n W)(\lambda\gamma^p W)\rangle. \tag{3.37}$$

Using (3.33) one converts (3.37) into a pure spinor superspace expression with three λ's,

$$\langle\big[(D\gamma_{mnp}\overline{\lambda})(\lambda A)\big](\lambda\gamma^m W)(\lambda\gamma^n W)(\lambda\gamma^p W)\rangle =$$

$$\langle[(D\gamma_{mnp}A)](\lambda\gamma^m W)(\lambda\gamma^n W)(\lambda\gamma^p W)\rangle + 24\langle[D_\alpha(\lambda A)](\gamma_{np}W)^\alpha(\lambda\gamma^n W)(\lambda\gamma^p W)\rangle$$
$$+\frac{1}{4}\langle[(\lambda\gamma_q\gamma_{mnp}D)A_\alpha](\gamma^q\gamma^m W)^\alpha(\lambda\gamma^n W)(\lambda\gamma^p W)\rangle$$
$$+\frac{1}{4}\langle[(\lambda\gamma_q\gamma_{mnp}D)(\lambda A)](W\gamma^m\gamma^q\gamma^n W)(\lambda\gamma^p W)\rangle, \tag{3.38}$$

where in the above it has to be understood that D_α acts only over the superfield A_β. Note that the last two terms of (3.38) come from the gamma matrix terms of (3.33).

Using

$$D_\alpha A_\beta + D_\beta A_\alpha = \gamma^q_{\alpha\beta}A_q \tag{3.39}$$

one can show that

$$\langle D_\alpha(\lambda A)(\gamma_{np}W)^\alpha(\lambda\gamma^n W)(\lambda\gamma^p W)\rangle =$$
$$-\langle(\lambda D)A_\alpha(\gamma_{np}W)^\alpha(\lambda\gamma^n W)(\lambda\gamma^p W)\rangle + \langle A_m(\lambda\gamma^m\gamma^{np}W)(\lambda\gamma^n W)(\lambda\gamma^p W)\rangle$$

where the last term is zero due to the pure spinor condition and the BRST charge in the first one can be integrated by parts to give

$$-\langle(\lambda D)A_\alpha(\gamma_{np}W)^\alpha(\lambda\gamma^n W)(\lambda\gamma^p W)\rangle = +\frac{1}{4}\langle(\lambda\gamma^{rs}\gamma_{np}A)(\lambda\gamma^n W)(\lambda\gamma^p W)\mathcal{F}_{rs}\rangle$$
$$= -2\langle(\lambda A)(\lambda\gamma^m W)(\lambda\gamma^n W)\mathcal{F}_{mn}\rangle.$$

Doing similar manipulations we also get

$$\langle(\lambda\gamma_q\gamma_{mnp})^\beta(D_\beta A_\alpha)(\gamma^q\gamma^m W)^\alpha(\lambda\gamma^n W)(\lambda\gamma^p W)\rangle = 4\langle\big[(\lambda D)A_\alpha\big](\gamma^q\gamma^m W)^\alpha(\lambda\gamma^m W)(\lambda\gamma^q W)\rangle$$
$$= \langle(\lambda\gamma^{rs}\gamma_{mq}A)(\lambda\gamma^m W)(\lambda\gamma^q W)\mathcal{F}_{rs}\rangle = -8\langle(\lambda A)(\lambda\gamma^m W)(\lambda\gamma^n W)\mathcal{F}_{mn}\rangle.$$

and

$$\langle(\lambda\gamma_q\gamma_{mnp})^\alpha[D_\alpha(\lambda A)](W\gamma^m\gamma^q\gamma^n W)(\lambda\gamma^p W)\rangle = -\langle Q\big[(\lambda\gamma_q\gamma_{mnp}A)\big](W\gamma^m\gamma^q\gamma^n W)(\lambda\gamma^p W)\rangle$$
$$+\langle(\lambda\gamma_{mnpqr}\lambda)A_r(W\gamma^{mnq}W)(\lambda\gamma^p W)\rangle. \tag{3.40}$$

The last line of (3.40) is zero due to the pure spinor condition, so (3.40) becomes

$$\langle(\lambda\gamma_q\gamma_{mnp})^\alpha D_\alpha(\lambda A)(W\gamma^m\gamma^q\gamma^n W)(\lambda\gamma^p W)\rangle = -\frac{1}{2}\langle(\lambda\gamma_q\gamma_{mnp}A)(\lambda\gamma^{rs}\gamma^m\gamma^q\gamma^n W)(\lambda\gamma^p W)\mathcal{F}_{rs}\rangle$$
$$= -32\langle(\lambda A)(\lambda\gamma^m W)(\lambda\gamma^n W)\mathcal{F}_{mn}\rangle, \tag{3.41}$$

where we integrated the BRST-charge by parts and went through a long list of gamma matrix manipulations.

Finally, the first term in the right hand side of (3.38) can be rewritten using the gamma matrix identity of $\eta_{mn}\gamma^m_{\alpha(\beta}\gamma^n_{\gamma\delta)} = 0$,

$$\langle[(D\gamma_{mnp}A)](\lambda\gamma^m W)(\lambda\gamma^n W)(\lambda\gamma^p W)\rangle =$$

$$= \langle(\gamma_m\gamma_n W)^\sigma(\gamma_p\gamma^n\lambda)^\rho D_\sigma A_\rho(\lambda\gamma^m W)(\lambda\gamma^p W)\rangle + \langle(\gamma_m\gamma_n\lambda)^\sigma(\gamma_p\gamma^n W)^\rho D_\sigma A_\rho(\lambda\gamma^m W)(\lambda\gamma^p W)\rangle$$

$$= +2\langle(W\gamma_n\gamma_m)^\alpha[D_\alpha(\lambda A)](\lambda\gamma^m W)(\lambda\gamma^n W)\rangle + 2\langle(\gamma^p\gamma^n W)^\sigma[(\lambda D)A_\sigma](\lambda\gamma^n W)(\lambda\gamma^p W)\rangle$$

Using (3.39) in the first term we get,

$$= -4\langle(W\gamma_n\gamma_m)^\alpha[D_\alpha(\lambda A)](\lambda\gamma^m W)(\lambda\gamma^n W)\rangle = -\langle(\lambda\gamma^{rs}\gamma_n\gamma_m A)(\lambda\gamma^m W)(\lambda\gamma^n W)\mathcal{F}_{rs}\rangle$$

and so

$$\langle[(D\gamma_{mnp}A)](\lambda\gamma^m W)(\lambda\gamma^n W)(\lambda\gamma^p W)\rangle = -8\langle(\lambda A)(\lambda\gamma^m W)(\lambda\gamma^n W)\mathcal{F}_{mn}\rangle \tag{3.42}$$

With all the above identities we finally arrive at

$$\langle\big[(D\gamma_{mnp}\overline{\lambda})(\lambda A)\big](\lambda\gamma^m W)(\lambda\gamma^n W)(\lambda\gamma^p W)\rangle = -58\langle(\lambda A)(\lambda\gamma^m W)(\lambda\gamma^n W)\mathcal{F}_{mn},$$

which finishes the covariant proof of equivalence of the massless four-point kinematic factor of the one-loop amplitude between the minimal and non-minimal pure spinor formalism.

We also observe that the terms containing gamma matrices in (3.33) – they are responsable for the traceless property of $T_\epsilon^{\alpha\beta\gamma\delta}$ – covariantly generate the term (3.28) whose existence was deduced through non-covariant means. This can be checked by noticing that (3.40) can also be written as,

$$\langle(\lambda\gamma_q\gamma_{mnp}D)(\lambda A)(W\gamma^m\gamma^q\gamma^n W)(\lambda\gamma^p W)\rangle = -4\langle\big[(\lambda\gamma_{mn}D)(\lambda A)\big](W\gamma^{mnp}W)(\lambda\gamma_p W)\rangle, \tag{3.43}$$

by using the pure spinor property of $(\lambda\gamma_p)_\alpha(\lambda\gamma^p)_\beta = 0$ to get rid of the γ_p inside of $(\lambda\gamma_q\gamma_{mnp}D)(\lambda A)$.

3.3.4 Yet another covariant proof of equivalence

There is yet another covariant proof of the equivalence between (3.12) and (3.21), which is perhaps more elegant than the proof presented in the previous section.

From (3.35) and (3.36) we know that

$$\langle(\overline{\lambda}\gamma_{mnp}D)[(\lambda A)(\lambda\gamma^m W)(\lambda\gamma^n W)(\lambda\gamma^p W)]\rangle = 36(\lambda\overline{\lambda})\langle(\lambda A)(\lambda\gamma^m W)(\lambda\gamma^n W)\mathcal{F}_{mn}\rangle$$

$$+\langle\big[(\overline{\lambda}\gamma_{mnp}D)(\lambda A)\big](\lambda\gamma^m W)(\lambda\gamma^n W)(\lambda\gamma^p W)\rangle. \tag{3.44}$$

Using $\eta_{mn}\gamma^m_{\alpha(\beta}\gamma^n_{\gamma\delta)} = 0$ and that the factor of $\big[(\overline{\lambda}\gamma_{mnp}D)(\lambda A)\big]$ can be substituted by $\big[(\overline{\lambda}\gamma_m\gamma_n\gamma_p D)(\lambda A)\big]$ we arrive at the following identity for the second term of (3.44)

$$\langle\big[(\overline{\lambda}\gamma_{mnp}D)(\lambda A)\big](\lambda\gamma^m W)(\lambda\gamma^n W)(\lambda\gamma^p W)\rangle = \langle(\overline{\lambda}\gamma_m\gamma_n W)\Big[(\lambda\gamma^n\gamma_p D)(\lambda A)\Big](\lambda\gamma^m W)(\lambda\gamma^p W)\rangle$$

$$+\langle(\overline{\lambda}\gamma_m\gamma_n\lambda)(W\gamma^n\gamma_p)^\sigma\Big[D_\sigma(\lambda A)\Big](\lambda\gamma^m W)(\lambda\gamma^p W)\rangle. \tag{3.45}$$

Using $\gamma^n\gamma_p = -\gamma_p\gamma^n + 2\delta^n_p$ and the equation of motion $Q(\lambda A) = 0$ the first term of (3.45) vanishes, while the second can be rewritten as

$$\langle(\overline{\lambda}\gamma_m\gamma_n\lambda)(W\gamma^n\gamma_p)^\sigma\Big[D_\sigma(\lambda A)\Big](\lambda\gamma^m W)(\lambda\gamma^p W)\rangle =$$

$$= -2(\lambda\overline{\lambda})\langle(W\gamma_m\gamma_p)^\sigma\Big[(\lambda D)A_\sigma\Big](\lambda\gamma^m W)(\lambda\gamma^p W)\rangle$$

$$= +4(\lambda\overline{\lambda})\langle(\lambda A)(\lambda\gamma^m W)(\lambda\gamma^n W)\mathcal{F}_{mn}\rangle,$$

where we used (3.39) and integrated the BRST-charge by parts. So we have just shown that (3.44) is equal to

$$\langle(\overline{\lambda}\gamma_{mnp}D)\Big[(\lambda A)(\lambda\gamma^m W)(\lambda\gamma^n W)(\lambda\gamma^p W)\Big]\rangle = 40(\lambda\overline{\lambda})\langle(\lambda A)(\lambda\gamma^m W)(\lambda\gamma^n W)\mathcal{F}_{mn}\rangle,$$

which finishes the proof.

3.3.5 On the pure spinor expression (1.15)

The pure spinor superspace expression (1.15)

$$K_c = \langle(\lambda\gamma^m\theta)(\lambda\gamma^n W^1)(\lambda\gamma^p W^2)(W^3\gamma_{mnp}W^4)\rangle \tag{3.46}$$

is an interesting one. The complete explanation of its origins remains unknown[3] to this date, but it may turn out to be useful as a hint to further studies.

First of all one should notice that (3.46) is manifestly gauge invariant but appears to be non-supersymmetric. Secondly, it is BRST-closed and finally it has the dimensions of a F^4 term. What does the component expression of (3.46) look like?

[3]It was found when trying to discover a prescription which uses only integrated vertices to compute the massless four-point amplitude at one-loop and can be roughly seen as the substitution rule of $\int bV \to (d^0\theta)\int U$ inside the kinematic factor.

Using the SYM superfield expansions of (B.24) one easily obtains the bosonic contribution

$$K_c = -\frac{1}{256}\langle(\lambda\gamma^m\theta)(\lambda\gamma^n\gamma^{m_1n_1}\theta)(\lambda\gamma^p\gamma^{m_2n_2}\theta)(\theta\gamma^{m_3n_3}\gamma_{mnp}\gamma^{m_4n_4}\theta)\rangle F^1_{m_1n_1}\ldots F^4_{m_4n_4}$$
$$= At_8^{m_1n_1\ldots m_4n_4}F^1_{m_1n_1}\ldots F^4_{m_4n_4} \tag{3.47}$$

whose proportionality with the four-point kinematic factor at one-loop seems surprising at first, and clearly deserves some kind of explanation.

Although (3.46) seems to be non-supersymmetric due to the explicit θ, one can show that its supersymmetry variation is a total derivative,

$$\delta_{susy}K_c = \langle(\lambda\gamma^m\epsilon)(\lambda\gamma^n W^1)(\lambda\gamma^p W^2)(W^3\gamma_{mnp}W^4)\rangle \tag{3.48}$$

because $\delta_{susy}K_c$ is proportional to the anomaly kinematic factor, which is known to be a total derivative[4].

One can understand the appearance of the t_8 tensor in (3.47) by noticing that in the bosonic θ-expansion of $(D\gamma^{qrs}A)$, due to the antisymmetry of γ^{qrs} in its spinor indices, has no components with zero thetas,

$$(D\gamma^{mnp}A) = \frac{1}{4}(\theta\gamma^{mnp}\gamma^{tu}\theta)F_{tu} + \frac{1}{4}\partial_t a_u(\theta\gamma^t\gamma^{mnp}\gamma^u\theta) + \ldots$$
$$= -(\theta\gamma^{mnp}W) + \frac{1}{4}\partial_t a_u(\theta\gamma^t\gamma^{mnp}\gamma^u\theta). \tag{3.49}$$

where the substitution in the second line is valid up to fermionic terms.

Now if one considers the bosonic computation of

$$L = \langle\big[(D\gamma_{mnp}A)\big](\lambda\gamma^m W)(\lambda\gamma^n W)(\lambda\gamma^p W)\rangle \tag{3.50}$$

there is only one contribution coming from the superfield $(D\gamma_{mnp}A)$, namely the two-thetas term of (3.49). So the pure spinor expression (3.50) becomes

$$L = -\langle(\theta\gamma_{mnp}W)(\lambda\gamma^m W)(\lambda\gamma^n W)(\lambda\gamma^p W)\rangle + \frac{1}{4}\partial_t a_u\langle(\theta\gamma^t\gamma^{mnp}\gamma^u\theta)(\lambda\gamma^m W)(\lambda\gamma^n W)(\lambda\gamma^p W)\rangle.$$

It is a straightforward exercise to show that the bosonic part of second term vanishes identically, leading us to

$$L = -\langle(\theta\gamma_{mnp}W)(\lambda\gamma^m W)(\lambda\gamma^n W)(\lambda\gamma^p W)\rangle.$$

[4] To make it clearer, just suppose that one of the W's in (1.13) is a constant spinor ϵ.

On the one hand, using that $(\lambda\gamma^m)_{\delta_1}(\lambda\gamma^n)_{\delta_2}(\lambda\gamma^p)_{\delta_3}(\gamma_{mnp})_{\delta_4\delta_5}$ is completely antisymmetric in its spinor indices one obtains,

$$L = -\langle(\lambda\gamma^m\theta)(\lambda\gamma^n W)(\lambda\gamma^p W)(W\gamma_{mnp}W)\rangle. \tag{3.51}$$

on the other hand, from (3.42) and (3.50) one is led to conclude

$$\langle(\lambda\gamma^m\theta)(\lambda\gamma^n W)(\lambda\gamma^p W)(W\gamma_{mnp}W)\rangle = 8\langle(\lambda A)(\lambda\gamma^m W)(\lambda\gamma^n W)\mathcal{F}_{mn}\rangle. \tag{3.52}$$

As it was shown in (3.48), the left hand side of (3.52) is supersymmetric. So it turns out that (3.52) is valid for all bosonic and fermionic components! Consequentely not only (3.47) is expected to happen at the level of pure spinor superspace but it is a completely equivalent way of computing the massless four-point kinematic factor at one-loop. Note that somehow we are exchanging manifest supersymmetry by manifest gauge invariance when computing either the left or right hand side of (3.52). This may be a hint on how to develop an amplitude prescription at one-loop which uses only integrated vertices, indicating that maybe manifest supersymmetry will be spoiled.

It is also interesting to note that the analogous generalization for the massless five-point amplitude,

$$K_5 = \langle(\lambda\gamma^m\theta)(\lambda\gamma^n\gamma^{rs}W^5)(\lambda\gamma^p W^1)(W^3\gamma_{mnp}W^4)\mathcal{F}^2_{rs}\rangle - (2\leftrightarrow 5)$$

leading to

$$K_5 = \langle(D\gamma_{mnp}A^1)(\lambda\gamma^m\gamma^{rs}W^5)(\lambda\gamma^n W^3)(\lambda\gamma^p W^4)\mathcal{F}^2_{rs}\rangle - (2\leftrightarrow 5)$$

also gives rise to the correct bosonic component expression (as we shall compute in section 3.9). Whether they are the correct full kinematic factor is presently under investigation with the colaboration of Christian Stahn.

3.4 Massless four-point amplitude at two-loops

In this section we will present the massless four-point amplitude at two-loops and as in the other amplitudes considered in this thesis, we will focus our attention in the form of the kinematic factor in pure spinor superspace.

There are two different ways of computing this particular amplitude, using the minimal and the non-minimal pure spinor formalism. As it turns out, the minimal version is more efficient in getting the final expression for the kinematic factor. With the non-minimal version what one obtains is a kinematic factor which contains also the pure spinor $\overline{\lambda}$'s, complicating things a bit. But in the end one can prove that both versions are equivalent to

$$K_2 = \langle(\lambda\gamma^{mnpqr}\lambda)\mathcal{F}^1_{mn}\mathcal{F}^2_{pq}\mathcal{F}^3_{rs}(\lambda\gamma^s W^4)\rangle\Delta(1,3)\Delta(2,4) + \text{perm.}(1234). \tag{3.53}$$

This is again a manifestation of the fact that both prescriptions were shown to be equivalent in [16].

3.4.1 Minimal computation

Following the prescription given in section 2.6 the massless four-point closed[5] superstring amplitude at genus two is given by

$$A = \int d^2\tau_1 d^2\tau_2 d^2\tau_3 \langle| \prod_{P=1}^{3} \int d^2u_P \mu_P(u_P)\tilde{b}_{B_P}(u_P, z_P)$$

$$\prod_{P=4}^{20g} Z_{B_P}(z_P) \prod_{R=1}^{2} Z_J(v_R) \prod_{I=1}^{11} Y_{C_I}(y_I)\ |^2\ \prod_{i=1}^{4} \int d^2z_i U^i(z_i)\rangle,$$

The explicit computation can be done by considering the 32 zero modes of d_α and the 16 zero modes of θ^α.

Seventeen zero modes of d_α come from the Z_B operators, while another two come from the Z_J's. Therefore the thirteen zero modes left to give a non-vanishing Berezin integration must come from the three b-ghosts and the external vertices, and there is only one possibility. This is because the b-ghost has only terms with zero, one, two or four d_α's. So suppose the three b-ghosts contribute with ten d's and the external vertices the remaining three through $(dW)^3$. In this case the amplitude vanishs because there is a factor of $\delta'(B^{mn}N_{mn})$ in the term with four d's in the b-ghost,

$$b_B\Big|_{4d} = B_{mn}B^{qr}(d\gamma^{mnp}d)(d\gamma_{pqr}d)\delta'(B^{st}N_{st})$$

[5]The closed string amplitude is computed as the square of the open string. We use this fact implicitly all the time in this thesis, by considering only one Riemann surface at each genus. However we always present the results in terms of open superstring kinematic factors K, from which the closed string kinematic factor can be obtained by the holomorphic square $K\overline{K}$. In terms of the effective action, the Riemann tensor is obtained by identifying R_{mnpq} with $F_{mn}\overline{F}_{pq}$, or $R_{mnpq} = k_{[m}h_{n][q}k_{q]}$, where we identified $e_n\tilde{e}_q \to h_{nq}$.

which makes the amplitude vanish because there is no N_{mn} coming from the external vertices to cancel the derivative of the delta function[6].

Going through all the possibilities one obtains a vanishing result for all but one case, when the three b-ghosts provide twelve zero modes of d_α and three $\delta'(B^{mn}N_{mn})$ while the four external vertices contribute with the required $(dW)(N\cdot F)^3$ term to give a non-vanishing result.

The integrations over the pure spinor measures have the effect of selecting only the components such that $\langle\lambda^3\mathcal{F}\mathcal{F}\mathcal{F}W\rangle$ is proportional to the pure spinor measure (2.74), and one can easily check that there is only one scalar which can be built out of this superfields, namely

$$\langle(\lambda\gamma^{mnpqr}\lambda)(\lambda\gamma^s W^4)\mathcal{F}^1_{mn}\mathcal{F}^2_{pq}\mathcal{F}^3_{rs}.\rangle \tag{3.54}$$

The computation of the moduli space part is summarized as follows. Each b-ghost has conformal weight +2 and no poles over the Riemann surface of genus two and their product has zeros when their position coincide. These conditions uniquely determine their Riemannian contribution to be given by the product of three quadri-holomorphic 1-forms $\Delta(u_1,u_2)\Delta(u_2,u_3)\Delta(u_3,u_1)$. Analogously, each external vertex has conformal weight +1 and therefore their contribution is given by some linear combination of the holomorphic 1-forms, $h^{IJKL}w_I(z_1)w_J(z_2)w_K(z_3)w_L(z_4)$. The only linear combination compatible with the symmetries of (3.54) is if the kinematic factor is the sum over permutations of

$$K_2 = \langle(\lambda\gamma^{mnpqr}\lambda)(\lambda\gamma^s W^4)\mathcal{F}^1_{mn}\mathcal{F}^2_{pq}\mathcal{F}^3_{rs}\Delta(z_1,z_3)\Delta(z_2,z_4)\rangle + \text{perm}(1234). \tag{3.55}$$

For example, due to the symmetry of $(\lambda\gamma^{mnpqr}\lambda)\mathcal{F}^1_{mn}\mathcal{F}^2_{pq}$ under $(1\leftrightarrow 2)$ there could be no factor of $\Delta(z_1,z_2)$ in the above combination.

And from the theory of Riemann surfaces one can show that

$$\int d^2\tau_1 d^2\tau_2 d^2\tau_3 |\prod_{j=1}^{3}\int d^2u_j\mu(u_j)\Delta(u_1,u_2)\Delta(u_2,u_3)\Delta(u_3,u_1)|^2 = \int d^2\Omega_{11}d^2\Omega_{12}d^2\Omega_{22},$$

where Ω_{IJ} is the 2×2 period matrix of the genus two Riemann surface.

Finally, the final amplitude is given by

$$\mathcal{A}_2 \propto \mathrm{e}^{2\phi}\tilde{K}_2\overline{\tilde{K}}_2\int_{\mathcal{M}_2}\frac{|d^3\Omega|^2}{(\det \mathrm{Im}\Omega)^5}F_2(\Omega,\mathcal{Y}),$$

[6]The derivative of the delta function satisfies $x\delta'(x) = -\delta(x)$.

where $K_2 = \tilde{K}_2 \mathcal{Y}$ and $F_2(\Omega, \mathcal{Y})$, apart from the factor of $|\mathcal{Y}|^2$,

$$\mathcal{Y}(s,t,u) = [(u-t)\Delta(1,2)\Delta(3,4) + (s-t)\Delta(1,3)\Delta(2,4) + (s-u)\Delta(1,4)\Delta(2,3)]$$

comes from the standard integration over zero modes of X^m and is given by

$$F_2(\Omega, \mathcal{Y}) = \int |\mathcal{Y}|^2 \prod_{i<j} G(z_i, z_j)^{k_i \cdot k_j},$$

where $G(z_i, z_j)$ is the scalar Green function.

3.4.2 Non-Minimal computation

The computation as dictated by the non-minimal pure spinor formalism gives rise to the same moduli space part described in the last subsection, with a final integration over the period matrix of the Riemann surface. The only difference comes from the computation of the kinematic factor, due to the different expression for the b-ghost and the additional functional integrations over the non-minimal variables.

If we restrict our attention to the kinematic factor, then the computation can be easily performed by zero mode counting, and goes as follows [26].

First note that there is only one place where the 22 zero modes of s^α can come from, the regulator $\mathcal{N}$ of (2.92). It must provide the whole 22 s^α zero modes, which come multiplied by 22 zero modes of d_α. So the remaining 10 d_α zero modes must come from the four integrated vertex operators and the three b_α ghosts. This is only possible if each integrated vertex operators provides a d_α zero mode through the term $(W^\alpha d_\alpha)$ and each b ghost provides two d_α zero modes through the term of (3.16).

After integrating over the zero modes of the conformal weight +1 fields $(w^I_\alpha, \overline{w}^{I\alpha}, d^I_\alpha, s^{I\alpha})$ using the measure factors described in [26], one is left with an expression proportional to

$$\int d^{16}\theta \int [d\lambda][d\overline{\lambda}][dr](\lambda\overline{\lambda})^{-6}(\lambda)^6(\overline{\lambda}\gamma^{mnp}r)^3 WWWW \exp(-\lambda\overline{\lambda} - r\theta) \tag{3.56}$$

$$= \int d^{16}\theta \int [d\lambda][d\overline{\lambda}][dr] \exp(-\lambda\overline{\lambda} - r\theta)(\lambda\overline{\lambda})^{-6}(\lambda)^6(\overline{\lambda}\gamma^{mnp}D)^3 WWWW \tag{3.57}$$

where the index contractions on

$$(\lambda)^6(\overline{\lambda}\gamma^{mnp}D)^3 WWWW \tag{3.58}$$

will be found in the discussion below. Note that the factor of $(\lambda)^6$ in (3.56) comes from the $11g$ factors of λ and $\overline{\lambda}$ which multiply the zero modes of d^I_α and s^I_α in $\mathcal{N}$, the factor of

$(\lambda)^{-8g}(\overline{\lambda})^{-8g}$ in the measure factor of w^I_α and $\overline{w}^{I\alpha}$, and the factor of $(\overline{\lambda})^{-3g}$ in the measure factor of $s^{I\alpha}$.

As in the one-loop four-point amplitude, there is fortunately a unique way of contracting the indices of (3.58) in a Lorentz-invariant manner. Choosing the Lorentz frame where λ^+ is the only non-zero component of λ^α, one finds that $(\lambda^+)^6$ contributes $+15$ $U(1)$ charge so that each $(\overline{\lambda}\gamma^{mnp}D)$ must contribute -3 charge and each W must contribute $-\frac{3}{2}$ charge. Since the -3 component of $(\overline{\lambda}\gamma^{mnp}D)$ is $(\overline{\lambda}^{[ab]}D_+ - \overline{\lambda}_+ D^{[ab]})$, and since D_+ annihilates the $-\frac{3}{2}$ component of W^α, the only contribution to (3.58) comes from a term of the form

$$(\lambda^+)^6(\overline{\lambda}_+)^3(D^{[ab]}D^{[cd]}D^{[ef]})(W^g W^h W^j W^k) \tag{3.59}$$

where the ten $SU(5)$ indices are contracted with two ϵ_{abcde}'s.

The term of (3.59) produces three types of terms depending on how the three D's act on the four W's. If all three D's act on the same W, one gets a term proportional to $(\lambda^+)^6(\overline{\lambda}_+)^3 WWW\partial\mathcal{F}$, which by $U(1)\times SU(5)$ invariance must have the form

$$(\lambda^+)^6(\overline{\lambda}_+)^3 W^a W^b W^c \partial_a \mathcal{F}_{bc}. \tag{3.60}$$

And if two D's act on the same W, one gets a term proportional to $(\lambda^+)^6(\overline{\lambda}_+)^3\mathcal{F}WW\partial W$, which by $U(1)\times SU(5)$ invariance must have the form

$$(\lambda^+)^6(\overline{\lambda}_+)^3 \mathcal{F}_{bc} W^a W^b \partial_a W^c. \tag{3.61}$$

Finally, if each D acts on a different W, one obtains a term that is proportional to $(\lambda^+)^6(\overline{\lambda}_+)^3 W\mathcal{F}\mathcal{F}\mathcal{F}$, which by $U(1)\times SU(5)$ invariance must have the form

$$(\lambda^+)^6(\overline{\lambda}_+)^3 \mathcal{F}_{ab}\mathcal{F}_{cd}\mathcal{F}_{ef} W^f \epsilon^{abcde}. \tag{3.62}$$

The first term in (3.60) vanishes by Bianchi identities. And the second term in (3.61) is proportional to the first term after integrating by parts with respect to ∂_a and using the equation of motion $\partial_a W^a = 0$. So the only contribution to the kinematic factor comes from the third term of (3.62), which can be written in Lorentz-covariant notation as

$$(\lambda\overline{\lambda})^3(\lambda\gamma^{mnpqr}\lambda)\mathcal{F}_{mn}\mathcal{F}_{pq}\mathcal{F}_{rs}(\lambda\gamma^s W). \tag{3.63}$$

So the non-minimal computation of the two-loop kinematic factor agrees with the minimal computation of (3.55).

3.5 Relating massless four-point kinematic factors

In this section the usefulness of having pure spinor superspace expressions for the kinematic factors will reveal itself, uncovering a long expected-to-be-true lore but never explicitly proved fact about the scattering of four massless strings.

3.5.1 Tree-level and one-loop

In the following sections we will make heavy use of the superfield equations of motion in the formulation of ten-dimensional Super-Yang-Mills theory in superspace (see review in Appendix A), namely

$$Q\mathcal{F}_{mn} = 2\partial_{[m}(\lambda\gamma_{n]}W), \quad QW^{\alpha} = \frac{1}{4}(\lambda\gamma^{mn})^{\alpha}\mathcal{F}_{mn}, \quad QA_m = (\lambda\gamma_m W) + \partial_m(\lambda A), \quad (3.64)$$

where $Q = \oint \lambda^{\alpha} d_{\alpha}$ is the pure spinor BRST operator With these relations in hand we will show that (expres arvore) holds true. To prove this we note that

$$\langle(\lambda A)\partial^m(\lambda A)(QA^n)F_{mn}\rangle = -\langle(\lambda A)\partial^m(\lambda A)A^n(QF_{mn})\rangle,$$

which upon use of (3.64) and momentum conservation becomes

$$\langle(\lambda A)\partial^m(\lambda A)(QA^n)F_{mn}\rangle = \langle(\lambda A)\partial^m(\lambda A)\partial_m A_n(\lambda\gamma^n W)\rangle$$
$$-\langle\partial_n(\lambda A)\partial_m(\lambda A)A^n(\lambda\gamma^m W)\rangle - \langle(\lambda A)\partial_n\partial_m(\lambda A)A^n(\lambda\gamma^m W)\rangle. \quad (3.65)$$

The second term can be rewritten like

$$\langle\partial_n(\lambda A)\partial_m(\lambda A)A^n(\lambda\gamma^m W)\rangle = -\langle(\lambda A)(\lambda\gamma^m W)\left[A^n\partial_m\partial_n(\lambda A) + \partial^n(\lambda A)\partial_m A_n\right]\rangle$$

as can be shown by integrating ∂^m by parts and using the equation of motion for W^{α}. So,

$$\langle(\lambda A)\partial^m(\lambda A)(QA^n)F_{mn}\rangle = \langle(\lambda A)\partial^m(\lambda A)(\lambda\gamma^n W)F_{mn}\rangle - 2\langle(\lambda A)\partial_n\partial_m(\lambda A)A^n(\lambda\gamma^m W)\rangle$$

which implies that $\langle(\lambda A)\partial^m(\lambda A)\partial^n(\lambda A)F_{mn}\rangle = -2\langle(\lambda A)\partial_n\partial_m(\lambda A)A^n(\lambda\gamma^m W)\rangle$, or equivalently,

$$\langle(\lambda A)\partial^m(\lambda A)\partial^n(\lambda A)F_{mn}\rangle = -2\langle(\lambda A)\partial_n(QA_m)A^n(\lambda\gamma^m W)\rangle. \quad (3.66)$$

Using $[Q, \partial^n] = 0$ and the decoupling of BRST-trivial operators, equation (3.66) becomes

$$\langle(\lambda A)\partial^m(\lambda A)\partial^n(\lambda A)F_{mn}\rangle = 2\langle(\lambda A)(\partial_n A_m)(QA^n)(\lambda\gamma^m W)\rangle$$

$$= \langle (\lambda A)(\lambda\gamma^m W)(\lambda\gamma^n W)F_{mn}\rangle + 2\langle (\partial_n A_m)(\lambda A)\partial^n(\lambda A)(\lambda\gamma^m W)\rangle. \tag{3.67}$$

Plugging (3.67) in the tree-level kinematic factor (3.10) we finally obtain

$$K_0 = -\langle (\lambda A)(\lambda\gamma^m W)(\lambda\gamma^n W)F_{mn}\rangle = -\frac{1}{3}K_1, \tag{3.68}$$

which finishes the proof and explicitly relates the tree-level and one-loop kinematic factors.

3.5.2 One- and two-loop

To obtain a relation between the one- and two-loop kinematic factors we first need to show that $\langle (\lambda A^1)(\lambda\gamma^m W^2)(\lambda\gamma^n W^3)\mathcal{F}^4_{mn}\rangle$ is completely symmetric in the labels (1234). This can be done by noting that

$$\langle (\lambda\gamma^{mnpqr}\lambda)(\lambda A^1)(W^2\gamma_{pqr}W^3)\mathcal{F}^4_{mn}\rangle = 4\langle (\lambda A^1)Q\left[(W^2\gamma_{pqr}W^3)\right](\lambda\gamma^{pqr}W^4)\rangle. \tag{3.69}$$

Together with the identities

$$(\lambda\gamma^{mn}\gamma^{pqr}W^2)(\lambda\gamma_{pqr}W^4) = -48(\lambda\gamma^{[m}W^2)(\lambda\gamma^{n]}W^4)$$

$$(\lambda\gamma^{mnpqr}\lambda)(W^2\gamma_{pqr}W^3) = -96(\lambda\gamma^{[m}W^2)(\lambda\gamma^{n]}W^3),$$

equation (3.69) implies that

$$\langle (\lambda A^1)(\lambda\gamma^m W^4)(\lambda\gamma^n W^2)\mathcal{F}^3_{mn}\rangle + \langle (\lambda A^1)(\lambda\gamma^m W^3)(\lambda\gamma^n W^4)\mathcal{F}^2_{mn}\rangle =$$

$$= 2\langle (\lambda A^1)(\lambda\gamma^m W^2)(\lambda\gamma^n W^3)\mathcal{F}^4_{mn}\rangle. \tag{3.70}$$

From (3.70) it follows that,

$$K_{1-\text{loop}} = 3\langle (\lambda A^1)(\lambda\gamma^m W^2)(\lambda\gamma^n W^3)\mathcal{F}^4_{mn}\rangle. \tag{3.71}$$

Furthermore, the independence of which vertex operator we choose to be non-integrated implies total symmetry of $\langle (\lambda A^1)(\lambda\gamma^m W^2)(\lambda\gamma^n W^3)\mathcal{F}^4_{mn}\rangle$ in the labels (1234).

Now we can relate the one- and two-loop kinematic factors by noting that

$$(\lambda\gamma^{mnpqr}\lambda)\mathcal{F}^1_{mn}\mathcal{F}^2_{pq}\mathcal{F}^3_{rs}(\lambda\gamma^s W^4) = -4Q\left[(\lambda\gamma^r\gamma^{mn}W^2)(\lambda\gamma^s W^4)\mathcal{F}^1_{mn}\mathcal{F}^3_{rs}\right]$$

$$-8ik^1_m(\lambda\gamma_n W^1)(\lambda\gamma^r\gamma^{mn}W^2)(\lambda\gamma^s W^4)\mathcal{F}^3_{rs}, \tag{3.72}$$

where the pure spinor constraint $(\lambda\gamma^m\lambda) = 0$ and the identity $\eta_{mn}\gamma^m_{\alpha(\beta}\gamma^n_{\gamma\delta)} = 0$ must be used to show the vanishing of terms containing factors of $(\lambda\gamma^m)_\alpha(\lambda\gamma_m)_\beta$. Furthermore, as BRST-exact terms decouple from pure spinor correlations $\langle\ldots\rangle$, equation (3.72) implies

$$\langle(\lambda\gamma^{mnpqr}\lambda)\mathcal{F}^1_{mn}\mathcal{F}^2_{pq}\mathcal{F}^3_{rs}(\lambda\gamma^s W^4)\rangle = +16ik^1_m\langle(\lambda\gamma^r W^1)(\lambda\gamma^m W^2)(\lambda\gamma^s W^4)\mathcal{F}^3_{rs},\rangle, \quad (3.73)$$

where we have used $k^1_m(\lambda\gamma_n W^1)(\lambda\gamma^r\gamma^{mn}W^2) = -2k^1_m(\lambda\gamma^r W^1)(\lambda\gamma^m W^2)$, which is valid when the equation of motion $k^1_m(\gamma^m W^1)_\alpha = 0$ is satisfied.

Using $(\lambda\gamma_m W^2) = QA^2_m - ik^2_m(\lambda A^2)$ and $\langle(\lambda\gamma^r W^1)Q(A^m_2)(\lambda\gamma^s W^4)\mathcal{F}^3_{rs}\rangle = 0$ we arrive at the following pure spinor superspace identity

$$\langle(\lambda\gamma^{mnpqr}\lambda)\mathcal{F}^1_{mn}\mathcal{F}^2_{pq}\mathcal{F}^3_{rs}(\lambda\gamma^s W^4)\rangle = -16(k^1\cdot k^2)\langle(\lambda A^2)(\lambda\gamma^r W^1)(\lambda\gamma^s W^4)\mathcal{F}^3_{rs}\rangle \quad (3.74)$$

Multiplying (3.74) by $\Delta(1,3)\Delta(2,4)$ and summing over permutations leads to the following identity,

$$K_2 = \frac{32}{3}K_1\left[(u-t)\Delta(1,2)\Delta(3,4) + (s-t)\Delta(1,3)\Delta(2,4) + (s-u)\Delta(1,4)\Delta(2,3)\right], \quad (3.75)$$

where we used (3.71) and the standard Mandelstam variables $s = -2(k^1\cdot k^2)$, $t = -2(k^1\cdot k^4)$, $u = -2(k^2\cdot k^4)$.

In view of the results from the next section, (3.75) not only provides a simple proof of two-loop equivalence with the (bosonic) RNS result of [13] but it also automatically implies the knowledge of the full amplitude, including fermionic external states (which has not been computed in the RNS yet).

3.6 The complete supersymmetric kinematic factors at tree-level, one- and two-loops in components

In this section the usefulness and the simplifying power of the pure spinor formalism become manifest. Using either the RNS or GS formalism, the computation of the kinematic factor for all possible external state combination allowed by supersymmetry for the massless four-point amplitude at tree-level, one-loop and two-loop order would be a daunting task. In fact, after more than two decades of effort, the kinematic factors for fermionic states have never been explicitly computed at two-loops. Using the pure spinor formalism derivation of the identities in section 3.5 this task becomes almost trivial, as shown below.

To obtain the complete supersymmetric kinematic factors for the massless four-point amplitudes at tree-level, one- and two-loops all one needs to do is to use the method of appendix A to evaluate the pure spinor superspace expression of

$$K_0 = \frac{1}{2}k_1^m k_2^n \langle (\lambda A^1)(\lambda A^2)(\lambda A^3)\mathcal{F}^4_{mn}\rangle - (k^1\cdot k^3)\langle A^1_m(\lambda A^2)(\lambda A^3)(\lambda\gamma^m W^4)\rangle + (1\leftrightarrow 2). \tag{3.76}$$

The first term doesn't contribute in the computation of $K_0(f_1f_2f_3f_4) \equiv K_0^{\rm 4F}$, while the second leads to[7]

$$K_0^{\rm 4F} = -\frac{1}{9}(k^1\cdot k^3)\langle(\lambda\gamma^a\theta)(\lambda\gamma^b\theta)(\lambda\gamma^c\chi^4)(\chi^3\gamma^b\theta)(\theta\gamma^c\chi^1)(\chi^2\gamma^a\theta)\rangle + (1\leftrightarrow 2),$$

$$= \frac{1}{5760}\left[(\chi^1\gamma^m\chi^2)(\chi^3\gamma_m\chi^4)\left[(k^2\cdot k^3)-(k^1\cdot k^3)\right] - \frac{1}{12}(k^3\cdot k^4)(\chi^1\gamma^{mnp}\chi^2)(\chi^3\gamma_{mnp}\chi^4)\right].$$

Using the following Fierz identity

$$(\chi^1\gamma^{mnp}\chi^2)(\chi^3\gamma_{mnp}\chi^4) = 24(\chi^1\gamma^m\chi^3)(\chi^2\gamma_m\chi^4) - 12(\chi^1\gamma^m\chi^2)(\chi^3\gamma_m\chi^4),$$

the four-fermion kinematic factor can be conveniently rewritten as

$$K_0^{\rm 4F} = -\frac{1}{2880}\left[(k^1\cdot k^3)(\chi^1\gamma^m\chi^2)(\chi^3\gamma_m\chi^4) + (k^3\cdot k^4)(\chi^1\gamma^m\chi^3)(\chi^2\gamma_m\chi^4)\right]. \tag{3.77}$$

Both terms of (3.76) contribute in the $K_0^{\rm 2B2F} \equiv K_0(f_1f_2b_3b_4)$ kinematic factor,

$$K_0^{\rm 2B2F} = -\frac{1}{36}k_1^m k_2^n F^4_{mn}e^3_p\langle(\lambda\gamma^t\theta)(\lambda\gamma^u\theta)(\lambda\gamma^p\theta)(\theta\gamma_t\chi^1)(\chi^2\gamma_u\theta)\rangle$$

$$-\frac{1}{24}(k^1\cdot k^3)F^4_{mn}e^3_p\langle(\lambda\gamma^t\theta)(\lambda\gamma^p\theta)(\lambda\gamma^q\gamma^{mn}\theta)(\theta\gamma_q\chi^1)(\chi^2\gamma_t\theta)\rangle + (1\leftrightarrow 2)$$

$$= \frac{1}{5760}F^4_{mn}e^3_p\left[k_1^m k_2^n(\chi^1\gamma^p\chi^2) + \frac{1}{2}(k^1\cdot k^3)(\chi^1\gamma^{mn}\gamma^p\chi^2)\right] + (1\leftrightarrow 2) \tag{3.78}$$

It is worth noticing that the explicit computation of $K_0^{\rm 2B2F}$ becomes easier if we use the identity (3.68) with a convenient choice for the labels in the right hand side, namely $K_0 = -\langle(\lambda A^1)(\lambda\gamma^m W^3)(\lambda\gamma^n W^4)\mathcal{F}^2_{mn}\rangle$, because now one can check that only one term contributes

$$K_0^{\rm 2B2F} = \frac{1}{24}\langle(\lambda\gamma^p\theta)(\lambda\gamma^{[m|}\gamma^{rs}\theta)(\lambda\gamma^{|n]}\gamma^{tu}\theta)(\theta\gamma_p\chi^1)(\chi^2\gamma_n\theta)\rangle k^2_m F^3_{rs}F^4_{tu}$$

$$= \frac{1}{5760}F^3_{mn}F^4_{rs}\left[-i(\chi^1\gamma^r\chi^2)\eta^{sm}k_2^n + \frac{i}{2}(\chi^1\gamma^{mnr}\chi^2)k_2^s\right] + (3\leftrightarrow 4). \tag{3.79}$$

[7] I acknowledge the use of GAMMA [28] and specially of FORM [14][15] in these computations.

One can verify that (3.78) and (3.79) are in fact equal and equivalent to the RNS result (see for example [27]). This equality can also be regarded as a check of identity (3.68), which is reassuring.

The computation of $K_0^{4\text{B}}$ is straightforward (and can also be deduced from the one-loop result of [3]). One can in fact check that

$$
\begin{aligned}
K_0^{4\text{B}} = \frac{1}{5760}\Big[& -\frac{1}{2}(e^1\cdot e^3)(e^2\cdot e^4)ts - \frac{1}{2}(e^1\cdot e^4)(e^2\cdot e^3)us - \frac{1}{2}(e^1\cdot e^2)(e^3\cdot e^4)tu \\
& +(k^4\cdot e^1)(k^2\cdot e^3)(e^2\cdot e^4)s + (k^3\cdot e^2)(k^1\cdot e^4)(e^1\cdot e^3)s \\
& +(k^3\cdot e^1)(k^2\cdot e^4)(e^2\cdot e^3)s + (k^4\cdot e^2)(k^1\cdot e^3)(e^1\cdot e^4)s \\
& +(k^1\cdot e^2)(k^3\cdot e^4)(e^1\cdot e^3)t + (k^4\cdot e^3)(k^2\cdot e^1)(e^2\cdot e^4)t \\
& +(k^4\cdot e^2)(k^3\cdot e^1)(e^3\cdot e^4)t + (k^1\cdot e^3)(k^2\cdot e^4)(e^1\cdot e^2)t \\
& +(k^2\cdot e^1)(k^3\cdot e^4)(e^2\cdot e^3)u + (k^4\cdot e^3)(k^1\cdot e^2)(e^1\cdot e^4)u \\
& +(k^4\cdot e^1)(k^3\cdot e^2)(e^3\cdot e^4)u + (k^2\cdot e^3)(k^1\cdot e^4)(e^1\cdot e^2)u \Big] \\
= & \frac{1}{2880} t_8^{m_1n_1m_2n_2m_3n_3m_4n_4} F^1_{m_1n_1} F^2_{m_2n_2} F^3_{m_3n_3} F^4_{m_4n_4}, \qquad (3.80)
\end{aligned}
$$

where we used the t_8 tensor definition of [33][34].

3.7 Anomaly kinematic factor

Type-I superstring theory is defined in ten dimensional space and contains gluinos of only one chirality. So it could be plagued by an anomaly which would reveal itself as the failure of the massless six-point amplitude to be gauge invariant, therefore making the theory inconsistent at the quantum level. One could even expect type-I theory to be anomalous, considering the fact that $N = 1$ super-Yang-Mills in $D = 10$ definitely has an anomaly and it is recovered as the low-energy limit of the type-I superstring. But Green and Schwarz showed in a seminal 1984 paper that superstring theory is anomaly-free [21], setting free the spark which lighted the wondrous fire of the so-called First Superstring Revolution.

One way to understand the absence of the anomaly it to note that to compute the massless six-point amplitude in type-I superstring theory one has to sum over three different manifolds, because at genus one there can be a planar and non-planar cilynders and finally a Möbius strip. So there is a chance of getting a vanishing gauge transformation if the sum of the non-vanishing partial amplitudes cancel out. And that is exactly what happens. So it is the extended nature of the string – its propagation through space-time forms a Riemann surface – which allows the cancelation of the gauge variation. And now one can understand why SYM is anomalous; being the low-energy limit of type-I superstring theory it looses the information about the size of the string to become a point, and the three different worldsheet configurations become only one Feynman graph along the way, making a cancelation impossible.

One can compute the superstring scattering for each worldsheet configuration individually and the kinematic factor turns out to be the same for all of them and is given by

$$K = \epsilon^{m_1 n_1 \ldots m_5 n_5} k^1_{m_1} e^1_{n_1} \ldots k^5_{m_5} e^5_{n_5}. \tag{3.81}$$

In the following sections it will be shown that the non-minimal pure spinor formalism computation of the hexagon gauge anomaly in the Type-I superstring is equivalent to the RNS result of [21]. As will be proved below, the kinematic factor of the hexagon gauge variation can be written as the pure spinor superspace integral

$$K = \left\langle (\lambda\gamma^m W^2)(\lambda\gamma^n W^3)(\lambda\gamma^p W^4)(W^5\gamma_{mnp}W^6) \right\rangle,$$

whose bosonic part will be computed to demonstrate that it is the well-known $\epsilon_{10}F^5$ RNS result of (3.81).

As discussed in [22][23], the anomaly can be computed as a surface term which contributes at the boundary of moduli space. The result can be separated in two parts: the kinematic factor depending only on momenta and polarizations, and the moduli space part which depends on the worldsheet surface. We will treat them separatedly, first we will compute the kinematic factor and afterwards we will say a few words about the moduli space part.

3.7.1 Kinematic factor computation

In the type-I superstring theory with gauge group SO(N), the massless open string six-point one-loop amplitude is given by

$$\mathcal{A} = \sum_{top=P,NP,N} G_{top} \int_0^\infty dt \langle \mathcal{N} \int dw b(w)(\lambda A_1) \prod_{r=2}^{6} \int dz_r U_r(z_r) \rangle \tag{3.82}$$

where P, NP, N denotes the three possible different world-sheet topologies, each of which has a different group-factor G_{top} [34]. When all particles are attached to one boundary, we have a cylinder with $G_P = N\text{tr}(t^{a_1}t^{a_2}t^{a_3}t^{a_4}t^{a_5}t^{a_6})$. When particles are attached to both boundaries, the diagram is a non-planar cylinder, where $G_{NP} = \text{tr}(t^{a_1}t^{a_2})\text{tr}(t^{a_3}t^{a_4}t^{a_5}t^{a_6})$. And finally, there is the non-orientable Möbius strip where $G_N = -\text{tr}(t^{a_1}t^{a_2}t^{a_3}t^{a_4}t^{a_5}t^{a_6})$.

We will be interested in the amplitude when all external states are massless gluons with polarization e^r_m *i.e.*, $a^r_m(x) = e^r_m e^{ik\cdot x}$, where $m = 0, \ldots 9$ is the space-time vector index and r is the particle label [8]. To probe the anomaly, one can compute (3.82) and substitute one of the external polarizations for its respective momentum. However, instead of first computing the six-point amplitude and substituting $e_m \to k_m$ in the answer, we will first make the gauge transformation in (3.82) and then compute the resulting correlation function. This will give us the anomaly kinematic factor directly.

Under the super-Yang-Mills gauge transformation (B.7)

$$\delta A_\alpha = D_\alpha \Omega, \quad \delta A_m = \partial_m \Omega, \tag{3.83}$$

the integrated vertex operator $\int dz U$ changes by the surface term $\int dz \delta U = \int dz \partial \Omega$, and the unintegrated vertex operator changes by the BRST-trivial quantity $\delta(\lambda A) = \lambda^\alpha D_\alpha \Omega = Q\Omega$. Choosing $\Omega(x, \theta) = e^{ik\cdot x}$ has the same effect as changing $e^m \to k^m$, which is the desired gauge transformation to probe the anomaly.

To compute the gauge anomaly, it will be convenient to choose the gauge transformation to act on the polarization e^1_m in the unintegrated vertex operator, so that the gauge variation of (3.82) is

$$\delta \mathcal{A} = \sum_{top=P,N,NP} G_{top} \int_0^\infty dt \langle \mathcal{N} \int dw b(w) (Q\Omega(z_1)) \prod_{r=2}^{6} \int dz_r U_r(z_r) \rangle. \tag{3.84}$$

Integrating Q by parts inside the correlation function will only get a contribution from the BRST variation of the b-ghost, which is a derivative with respect to the modulus [25][23]. This is due to the fact that $\{Q, b\} = T$ together with the definition of the Beltrami differential as parametrizing the violation of the conformal gauge under a variation of the metric tensor

$$\delta g = \delta g_{z\overline{z}} dz d\overline{z} + \delta g_{zz} dz dz$$

[8] We will omit the adjoint gauge group index from the polarizations and field-strengths for the rest of this section.

where if τ is the modulus parameter then

$$\delta g_{zz} = \mu_z^{\ \overline{z}} g_{z\overline{z}} \delta\tau.$$

In this way, having the insertion of $\int d^2z T_{z\overline{z}} \mu_z^{\ \overline{z}}$ in a correlation function is equivalent to derive it with respect to the modulus because

$$\frac{\delta S}{\delta \tau} = \frac{\delta S}{\delta g^{zz}} \frac{\delta g^{zz}}{\delta \tau} = \int d^2 z T \cdot \mu.$$

So (3.84) becomes

$$\delta\mathcal{A} = -\sum_{top} G_{top} \int_0^\infty dt \frac{d}{dt} \langle \Omega(z_1) \mathcal{N} \prod_{r=2}^{6} \int dz_r U_r(z_r) \rangle \tag{3.85}$$

$$\equiv -K \sum_{top} G_{top} \Big[B_{top}(\infty) - B_{top}(0) \Big], \tag{3.86}$$

where the moduli space part of the anomaly is encoded in the function

$$B_{top}(t) \equiv \int_0^t dz_6 \int_0^{z_6} dz_5 \int_0^{z_5} dz_4 \int_0^{z_4} dz_3 \int_0^{z_3} dz_2 \, \langle \prod_{r=1}^{6} : e^{ik_r \cdot x_r} : \rangle_{top},$$

and $K = \langle \mathcal{N} U_2 U_3 U_4 U_5 U_6 \rangle$. From (3.85), it is clear that the anomaly comes from the boundary of moduli space.

To compute the kinematic factor K, observe that there is an unique way to absorb the 16 zero modes of d_α, 11 of s^α and 11 of r_α. The regularization factor $\mathcal{N}$ must provide 11 d_α, 11 s^α and 11 r_α zero modes. The five remaining d_α zero modes must come from the external vertices[9] through $(d\mathcal{W})^5$. As in the computations of the previous section, the kinematic factor is thus given by a pure spinor superspace integral involving 3 λ's and 5 W's, as can be easily verified by integrating all the zero mode measures except $[d\lambda], [d\overline{\lambda}]$ and $[dr]$. To find out how the indices are contracted in K, choose the reference frame where only $\lambda^+ \neq 0$. Then one can easily check that the unique $U(1) \times SU(5)$-invariant contraction is

$$K = \left\langle (\lambda^+)^3 \epsilon_{abcde} W_2^a W_3^b W_4^c W_5^d W_6^e \right\rangle,$$

which in SO(10)-covariant notation translates into

$$K = \langle (\lambda\gamma^m W_2)(\lambda\gamma^n W_3)(\lambda\gamma^p W_4)(W_5 \gamma_{mnp} W_6) \rangle. \tag{3.87}$$

[9] It follows from this zero mode counting that the anomaly trivially vanishes for amplitudes with less than six external massless particles.

Now one needs to check if the pure spinor expression (3.87) reproduces the known $\epsilon_{10}F^5$ contraction computed with the RNS formalism [21].

When all external states are gluons, there is only one possibility to obtain the required five θ's from the pure spinor measure $\langle\lambda^3\theta^5\rangle$. Using the superfield expansions from appendix B we see that each superfield $W^\alpha(\theta)$ must contribute one θ through the term $-\frac{1}{4}(\gamma^{mn}\theta)^\alpha F_{mn}$. Therefore the kinematic factor (3.87) is equal to

$$= -\frac{1}{1024}\,\langle(\lambda\gamma^p\gamma^{m_2n_2}\theta)(\lambda\gamma^q\gamma^{m_3n_3}\theta)(\lambda\gamma^r\gamma^{m_4n_4}\theta)(\theta\gamma^{m_5n_5}\gamma_{pqr}\gamma^{m_6n_6}\theta)\rangle\, F^2_{m_2n_2}\ldots F^6_{m_6n_6}. \tag{3.88}$$

We will now demonstrate the equivalence with the RNS anomaly result of [21] by proving that

$$\langle(\lambda\gamma^p\gamma^{m_1n_1}\theta)(\lambda\gamma^q\gamma^{m_2n_2}\theta)(\lambda\gamma^r\gamma^{m_3n_3}\theta)(\theta\gamma^{m_4n_4}\gamma_{pqr}\gamma^{m_5n_5}\theta)\rangle = \frac{1}{45}\epsilon^{m_1n_1\ldots m_5n_5}. \tag{3.89}$$

We will first show that the correlation in (3.89) is proportional to ϵ_{10} by checking its behavior under a parity transformation. Using the language of [7], we can rewrite (3.89) as

$$(T^{-1})^{(\alpha\beta\gamma)[\rho_1\rho_2\rho_3\rho_4\rho_5]}T_{(\alpha\beta\gamma)[\delta_1\delta_2\delta_3\delta_4\delta_5]}(\gamma^{m_1n_1})^{\delta_1}_{\ \rho_1}(\gamma^{m_2n_2})^{\delta_2}_{\ \rho_2}(\gamma^{m_3n_3})^{\delta_3}_{\ \rho_3}(\gamma^{m_4n_4})^{\delta_4}_{\ \rho_4}(\gamma^{m_5n_5})^{\delta_5}_{\ \rho_5}, \tag{3.90}$$

where T and T^{-1} are defined by

$$(T^{-1})^{(\alpha_1\alpha_2\alpha_3)[\delta_1\delta_2\delta_3\delta_4\delta_5]} = (\gamma^m)^{\alpha_1\delta_1}(\gamma^n)^{\alpha_2\delta_2}(\gamma^p)^{\alpha_3\delta_3}(\gamma_{mnp})^{\delta_4\delta_5} \tag{3.91}$$

$$T_{(\alpha_1\alpha_2\alpha_3)[\delta_1\delta_2\delta_3\delta_4\delta_5]} = \gamma^m_{\alpha_1\delta_1}\gamma^n_{\alpha_2\delta_2}\gamma^p_{\alpha_3\delta_3}(\gamma_{mnp})_{\delta_4\delta_5},$$

and the α-indices are symmetric and gamma matrix traceless, and the δ-indices are antisymmetric. Since a parity transformation has the effect of changing a Weyl spinor ψ^α to an anti-Weyl spinor ψ_α, it follows from the definitions of (3.91) that a parity transformation exchanges $T \leftrightarrow T^{-1}$. Furthermore, since a parity transformation also changes

$$(\gamma^{mn})^{\delta}_{\ \rho} \to (\gamma^{mn})_{\delta}^{\ \rho} = -(\gamma^{mn})^{\rho}_{\ \delta},$$

it readily follows that the kinematic factor (3.90) is odd under parity, so it is proportional to ϵ_{10}. Finally, the proportionality constant of $\frac{1}{45}$ in (3.89) can be explicitly computed using the identities listed in Appendix B.

3.7.2 The evaluation of the moduli space part

In this section we compute the limits of $B_{top}(t)$ when the topology of the open string world-sheet is a cylinder, where $B_{top}(t)$ is given by

$$B_{top}(t) \equiv \int_0^\theta dw_5 \int_0^{w_5} dw_4 \int_0^{w_4} dw_3 \int_0^{w_3} dw_2 \int_0^{w_2} dw_1 \, \langle \prod_{r=1}^{6} :e^{ik_r \cdot X_r}: \rangle_{top}, \tag{3.92}$$

and $\theta = it$. Note that to obtain (3.92) we did not need to introduce any regularization factor such as the Pauli-Villars used in [34][10]. Furthermore, in the RNS computations of [34] it is unclear that the anomaly is an effect from the boundary of moduli space, which was first pointed out in [23]. This was later verified to be the case in [22] and [24]. Using the non-minimal pure spinor formalism it is evident from (3.86) that the anomaly comes from the boundary of moduli space, and this result is obtained without the need of a special regularization scheme. Although reference [24] uses a regularization scheme, the evaluation of the moduli space limits of (3.92) require almost identical manipulations as the ones described there, so this subsection is heavily based on it. The final result is obviously the same, concluding the proof that the non-minimal pure spinor formalism analysis of the gauge anomaly in type-I superstring is equivalent to the RNS.

When the topology is a cylinder (P) we have [24]

$$\langle \prod_{r=1}^{6} :e^{ik_r \cdot X_r}: \rangle_P \sim \frac{1}{t^5} \prod_{i<j} \Big[\psi_C(w_{ij}|\tau) \Big]^{k_i \cdot k_j},$$

where

$$\psi_C(w_{ij}|\tau) = -ie^{(i\pi w_{ij}^2/\tau)} \frac{\vartheta_1(w_{ij}|\tau)}{\eta^3(\tau)},$$

and

$$\vartheta_1(\nu|\tau) = 2q^{1/8} \sin(\pi\nu) \prod_{n=1}^{\infty} (1-q^n)(1-q^n e^{2\pi i\nu})(1-q^n e^{-2\pi i\nu}),$$

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n),$$

with $q = \exp(2\pi i\tau)$. Note that

$$\lim_{t\to\infty} \psi_C(w_{ij}|\tau) = 2\sin(\pi w_{ij}), \tag{3.93}$$

[10] Note that the superfluous nature of a regularization scheme was first noted in [20]. There it was explicitly shown that the massless six-point amplitude in type-I superstring is *finite* for arbitrary gauge group but anomaly cancellation requires SO(32).

implies the vanishing of $\lim_{t\to\infty} B(t)$. To obtain the limit when $t \to 0$ we make the following change of variables $\tilde{w} = \frac{w}{\tau}$ and $\tilde{\tau} \equiv i\tilde{t} = -\frac{1}{\tau}$ to get

$$B_P(t) \equiv \int_0^1 d\tilde{w}_5 \int_0^{\tilde{w}_5} d\tilde{w}_4 \int_0^{\tilde{w}_4} d\tilde{w}_3 \int_0^{\tilde{w}_3} d\tilde{w}_2 \int_0^{\tilde{w}_2} d\tilde{w}_1 \prod_{i\leq j} \Big[\psi_C(\tilde{w}_{ij}|\tilde{\tau})\Big]^{k_i\cdot k_j},$$

where

$$\begin{aligned}\psi_C(\tilde{w}_{ij}|\tilde{\tau}) &= -i\exp\left(-i\pi\tilde{w}_{ij}/\tilde{\tau}\right)\frac{\vartheta_1(-\frac{\tilde{w}_{ij}}{\tilde{\tau}}|-\frac{1}{\tilde{\tau}})}{\eta^3(-\frac{1}{\tilde{\tau}})} \\ &= \frac{i}{\tilde{\tau}}\frac{\vartheta_1(\tilde{w}_{ij}|\tilde{\tau})}{\eta^3(\tilde{\tau})}.\end{aligned} \tag{3.94}$$

To obtain (3.94) we used the well-known modular transformation properties,

$$\vartheta_1(-\frac{\nu}{\tau}|-\frac{1}{\tau}) = -i\sqrt{-i\tau}\exp\left(i\pi\nu^2/\tau\right)\vartheta_1(-\nu|\tau)$$

$$\eta(-\frac{1}{\tau}) = \sqrt{-i\tau}\eta(\tau),$$

and $\vartheta_1(-\nu|\tau) = -\vartheta_1(\nu|\tau)$. Note that the factor of $i/\tilde{\tau}$ in (3.94) will not contribute because of momentum conservation and masslessness of the external particles,

$$\prod_{i<j}\left(\frac{i}{\tilde{\tau}}\right)^{k_i\cdot k_j} = \left(\frac{i}{\tilde{\tau}}\right)^{\sum_{i<j}k_i\cdot k_j} = 1.$$

Noting that the limit is now $\tilde{t} \to \infty$, we can use the previous result (3.93) to finally obtain

$$\lim_{t\to 0} B_P(t) = \int_0^1 d\tilde{w}_5 \int_0^{\tilde{w}_5} d\tilde{w}_4 \int_0^{\tilde{w}_4} d\tilde{w}_3 \int_0^{\tilde{w}_3} d\tilde{w}_2 \int_0^{\tilde{w}_2} d\tilde{w}_1 \prod_{i<j}\Big[2\sin(\pi w_{ij})\Big]^{k_i\cdot k_j}. \tag{3.95}$$

The final expression for the gauge variation of the amplitude in the planar cylinder is finally

$$\delta\mathcal{A} = KG_P \int_0^1 d\tilde{w}_5 \int_0^{\tilde{w}_5} d\tilde{w}_4 \int_0^{\tilde{w}_4} d\tilde{w}_3 \int_0^{\tilde{w}_3} d\tilde{w}_2 \int_0^{\tilde{w}_2} d\tilde{w}_1 \prod_{i<j}\Big[\sin(\pi\tilde{w}_{ij})\Big]^{k_i\cdot k_j}, \tag{3.96}$$

where K is given by (1.13). When the world-sheet is a Möbius strip the kinematical factor is exactly the same. The limits of (3.92) in the boundary of moduli space are however different. We will not repeat the computation here and merely quote the result of [34][24] that $\lim_{t\to 0} B_N(t) = 32\lim_{t\to 0} B_P(t)$. From this it follows that the sum of the anomalies for the planar and non-orientable diagrams vanish if $N = 32$, i.e., if the gauge group is SO(32). For the non-planar cylinder one can show that (3.92) vanishes [34].

3.7.3 The fermionic expansion

From the explicit bosonic computation of the anomaly kinematic factor of section 3.7.1 we know that it is a total derivative, because

$$\epsilon_{10}^{mnpqrstuvx} F_{mn}F_{pq}F_{rs}F_{tu}F_{vx} = 2\partial_m\big[\epsilon_{10}^{mnpqrstuvx} A_n F_{pq}F_{rs}F_{tu}F_{vx}\big].$$

What about the fermionic components? Based on supersymmetry we expect them to be total derivatives also, and in this section we explicitly compute the $2F3B$ components of (3.87) to show that this is indeed true.

If we choose the particles labeled by "0" and "1" to be the fermions and the others to be bosons we obtain for

$$K = \langle(\lambda\gamma^m W_2)(\lambda\gamma^n W_3)(\lambda\gamma^p W_4)(W_0\gamma_{mnp}W_1)\rangle$$

the following distribution of thetas

$W^2(\theta)$	$W^3(\theta)$	$W^4(\theta)$	$W^0(\theta)$	$W^1(\theta)$
3	1	1	0	0
1	1	3	0	0
1	3	1	0	0
1	1	1	2	0
1	1	1	0	2

Using the component expansion of Appendix B and the identity

$$\psi^\alpha\chi^\beta = \frac{1}{16}\gamma_m^{\alpha\beta}(\psi\gamma^m\chi) + \frac{1}{96}\gamma_{mnp}^{\alpha\beta}(\psi\gamma^{mnp}\chi) + \frac{1}{3840}\gamma_{mnpqr}^{\alpha\beta}(\psi\gamma^{mnpqr}\chi)$$

the above table translates to

$$
\begin{aligned}
256K = \\
&+\Big[\frac{1}{3}\langle(\lambda\gamma^m\gamma^{rt}\theta)(\lambda\gamma^n\gamma^{m_3n_3}\theta)(\lambda\gamma^p\gamma^{m_4n_4}\theta)(\theta\gamma^{tm_2n_2}\theta)\rangle\ (\chi^0\gamma_{mnp}\chi^1)k^2_r\\
&+\frac{1}{3}\langle(\lambda\gamma^m\gamma^{rt}\theta)(\lambda\gamma^n\gamma^{m_2n_2}\theta)(\lambda\gamma^p\gamma^{m_4n_4}\theta)(\theta\gamma^{tm_3n_3}\theta)\rangle\ (\chi^0\gamma_{mnp}\chi^1)k^3_r\\
&+\frac{1}{3}\langle(\lambda\gamma^m\gamma^{rt}\theta)(\lambda\gamma^n\gamma^{m_2n_2}\theta)(\lambda\gamma^p\gamma^{m_3n_3}\theta)(\theta\gamma^{tm_4n_4}\theta)\rangle\ (\chi^0\gamma_{mnp}\chi^1)k^4_r\\
&-\frac{1}{16}\langle(\lambda\gamma^m\gamma^{m_2n_2}\theta)(\lambda\gamma^n\gamma^{m_3n_3}\theta)(\lambda\gamma^p\gamma^{m_4n_4}\theta)(\theta\gamma^{ka}\gamma_{mnp}\gamma^r\gamma_a\theta)\rangle\ k^0_k(\chi^0\gamma_r\chi^1)\\
&+\frac{1}{16}\langle(\lambda\gamma^m\gamma^{m_2n_2}\theta)(\lambda\gamma^n\gamma^{m_3n_3}\theta)(\lambda\gamma^p\gamma^{m_4n_4}\theta)(\theta\gamma^{ka}\gamma_{mnp}\gamma^r\gamma_a\theta)\rangle\ k^1_k(\chi^0\gamma_r\chi^1)
\end{aligned}
$$

$$+\frac{1}{96}\langle(\lambda\gamma^m\gamma^{m_2n_2}\theta)(\lambda\gamma^n\gamma^{m_3n_3}\theta)(\lambda\gamma^p\gamma^{m_4n_4}\theta)(\theta\gamma^{ka}\gamma_{mnp}\gamma^{rst}\gamma_a\theta)\rangle\ k^0_k(\chi^0\gamma_{rst}\chi^1)$$
$$+\frac{1}{96}\langle(\lambda\gamma^m\gamma^{m_2n_2}\theta)(\lambda\gamma^n\gamma^{m_3n_3}\theta)(\lambda\gamma^p\gamma^{m_4n_4}\theta)(\theta\gamma^{ka}\gamma_{mnp}\gamma^{rst}\gamma_a\theta)\rangle\ k^1_k(\chi^0\gamma_{rst}\chi^1)$$
$$-\frac{1}{3840}\langle(\lambda\gamma^m\gamma^{m_2n_2}\theta)(\lambda\gamma^n\gamma^{m_3n_3}\theta)(\lambda\gamma^p\gamma^{m_4n_4}\theta)(\theta\gamma^{ka}\gamma_{mnp}\gamma^{rstuv}\gamma_a\theta)\rangle\ k^0_k(\chi^0\gamma_{rstuv}\chi^1)$$
$$+\frac{1}{3840}\langle(\lambda\gamma^m\gamma^{m_2n_2}\theta)(\lambda\gamma^n\gamma^{m_3n_3}\theta)(\lambda\gamma^p\gamma^{m_4n_4}\theta)(\theta\gamma^{ka}\gamma_{mnp}\gamma^{rstuv}\gamma_a\theta)\rangle\ k^1_k(\chi^0\gamma_{rstuv}\chi^1)\Big]$$
$$\times F^2_{m_2n_2}F^3_{m_3n_3}F^4_{m_4n_4}. \qquad (3.97)$$

It will be instructive to compute the correlator by parts, separating them according to the factors of $(\chi^0\gamma_r\chi^1)$, $(\chi^0\gamma_{rst}\chi^1)$ or $(\chi^0\gamma_{rstuv}\chi^1)$.

The terms containing $(\chi^0\gamma_r\chi^1)$ are given by $K_r =$

$$-\frac{1}{16}(\chi^0\gamma_r\chi^1)(k^0_k-k^1_k)F^2F^3F^4\langle(\lambda\gamma^m\gamma^{m_2n_2}\theta)(\lambda\gamma^n\gamma^{m_3n_3}\theta)(\lambda\gamma^p\gamma^{m_4n_4}\theta)(\theta\gamma^{ka}\gamma_{mnp}\gamma^r\gamma_a\theta)\rangle$$

which can be rewritten using the tensor (3.101) computed in section 3.8,

$$K_r = +\frac{1}{16}(\chi^0\gamma_r\chi^1)(k^0_k-k^1_k)t_{10}^{arm_2n_2m_3n_3m_4n_4kb}\,\delta^a_b$$
$$= -\frac{1}{360}t_8^{krm_2n_2m_3n_3m_4n_4}(k^0_k-k^1_k)(\chi^0\gamma_r\chi^1)F^2_{m_2n_2}F^3_{m_3n_3}F^4_{m_4n_4}. \qquad (3.98)$$

To compare (3.98) with the terms proportional to $(\chi^0\gamma_{rst}\chi^1)$ in (3.97) it is convenient to use Dirac's equation such that

$$k^0_k(\chi^0\gamma^{kmn}\chi^1) = -2k^0_{[m}(\chi^0\gamma_{n]}\chi^1), \quad k^1_k(\chi^0\gamma^{kmn}\chi^1) = +2k^1_{[m}(\chi^0\gamma_{n]}\chi^1)$$

which allows one to rewrite (3.98) as follows

$$256K_r = +\frac{1}{720}t_8^{mnm_2n_2m_3n_3m_4n_4}(k^0_k+k^1_k)(\chi^0\gamma_{kmn}\chi^1)F^2_{m_2n_2}F^3_{m_3n_3}F^4_{m_4n_4}.$$

The terms in (3.97) proportional to $(\chi^0\gamma_{rst}\chi^1)$ can be computed similarly and we obtain

$$256K_{rst} = +\frac{1}{720}t_8^{mnm_2n_2m_3n_3m_4n_4}(k^2_k+k^3_k++k^4_k)(\chi^0\gamma_{kmn}\chi^1)F^2_{m_2n_2}F^3_{m_3n_3}F^4_{m_4n_4},$$

and therefore

$$256(K_r+K_{rst}) = +\frac{1}{720}t_8^{mnm_2n_2m_3n_3m_4n_4}\sum_{i=0}^{4}k^i_k(\chi^0\gamma_{kmn}\chi^1)F^2_{m_2n_2}F^3_{m_3n_3}F^4_{m_4n_4}. \qquad (3.99)$$

Another long computation for the terms proportional to $(\chi^0\gamma_{rstuv}\chi^1)$ results in the following

$$256K_{rstuv} = -\frac{53}{25200}\sum_{i=0}^{4}k^i_k(\chi^0\gamma^{km_2n_2m_3n_3m_4n_4}\chi^1)F^2_{m_2n_2}F^3_{m_3n_3}F^4_{m_4n_4},$$

therefore we have show that the terms containing two fermions and three bosons in (3.97) combine into a total derivative.

3.8 t_8 and ϵ_{10} from pure spinor superspace

In this section we digress about an interesting identity which involves both the t_8 and ϵ_{10} tensors, showing how closely related they are when obtained from pure spinor superspace integrals. This is different from computations in the RNS formalism where t_8 and ϵ_{10} come from correlation functions with different spin structures.

Since the one-loop $t_8 F^4$ and $\epsilon_{10} B F_4$ terms are expected to be related by non-linear supersymmetry, there might be a common superspace origin for the t_8 and ϵ_{10} tensors. This suggests looking for a BRST-closed pure spinor superspace integral involving four super-Yang-Mills superfields whose bosonic part involves both the t_8 and ϵ_{10} tensors. One such BRST-closed expression is

$$\langle(\lambda\gamma^r W^1)(\lambda\gamma^s W^2)(\lambda\gamma^t W^3)(\theta\gamma^m\gamma^n\gamma_{rst}W^4)\rangle. \tag{3.100}$$

Although (3.100) is not spacetime supersymmetric because of the explicit θ, it might be related to a supersymmetric expression in a constant background where the $N=1$ supergravity superfield $G_{m\alpha}$ satisfies $G_{m\alpha} = \gamma_{m\alpha\beta}\theta^\beta + b_{mn}(\gamma^n\theta)_\alpha$ for constant b_{mn}.

When restricted to its purely bosonic part, (3.100) defines the following 10-dimensional tensor:

$$t_{10}^{mnm_1n_1m_2n_2m_3n_3m_4n_4} = \left\langle(\lambda\gamma^a\gamma^{m_1n_1}\theta)(\lambda\gamma^b\gamma^{m_2n_2}\theta)(\lambda\gamma^c\gamma^{m_3n_3}\theta)(\theta\gamma^m\gamma^n\gamma_{abc}\gamma^{m_4n_4}\theta)\right\rangle. \tag{3.101}$$

Using $\gamma^m\gamma^n = \gamma^{mn} + \eta^{mn}$ we obtain

$$\begin{aligned} t_{10}^{mnm_1n_1m_2n_2m_3n_3m_4n_4} &= +\langle(\lambda\gamma^a\gamma^{m_1n_1}\theta)(\lambda\gamma^b\gamma^{m_2n_2}\theta)(\lambda\gamma^c\gamma^{m_3n_3}\theta)(\theta\gamma^{mn}\gamma_{abc}\gamma^{m_4n_4}\theta)\rangle \\ &\quad +\eta^{mn}\langle(\lambda\gamma^a\gamma^{m_1n_1}\theta)(\lambda\gamma^b\gamma^{m_2n_2}\theta)(\lambda\gamma^c\gamma^{m_3n_3}\theta)(\theta\gamma_{abc}\gamma^{m_4n_4}\theta)\rangle. \end{aligned} \tag{3.102}$$

And using the identities listed in appendix B, one can check that[11]

$$t_{10}^{mnm_1n_1m_2n_2m_3n_3m_4n_4} = -\frac{2}{45}\Big[\eta^{mn}t_8^{m_1n_1m_2n_2m_3n_3m_4n_4} - \frac{1}{2}\epsilon^{mnm_1n_1m_2n_2m_3n_3m_4n_4}\Big]. \tag{3.103}$$

It is also interesting to contrast the similarity between ϵ_{10} and t_8 when written in terms of the T and T^{-1} tensors:

$$\epsilon^{mnm_1n_1\ldots m_4n_4} \propto (T^{-1})^{(\alpha\beta\gamma)[\rho_0\rho_1\rho_2\rho_3\rho_4]}T_{(\alpha\beta\gamma)[\delta_0\delta_1\delta_2\delta_3\delta_4]}(\gamma^{mn})^{\delta_0}{}_{\rho_0}\ldots(\gamma^{m_4n_4})^{\delta_4}{}_{\rho_4}$$

[11] The sign in front of ϵ_{10} depends on the chirality of θ. For an anti-Weyl θ_α, the sign is "+".

$$t_8^{m_1n_1\dots m_4n_4} \propto (T^{-1})^{(\alpha\beta\gamma)[\kappa\rho_1\rho_2\rho_3\rho_4]}T_{(\alpha\beta\gamma)[\kappa\delta_1\delta_2\delta_3\delta_4]}(\gamma^{m_1n_1})^{\delta_1}_{\rho_1}\dots(\gamma^{m_4n_4})^{\delta_4}_{\rho_4},$$

which shows, in a pure spinor superspace language, how one can "obtain" the t_8 tensor from ϵ_{10}: it is a matter of removing $(\gamma^{mn})^{\delta_0}_{\rho_0}$ and contracting the associated spinorial indices in T and T^{-1}. So when using pure spinors, there is a close relation between these two different-looking tensors.

3.9 Massless open string five-point amplitude at one-loop

In this section we will describe some of the ongoing work with Christian Stahn to obtain the kinematic factor for the massless five-point amplitude for open strings.

3.9.1 Modified NMPS computation

First of all I will show how one can get the correct answers for the bosonic components using the following substitution rule

$$\int d^2zb\cdot\mu V^1 \longrightarrow (d^0\theta)\int dzU^1(z) \tag{3.104}$$

where d^0_α means only the zero mode part. Using (3.104) the amplitude prescription of

$$\mathcal{A} = \int dt\langle\mathcal{N}(\int b\cdot\mu)V^1U^2U^3U^4U^5\rangle \tag{3.105}$$

becomes

$$\mathcal{A} = \int dt\langle\mathcal{N}(d\theta)U^1U^2U^3U^4U^5\rangle \tag{3.106}$$

where the regularization factor is given by (2.92)

$$\mathcal{N} = \exp\big[-(\lambda\overline{\lambda})-(w\overline{w})-(r\theta)+(sd)\big].$$

To saturate the 16 zero-modes of d_α and 11 of s^α which are present at the one-loop level there is only one possibility using the modified prescription of (3.106). The regularization factor $\mathcal{N}$ provides 11 zero-modes of s^α and d_α. Note that there is already a zero mode in the factor $(d^0\theta)$ so the remaining four d_α zero modes can come from the five external vertices through two different ways, depending on which term of the integrated vertex is chosen to act by its OPE over the others. The kinematic factor then becomes

$$K = \langle(d\theta)(dW^1)(dW^2)(dW^3)(dW^4)\Big(A^5_m\Pi^m + \hat{d}_\alpha W^\alpha_5\Big)\rangle + \text{perm}(2345) \tag{3.107}$$

$$= K^a + K^b,$$

where, following the notation of [27], we defined

$$K^a = \sum_{i=1}^{4} G^\theta(z_5, z_2)\langle(\lambda\gamma^m\theta)(\lambda\gamma^n W^1)(\lambda\gamma^p\gamma^{rs}W^5)(W^3\gamma_{mnp}W^4)F^2_{rs}\rangle + \text{perm}(2345) \quad (3.108)$$

$$K^b = \sum_{i=1}^{4} G^X(z_5, z_i)k^r_i\langle A^5_r(\lambda\gamma^m\theta)(\lambda\gamma^n W^1)(\lambda\gamma^p W^2)(W^3\gamma_{mnp}W^4)\rangle + \text{perm}(2345) \quad (3.109)$$

To go from (3.107) to arrive at (3.108) and (3.109) we used the following substitution

$$d_{\delta_1}d_{\delta_2}d_{\delta_3}d_{\delta_4}d_{\delta_5} \to (\lambda\gamma^m)_{[\delta_1}(\lambda\gamma^n)_{\delta_2}(\lambda\gamma^p)_{\delta_3}(\gamma_{mnp})_{\delta_4\delta_5]}, \quad (3.110)$$

which can be justified by a ghost number conservation argument using the integrations over the measures $[dw], [d\overline{w}]$ and $[ds]$, as explained in section 2.8.2.

To evaluate the bosonic components of (3.109) one notices that the superfield A^5_r doesn't contribute any θ's, so it can be moved out of the pure spinor brackets $\langle\ \rangle$. By making use of identity (3.52) we get

$$K^b = 8\sum_{i=1}^{4} G^X(z_5, z_i)(k^i \cdot e^5)\langle(\lambda A^1)(\lambda\gamma^m W^2)(\lambda\gamma^n W^3)\mathcal{F}^4_{mn}\rangle + \text{perm}(1234),$$

which is equivalent to the result computed in the RNS and GS formalisms [27].

In the kinematic factor K^b we will restrict our attention to the term proportional to $G^\theta(z_5, z_2)$, which is antisymmetric in its arguments. Consequently,

$$K^a = G^\theta(z_5, z_2)\Big[\langle(\lambda\gamma^m\theta)(\lambda\gamma^n W^1)(\lambda\gamma^p\gamma^{rs}W^5)(W^3\gamma_{mnp}W^4)F^2_{rs}\rangle - (2 \leftrightarrow 5)\Big].$$

After checking that the following is true for the bosonic components

$$\langle k^1_t e^1_u(\lambda\gamma^m W^3)(\lambda\gamma^n W^4)(\lambda\gamma^p\gamma^{rs}W^5)(\theta\gamma^t\gamma_{mnp}\gamma^u\theta)\mathcal{F}^2_{rs}\rangle - (2 \leftrightarrow 5) = 0,$$

one can repeat the steps in the proof of section 3.3.5 to arrive at the following

$$K^a = -G^\theta(z_5, z_2)\Big[\langle(D\gamma_{mnp}A^1)(\lambda\gamma^m W^3)(\lambda\gamma^n W^4)(\lambda\gamma^p\gamma^{rs}W^5)F^2_{rs}\rangle - (2 \leftrightarrow 5)\Big]. \quad (3.111)$$

The computation of the bosonic components of (3.111) is straightforward,

$$K^a = -\frac{1}{256}F^1_{m_1n_1}\ldots F^5_{m_5n_5}\Big[$$

$$\langle(\lambda\gamma^m\theta)(\lambda\gamma^n\gamma^{m_1n_1}\theta)(\lambda\gamma^p\gamma^{m_2n_2}\gamma^{m_5n_5}\theta)(\theta\gamma^{m_3n_3}\gamma_{mnp}\gamma^{m_4n_4}\theta)\rangle - (2\leftrightarrow 5)\Big]$$

$$= \frac{1}{360}\Big[+ 2F^1_{mn}F^2_{mp}F^3_{nq}F^4_{qr}F^5_{pr} + 2F^1_{mn}F^2_{mp}F^3_{qr}F^4_{nq}F^5_{pr} - F^1_{mn}F^2_{mp}F^3_{qr}F^4_{qr}F^5_{np}$$
$$+ F^1_{mn}F^2_{pq}F^3_{mn}F^4_{pr}F^5_{qr} - 2F^1_{mn}F^2_{pq}F^3_{mp}F^4_{nr}F^5_{qr} + 2F^1_{mn}F^2_{pq}F^3_{mr}F^4_{np}F^5_{qr}$$
$$+ 2F^1_{mn}F^2_{pq}F^3_{mr}F^4_{pr}F^5_{nq} + F^1_{mn}F^2_{pq}F^3_{pr}F^4_{mn}F^5_{qr} + 2F^1_{mn}F^2_{pq}F^3_{pr}F^4_{mr}F^5_{nq}\Big]. \qquad (3.112)$$

This result is equivalent to what was found[12] using the the RNS and GS formalisms [27]. So the modified prescription of (3.106) is able to quickly reproduce the same *bosonic* results while being far simpler to obtain than using the "unmodified" non-minimal pure spinor prescription of 2.8.2, as we shall see in the next subsection.

However until one can prove that (3.108) is supersymmetric or that (3.111) is BRST-closed we cannot guarantee that (3.111) will also correctly reproduce the scattering involving fermions.

3.9.2 Unmodified NMPS computation

Using the unmodified NMPS prescription of (3.105), with the b-ghost being

$$b = s^\alpha\partial\overline{\lambda}_\alpha + \frac{2\Pi^m(\overline{\lambda}\gamma_m d) - N_{mn}(\overline{\lambda}\gamma^{mn}\partial\theta) - J(\overline{\lambda}\partial\theta) - (\overline{\lambda}\partial^2\theta)}{4(\overline{\lambda}\lambda)} \qquad (3.113)$$

$$+\frac{(\overline{\lambda}\gamma^{mnp}r)(d\gamma_{mnp}d + 24N_{mn}\Pi_p)}{192(\overline{\lambda}\lambda)^2} - \frac{(r\gamma^{mnp}r)(\overline{\lambda}\gamma_m d)N_{np}}{16(\overline{\lambda}\lambda)^3} + \frac{(r\gamma^{mnp}r)(\overline{\lambda}\gamma_{pqr}r)N_{mn}N^{qr}}{128(\overline{\lambda}\lambda)^4},$$

the saturation of the 16 d_α and 11 s^α zero modes implies these four different expressions for the kinematic factor[13].

$$I_1 = \frac{1}{2}\langle\frac{\Pi^m}{(\lambda\overline{\lambda})}(\overline{\lambda}\gamma_m d)(\lambda A^1)(dW^2)(dW^3)(dW^4)(dW^5)\rangle \qquad (3.114)$$

$$I_2 = -\frac{1}{16}\langle\frac{(r\gamma_{mnp}r)}{(\lambda\overline{\lambda})^3}(\overline{\lambda}\gamma^m d)N^{np}(\lambda A^1)(dW^2)(dW^3)(dW^4)(dW^5)\rangle \qquad (3.115)$$

$$I_3 = \frac{1}{96}\langle\frac{(\overline{\lambda}\gamma_{mnp}r)}{(\lambda\overline{\lambda})^2}(d\gamma^{mnp}\hat{d})(\lambda A^1)(dW^2)(dW^3)(dW^4)(dW^5)\rangle \qquad (3.116)$$

$$I_4 = \langle\frac{(\overline{\lambda}\gamma_{mnp}r)}{192(\lambda\overline{\lambda})^2}(d\gamma^{mnp}d)(\lambda A^1)(dW^2)(dW^3)(dW^4)(A^5_q\Pi^q + (\hat{d}W^5) + \frac{1}{2}N\cdot F^5)\rangle$$

[12]In the abelian case, this interaction is identically zero.

[13]The hat notation means that the variable acts via its OPE.

$$+\mathrm{perm}(2345)\rangle \tag{3.117}$$

We will restrict out attention to I_4, which is the only case where the b-ghost doesn't act through OPE's.

The term proportional to $G^\theta(z_5, z_2)$ in (3.117) is

$$G^\theta(z_5,z_2)\langle(\overline{\lambda}\gamma_{mnp}r)(d\gamma^{mnp}d)(\lambda A^1)(d\gamma^{rs}W^5)(dW^3)(dW^4)\mathcal{F}^2_{rs}\rangle - (2\leftrightarrow 5),$$

which becomes

$$K^a = G^\theta(z_5,z_2)\langle(\overline{\lambda}\gamma_{mnp}D)\Big[(\lambda A^1)(\lambda\gamma^m\gamma^{rs}W^5)(\lambda\gamma^n W^3)(\lambda\gamma^p W^4)\mathcal{F}^2_{rs}\Big]\rangle - (2\leftrightarrow 5) + (3\leftrightarrow 4),$$

where we summed over $(3\leftrightarrow 4)$ for reasons which will soon become clear. After a long and tedious computation we get

$$\begin{aligned}
K^a/G^\theta(z_5,z_2) = {} & \langle(\overline{\lambda}\gamma_{mnp}D)\big[(\lambda A^1)\mathcal{F}^2_{rs}\big](\lambda\gamma^m\gamma^{rs}W^5)(\lambda\gamma^n W^3)(\lambda\gamma^p W^4)\rangle \\
& -48(\lambda\overline{\lambda})\langle(\lambda A^1)(\lambda\gamma^{[m}W^3)(\lambda\gamma^{n]}W^4)\mathcal{F}^2_{mu}F^5_{nu}\rangle \\
& +12(\lambda\overline{\lambda})\langle(\lambda A^1)(\lambda\gamma^m\gamma^{rs}W^5)(\lambda\gamma^n W^4)\mathcal{F}^2_{rs}\mathcal{F}^3_{mn}\rangle \\
& +12(\lambda\overline{\lambda})\langle(\lambda A^1)(\lambda\gamma^m\gamma^{rs}W^5)(\lambda\gamma^n W^3)\mathcal{F}^2_{rs}\mathcal{F}^4_{mn}\rangle \\
& +4\langle(\lambda\gamma^{rs}\overline{\lambda})(\lambda A^1)(\lambda\gamma^m W^3)(\lambda\gamma^n W^4)\mathcal{F}^2_{rs}\mathcal{F}^5_{mn}\rangle \\
& +4\langle(\lambda\gamma^{rs}\overline{\lambda})(\lambda A^1)(\lambda\gamma^m W^3)(\lambda\gamma^n W^4)\mathcal{F}^5_{rs}\mathcal{F}^2_{mn}\rangle \\
& -16\langle(\lambda\gamma^{tu}\overline{\lambda})(\lambda A^1)(\lambda\gamma^{[m}W^3)(\lambda\gamma^{n]}W^4)\mathcal{F}^2_{mt}\mathcal{F}^5_{nu}\rangle \\
& -(2\leftrightarrow 5) + (3\leftrightarrow 4).
\end{aligned} \tag{3.118}$$

However the last three lines of (3.118) vanish after antisymmetrization over [25]. The second line can be rewritten more conveniently as

$$-48(\lambda\overline{\lambda})\langle(\lambda A^1)(\lambda\gamma^{[m}W^3)(\lambda\gamma^{n]}W^4)\mathcal{F}^2_{mu}F^5_{nu}\rangle = \frac{1}{2}(\lambda\overline{\lambda})\langle(\lambda\gamma^{mnpqr}\lambda)(\lambda A^1)(W^3\gamma_{pqr}W^4)\mathcal{F}^2_{mu}\mathcal{F}^5_{nu}\rangle.$$

Therefore (3.118) becomes, after explicitly summing over $(3\leftrightarrow 4)$,

$$\begin{aligned}
K^a/G^\theta(z_5,z_2) = {} & \\
& +\langle(\overline{\lambda}\gamma_{mnp}D)\big[(\lambda A^1)\mathcal{F}^2_{rs}\big](\lambda\gamma^m\gamma^{rs}W^5)(\lambda\gamma^n W^3)(\lambda\gamma^p W^4)\rangle \\
& +\langle(\overline{\lambda}\gamma_{mnp}D)\big[(\lambda A^1)\mathcal{F}^2_{rs}\big](\lambda\gamma^m\gamma^{rs}W^5)(\lambda\gamma^n W^4)(\lambda\gamma^p W^3)\rangle
\end{aligned}$$

$$+(\lambda\overline{\lambda})\langle(\lambda\gamma^{mnpqr}\lambda)(\lambda A^1)(W^3\gamma_{pqr}W^4)\mathcal{F}^2_{mu}\mathcal{F}^5_{nu}\rangle$$

$$+24(\lambda\overline{\lambda})\langle(\lambda A^1)(\lambda\gamma^m\gamma^{rs}W^5)(\lambda\gamma^n W^4)\mathcal{F}^2_{rs}\mathcal{F}^3_{mn}\rangle$$

$$+24(\lambda\overline{\lambda})\langle(\lambda A^1)(\lambda\gamma^m\gamma^{rs}W^5)(\lambda\gamma^n W^3)\mathcal{F}^2_{rs}\mathcal{F}^4_{mn}\rangle$$

$$-(2\leftrightarrow 5).$$

The first two lines can be rewritten using $\gamma^n_{\alpha(\beta}(\gamma_n)_{\gamma\delta)}=0$, for example

$$(\overline{\lambda}\gamma_{mnp}D)\big[(\lambda A^1)\mathcal{F}^2_{rs}\big](\lambda\gamma^m\gamma^{rs}W^5)(\lambda\gamma^n W^3)(\lambda\gamma^p W^4)= \tag{3.119}$$

$$=-D_\sigma\big[(\lambda A^1)\mathcal{F}^2_{rs}\big](\overline{\lambda}\gamma^m\gamma^n\gamma^p)^\sigma(\lambda\gamma^n W^3)(\lambda\gamma^m\gamma^{rs}W^5)(\lambda\gamma^p W^4)$$

$$=(\lambda\gamma_n\gamma^p D)\big[(\lambda A^1)\mathcal{F}^2_{rs}\big](\overline{\lambda}\gamma^m\gamma^n W^3)(\lambda\gamma^m\gamma^{rs}W^5)(\lambda\gamma^p W^4)$$

$$+D_\sigma\big[(\lambda A^1)\mathcal{F}^2_{rs}\big](\overline{\lambda}\gamma_m\gamma^n\lambda)(W^3\gamma^n\gamma^p)^\sigma(\lambda\gamma^m\gamma^{rs}W^5)(\lambda\gamma_p W^4)$$

$$=2(\lambda D)\big[(\lambda A^1)\mathcal{F}^2_{rs}\big](\overline{\lambda}\gamma^m\gamma^n W^3)(\lambda\gamma^m\gamma^{rs}W^5)(\lambda\gamma^n W^4) \tag{3.120}$$

$$+2(\lambda\overline{\lambda})(W^3\gamma_m\gamma_p D)\big[(\lambda A^1)\mathcal{F}^2_{rs}\big](\lambda\gamma^m\gamma^{rs}W^5)(\lambda\gamma^p W^4). \tag{3.121}$$

Using the equation of motion $(\lambda D)(\lambda A)=0$ and a few gamma matrix identities we obtain for (3.120) (and its permutation over $(3\leftrightarrow 4)$)

$$2\langle(\lambda D)\big[(\lambda A^1)\mathcal{F}^2_{rs}\big](\overline{\lambda}\gamma^m\gamma^n W^3)(\lambda\gamma^m\gamma^{rs}W^5)(\lambda\gamma^n W^4)\rangle+(3\leftrightarrow 4)= \tag{3.122}$$

$$=-8k^2_r\langle(\lambda A^1)(\lambda\gamma^m W^2)(\lambda\gamma^r W^5)\big[(\overline{\lambda}\gamma_{mn}W^3)(\lambda\gamma^n W^4)+(\overline{\lambda}\gamma_{mn}W^4)(\lambda\gamma^n W^3)\big]\rangle$$

$$=-4k^2_r\langle(\lambda\gamma_{ab}\overline{\lambda})(\lambda A^1)(\lambda\gamma_m W^2)(\lambda\gamma^r W^5)(W^3\gamma^{mab}W^4)\rangle,$$

where to arrive at the last line we used

$$W^\alpha_3 W^\beta_4=\frac{1}{16}(W^3\gamma^m W^4)\gamma^{\alpha\beta}_m+\frac{1}{96}(W^3\gamma^{mnp}W^4)\gamma^{\alpha\beta}_{mnp}+\frac{1}{3840}(W^3\gamma^{mnpqr}W^4)\gamma^{\alpha\beta}_{mnpqr}.$$

To rewrite (3.121) in a more convenient way we use $D_\alpha(\lambda A^1)=-(\lambda D)A^1+(\lambda\gamma^q)_\alpha A_q$ and (B.10),

$$2(\lambda\overline{\lambda})(W^3\gamma_m\gamma_p D)\big[(\lambda A^1)\mathcal{F}^2_{rs}\big](\lambda\gamma^m\gamma^{rs}W^5)(\lambda\gamma^p W^4)= \tag{3.123}$$

$$=-2(\lambda\overline{\lambda})(W^3\gamma_m\gamma_p)^\alpha(QA^1_\alpha)\mathcal{F}^2_{rs}(\lambda\gamma^m\gamma^{rs}W^5)(\lambda\gamma^p W^4)$$

$$+2(\lambda\overline{\lambda})A^1_q(W^3\gamma^m\gamma^p\gamma^q\lambda)\mathcal{F}^2_{rs}(\lambda\gamma^m\gamma^{rs}W^5)(\lambda\gamma^p W^4)$$

$$+4k^2_r(\lambda\overline{\lambda})(\lambda A^1)(\lambda\gamma^m\gamma^{rs}W^5)(\lambda\gamma^p W^4)(W^3\gamma^m\gamma^p\gamma^s W^2).$$

The second line is zero due to the pure spinor condition. Integrating the BRST-charge by parts (3.123) becomes

$$= -\frac{1}{2}(\lambda\overline{\lambda})\langle(\lambda\gamma^{tu}\gamma^m\gamma^p A^1)F^3_{tu}F^2_{rs}(\lambda\gamma^m\gamma^{rs}W^5)(\lambda\gamma^p W^4)\rangle \tag{3.124}$$

$$+\frac{1}{2}(\lambda\overline{\lambda})\langle(W^3\gamma_m\gamma_p A^1)(\lambda\gamma^{mrstu}\lambda)(\lambda\gamma^p W^4)\mathcal{F}^2_{rs}\mathcal{F}^5_{tu}\rangle \tag{3.125}$$

$$+8k^2_r(\lambda\overline{\lambda})\langle(W^3\gamma_m\gamma_p A^1)(\lambda\gamma^m W^2)(\lambda\gamma^r W^5)(\lambda\gamma^p W^4)\rangle$$

$$+4k^2_r(\lambda\overline{\lambda})\langle(\lambda A^1)(\lambda\gamma^m\gamma^{rs}W^5)(\lambda\gamma^p W^4)(W^3\gamma^m\gamma^p\gamma^s W^2)\rangle.$$

The first line can be rewritten as

$$-\frac{1}{2}(\lambda\overline{\lambda})\langle(\lambda\gamma^{tu}\gamma^m\gamma^p A^1)F^3_{tu}F^2_{rs}(\lambda\gamma^m\gamma^{rs}W^5)(\lambda\gamma^p W^4)\rangle =$$

$$= 4(\lambda\overline{\lambda})(\lambda A^1)(\lambda\gamma^m\gamma^{rs}W^5)(\lambda\gamma^n W^4)\mathcal{F}^2_{rs}\mathcal{F}^3_{mn},$$

and (3.125) vanishes after antisymmetrization in [25]. Therefore

$$2(\lambda\overline{\lambda})(W^3\gamma_m\gamma_p D)\big[(\lambda A^1)\mathcal{F}^2_{rs}\big](\lambda\gamma^m\gamma^{rs}W^5)(\lambda\gamma^p W^4) + (3\leftrightarrow 4) = \tag{3.126}$$

$$+4(\lambda\overline{\lambda})(\lambda A^1)(\lambda\gamma^m\gamma^{rs}W^5)(\lambda\gamma^n W^4)\mathcal{F}^2_{rs}\mathcal{F}^3_{mn}$$

$$+4(\lambda\overline{\lambda})(\lambda A^1)(\lambda\gamma^m\gamma^{rs}W^5)(\lambda\gamma^n W^3)\mathcal{F}^2_{rs}\mathcal{F}^4_{mn}$$

$$-8k^2_r(\lambda\overline{\lambda})\langle(\lambda\gamma^r W^5)(\lambda\gamma^m W^2)(\lambda\gamma^n W^4)(W^3\gamma_{mn}A^1)\rangle$$

$$-8k^2_r(\lambda\overline{\lambda})\langle(\lambda\gamma^r W^5)(\lambda\gamma^m W^2)(\lambda\gamma^n W^3)(W^4\gamma_{mn}A^1)\rangle$$

$$+4k^2_r(\lambda\overline{\lambda})\langle(\lambda A^1)(\lambda\gamma^m\gamma^{rs}W^5)(\lambda\gamma^p W^4)(W^3\gamma^{mp}\gamma^s W^2)\rangle.$$

$$+4k^2_r(\lambda\overline{\lambda})\langle(\lambda A^1)(\lambda\gamma^m\gamma^{rs}W^5)(\lambda\gamma^p W^3)(W^4\gamma^{mp}\gamma^s W^2)\rangle.$$

So that finally from (3.119), (3.122) and (3.126) we get

$$\langle(\overline{\lambda}\gamma_{mnp}D)\Big[(\lambda A^1)\mathcal{F}^2_{rs}(\lambda\gamma^m\gamma^{rs}W^5)(\lambda\gamma^n W^3)(\lambda\gamma^p W^4)\Big]\rangle - (2\leftrightarrow 5) + (3\leftrightarrow 4) =$$

$$+(\lambda\overline{\lambda})\langle(\lambda\gamma^{mnpqr}\lambda)(\lambda A^1)(W^3\gamma_{pqr}W^4)\mathcal{F}^2_{mu}\mathcal{F}^5_{nu}\rangle$$

$$+28(\lambda\overline{\lambda})\langle(\lambda A^1)(\lambda\gamma^m\gamma^{rs}W^5)(\lambda\gamma^n W^4)\mathcal{F}^2_{rs}\mathcal{F}^3_{mn}\rangle$$

$$+28(\lambda\overline{\lambda})\langle(\lambda A^1)(\lambda\gamma^m\gamma^{rs}W^5)(\lambda\gamma^n W^3)\mathcal{F}^2_{rs}\mathcal{F}^4_{mn}\rangle$$

$$-8k^2_r(\lambda\overline{\lambda})\langle(\lambda\gamma^r W^5)(\lambda\gamma^m W^2)(\lambda\gamma^n W^4)(W^3\gamma_{mn}A^1)\rangle$$

$$-8k^2_r(\lambda\overline{\lambda})\langle(\lambda\gamma^r W^5)(\lambda\gamma^m W^2)(\lambda\gamma^n W^3)(W^4\gamma_{mn}A^1)\rangle$$

$$+4k_r^2(\lambda\overline{\lambda})\langle(\lambda A^1)(\lambda\gamma^m\gamma^{rs}W^5)(\lambda\gamma^n W^4)(W^3\gamma_{mn}\gamma_s W^2)\rangle$$

$$+4k_r^2(\lambda\overline{\lambda})\langle(\lambda A^1)(\lambda\gamma^m\gamma^{rs}W^5)(\lambda\gamma^n W^3)(W^4\gamma_{mn}\gamma_s W^2)\rangle$$

$$-4k_r^2\langle(\lambda\gamma_{ab}\overline{\lambda})(\lambda A^1)(\lambda\gamma_m W^2)(\lambda\gamma^r W^5)(W^3\gamma^{abm}W^4)\rangle - (2\leftrightarrow 5), \tag{3.127}$$

where we used $(\lambda\gamma^m)_\alpha(\lambda\gamma_m)_\beta = 0$ to convert $\gamma_m\gamma_p$ into γ_{mp} in the fourth and fifth lines.

Using the tensor (3.32) we obtain

$$\langle(\lambda\gamma_{ab}\overline{\lambda})(\lambda A^1)(\lambda\gamma_m W^2)(\lambda\gamma^r W^5)(W^3\gamma^{abm}W^4)\rangle =$$

$$+\frac{3}{2}\langle(\lambda\gamma^{ab}A^1)(\lambda\gamma^m W^2)(\lambda\gamma^r W^5)(W^3\gamma_{abm}W^4)\rangle$$

$$+\frac{3}{2}\langle(\lambda A^1)(\lambda\gamma^{abm}W^2)(\lambda\gamma^r W^5)(W^3\gamma_{abm}W^4)\rangle$$

$$+\frac{3}{2}\langle(\lambda A^1)(\lambda\gamma^m W^2)(\lambda\gamma^{ab}\gamma^r W^5)(W^3\gamma_{abm}W^4)\rangle,$$

and therefore the last line in (3.127) can be evaluated using the standard methods.

Using the identity $\gamma^n_{\alpha(\beta}(\gamma_n)_{\gamma\delta)} = 0$ one can show that (3.111) – which gives us the right answer for the kinematic factor – can be rewritten like

$$\langle(D\gamma_{mnp}A^1)(\lambda\gamma^m W^3)(\lambda\gamma^n W^4)(\lambda\gamma^p\gamma^{rs}W^5)F^2_{rs}\rangle - (2\leftrightarrow 5) = \tag{3.128}$$

$$= -8\langle(\lambda A^1)(\lambda\gamma^m\gamma^{rs}W^5)(\lambda\gamma^n W^4)\mathcal{F}^2_{rs}\mathcal{F}^3_{mn}$$

$$+16k_r^2\langle(\lambda\gamma^r W^5)(\lambda\gamma^m W^2)(\lambda\gamma^n W^4)(A^1\gamma_{mn}W^3) - (2\leftrightarrow 5),$$

therefore we conclude that if

$$+(\lambda\overline{\lambda})\langle(\lambda\gamma^{mnpqr}\lambda)(\lambda A^1)(W^3\gamma_{pqr}W^4)\mathcal{F}^2_{mu}\mathcal{F}^5_{nu}\rangle$$

$$+24(\lambda\overline{\lambda})\langle(\lambda A^1)(\lambda\gamma^m\gamma^{rs}W^5)(\lambda\gamma^n W^4)\mathcal{F}^2_{rs}\mathcal{F}^3_{mn}\rangle$$

$$+24(\lambda\overline{\lambda})\langle(\lambda A^1)(\lambda\gamma^m\gamma^{rs}W^5)(\lambda\gamma^n W^3)\mathcal{F}^2_{rs}\mathcal{F}^4_{mn}\rangle$$

$$+4k_r^2(\lambda\overline{\lambda})\langle(\lambda A^1)(\lambda\gamma^m\gamma^{rs}W^5)(\lambda\gamma^n W^4)(W^3\gamma_{mn}\gamma_s W^2)\rangle$$

$$+4k_r^2(\lambda\overline{\lambda})\langle(\lambda A^1)(\lambda\gamma^m\gamma^{rs}W^5)(\lambda\gamma^n W^3)(W^4\gamma_{mn}\gamma_s W^2)\rangle$$

$$-4k_r^2\langle(\lambda\gamma_{ab}\overline{\lambda})(\lambda A^1)(\lambda\gamma_m W^2)(\lambda\gamma^r W^5)(W^3\gamma^{abm}W^4)\rangle - (2\leftrightarrow 5), \tag{3.129}$$

is proportional to (3.128) then the NMPS kinematic factor computation described above is equivalent to the RNS result listed in [27].

Chapter 4

Conclusions

In this thesis we have shown how the pure spinor formalism can be used to obtain in a manifestly SO(1,9)-covariant and supersymmetric way various scattering amplitudes for massless particles. We emphasized the simple pure spinor superspace expressions for their kinematic factors and how they encode the complete results for all possible external state combination related by supersymmetry, and computed them explicitly in components. The results which were known from the computations of the RNS and GS formalisms were shown to be correctly reproduced. But also new results were obtained, namely the explicit computation at two-loops involving external fermionic states.

What was also accomplished is a simple proof which explicitly relates the massless four-point kinematic factors at tree-level with those at one- and two-loops. The proof can be summarized as follows,

$$K_0 = -\frac{1}{3}K_1, \quad K_2 = -32 K_0\, \mathcal{Y}(s,t,u), \tag{4.1}$$

where K_0, K_1 and K_2 denote the *complete* supersymmetric massless four-point kinematic factors at tree-level, one-loop and two-loops, respectively. The function $\mathcal{Y}(s,t,u)$ is called quadri-holomorphic 1-form and is quadratic in momenta. It is given by

$$\mathcal{Y}(s,t,u) = \left[(u-t)\Delta(1,2)\Delta(3,4) + (s-t)\Delta(1,3)\Delta(2,4) + (s-u)\Delta(1,4)\Delta(2,3)\right].$$

But there is more to the scattering amplitude than just its kinematic factor. The amplitudes also depend on the moduli space of the Riemann surface. Considering the amplitudes for closed strings they read

$$\mathcal{A}_0 \propto \mathrm{e}^{-2\phi} K_0 \overline{K}_0 \frac{\Gamma(-\frac{\alpha' t}{4})\Gamma(-\frac{\alpha' u}{4})\Gamma(-\frac{\alpha' s}{4})}{\Gamma(1+\frac{\alpha' t}{4})\Gamma(1+\frac{\alpha' u}{4})\Gamma(1+\frac{\alpha' s}{4})},$$

$$\mathcal{A}_1 \propto K_1 \overline{K}_1 \int_{\mathcal{M}_1} \frac{|d\tau|^2}{(\mathrm{Im}\tau)^2} F_1(\tau),$$

$$\mathcal{A}_2 \propto \mathrm{e}^{2\phi} \tilde{K}_2 \overline{\tilde{K}}_2 \int_{\mathcal{M}_2} \frac{|d^3\Omega|^2}{(\det \mathrm{Im}\Omega)^5} F_2(\Omega, \mathcal{Y}), \tag{4.2}$$

where $F_1(\tau)$ and $F_2(\Omega, \mathcal{Y})$ are modular invariant functions given by

$$F_1(\tau) = \frac{1}{(\mathrm{Im}\tau)^3} \int d^2 z_2 \int d^2 z_3 \int d^2 z_4 \prod_{i<j} G(z_i, z_j)^{k_i \cdot k_j}.$$

$$F_2(\Omega, \mathcal{Y}) = \int |\mathcal{Y}|^2 \prod_{i<j} G(z_i, z_j)^{k_i \cdot k_j}$$

and for convenience we defined $K_2 = \tilde{K}_2 \mathcal{Y}$. It is not hard to say that, up to overall coefficients, equations (4.1) – (4.2) encode the state-of-the-art of what is known about massless four-point scattering amplitudes in superstring theory after more than two decades of heavy development.

Much more work remains to be done. For example, the coefficients in the above amplitudes can be derived by factorization [42] , but it would certainly be interesting to obtain them directly from the pure spinor amplitude prescription[1]. That requires integrations over the compact pure spinor space $SO(10)/U(5)$ like

$$\int_{SO(10)/U(5)} [d\lambda][d\overline{\lambda}]\mathrm{e}^{-(\lambda\overline{\lambda})}$$

to be performed. The most efficient way to obtain the overall coefficients is still an open question in the pure spinor formalism, but there is work in progress in this direction with Humberto Zuñiga.

Another line of research which is being followed is the pursuit of efficient ways to obtain higher-point amplitudes, where the beginning of this work has been briefly described in 3.9. The idea is to derive formulæ like (3.110) in such a way as to allow one to compute the amplitude as quickly as possible.

There is yet another five-point computation being worked out at this moment, which is related to finding the supersymmetric completion of the $B\mathrm{tr}F^4$ interaction of the heterotic string. Partial results led us[2] to consider the following pure spinor superspace proposal

[1] The analogous task has not yet been done with the RNS at two-loops.

[2] Joint work with Nathan Berkovits.

for the effective action containing $B\mathrm{tr}F^4$,

$$K_B = \langle(\lambda B_m)(\lambda\gamma_n W)(\lambda\gamma_p W)(W\gamma^{mnp}W)\rangle, \tag{4.3}$$

where $B_{m\alpha}$ is the supergravity superfield which has the graviton multiplet as the lowest order component in its θ-expansion.

The proposal (4.3) correctly reproduces the $\epsilon_{10}BF^4$ interaction in components and it has the interesting "factorization" property under the action of the BRST-charge

$$QK_B \propto \langle(\lambda B_m)(\lambda\gamma^m W)(\lambda\gamma^n W)(\lambda\gamma^p W)\mathcal{F}_{mn}\rangle$$

$$\propto \langle(\lambda B_m)(\lambda\gamma^m W)(\lambda A)\rangle\frac{1}{Q}\langle(\lambda A)(\lambda\gamma^n W)(\lambda\gamma^p W)\mathcal{F}_{np}\rangle,$$

where $(\lambda A)\frac{1}{Q}(\lambda A)$ could be regarded as the propagator which connects the three-point tree-level $\langle(\lambda B_m)(\lambda\gamma^m W)(\lambda A)\rangle$ amplitude (which correctly reproduces the $dBAF$ interaction) with the four-point at one-level $\langle(\lambda A)(\lambda\gamma^n W)(\lambda\gamma^p W)\mathcal{F}_{np}\rangle$.

As an interesting exercise yet to be done is to find the explicit pure spinor superspace expression for L_m, where

$$\langle(\lambda\gamma^m W)(\lambda\gamma^n W)(\lambda\gamma^p W)(W\gamma_{mnp}W)\rangle = \langle\partial^m L_m\rangle.$$

As shown in section 3.7.3 it is known through explicit component computations that the anomaly kinematic factor (the left hand side of the above equation) is a total derivative. Finding the superspace expression for L_m would be interesting and could be used to start working in the direction of [43].

As a more ambitious project would be to compute higher-loop amplitudes for massless four-point amplitudes, the genus three surface being the next one in line to be explored. The pure spinor formalism can certainly give us some clues about what to expect from the kinematic factors at higher-loop order. Some hours of thinking are being spent to try to make rigorous the statement that the massless four-point kinematic factors at higher-loop order have the Lorentz structure given by $Y(s,t,u)K_0$, where $Y(s,t,u)$ is a polinomial in the Mandelstam variables. That conjecture is motivated by the observation that it is impossible to construct pure spinor superspace expressions using only the superfields $A_\alpha, A_m, W^\alpha, \mathcal{F}_{mn}$ to have mass dimension more higher than $\partial^3 F^4$. Together with the observation that the massless four-point kinematic factors in pure spinor superspace are BRST-closed[3] and that the BRST-charge comutes the derivatives ∂_m leads to the conjecture mentioned above,

$$K_n \propto Y(s,t,u)\langle(\lambda A)(\lambda\gamma^m W)(\lambda\gamma^n W)\mathcal{F}_{mn}\rangle.$$

For the amplitudes up to two-loops it has already been verified to be true. To finish the proof one would need to consider the case where K_n is composed by various terms which are not BRST-closed but whose sum is (like the tree-level expression). Work is in progress to complete the sketch of the proof presented here.

Therefore there remains some avenues of research exploring the pure spinor formalism's aptitude to compute superstring scattering amplitudes.

[3]They can't be factorized into lower-loop order because such terms are prohibited by the non-renormalization theorem, which by the way was also elegantly proved in the pure spinor formalism [7].

Appendix A

Evaluating Pure Spinor Superspace Expressions

In the previous chapter we have encountered many pure spinor superspace expressions of the form

$$\langle \lambda^\alpha \lambda^\beta \lambda^\gamma \theta^{\delta_1} \theta^{\delta_2} \theta^{\delta_3} \theta^{\delta_4} \theta^{\delta_5} f_{\alpha\beta\gamma}(\theta) \rangle \tag{A.1}$$

where $f_{\alpha\beta\gamma}(\theta)$ was composed by some combination of super-Yang-Mills superfields and the angle brackets $\langle \quad \rangle$ is defined in such a way that the only non-vanishing component is proportional to

$$\langle (\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta) \rangle = 1. \tag{A.2}$$

We will now proceed to show how they can be explicitly computed, obtaining as a result an expression which depends only on polarizations e_m, ξ^α and momenta k^m. That this is possible can be seen by checking that there is only one scalar built out of three pure spinors λ^α and five unconstrained θ's

$$[0,0,0,0,3] \otimes \Big([0,0,0,3,0] + [1,1,0,1,0] \Big) = 1X[0,0,0,0,0] + 2X[0,0,0,1,1] + \ldots,$$

so that an arbitrarily complicated pure spinor correlator written in terms of three λ's and five θ's can be written entirely in terms of Kronecker deltas and Levi-Civita ϵ_{10} tensors.

After expanding the superfields appearing in the generic correlator (A.1) and taking the terms which contain five θ's one will get

$$\theta^{\delta_1} \theta^{\delta_2} \theta^{\delta_3} \theta^{\delta_4} \theta^{\delta_5} f_{\alpha\beta\gamma\delta_1\delta_2\delta_3\delta_4\delta_5}$$

where $f_{\alpha\beta\gamma\delta_1\delta_2\delta_3\delta_4\delta_5}$ is composed by a string of gamma matrices with several indices. Each one of those terms can be easily evaluated using the rule (A.2).

Suppose one wants to compute the following pure spinor correlation

$$\langle(\lambda\gamma^m\theta)(\lambda\gamma^n\gamma^{rs}\theta)(\lambda\gamma^p\gamma^{tu}\theta)(\theta\gamma_{fgh}\theta)\rangle. \tag{A.3}$$

To use the argument after (A.2) it is better to write (A.3) in a form in which the symmetries over the vector indices are manifest. In this case we can do it by using the gamma matrix identity

$$\gamma^m\gamma^{np} = \gamma^{mnp} + \eta^{mn}\gamma^p - \eta^{mp}\gamma^n. \tag{A.4}$$

to obtain

$$\langle(\lambda\gamma^m\theta)(\lambda\gamma^n\gamma^{rs}\theta)(\lambda\gamma^p\gamma^{tu}\theta)(\theta\gamma_{fgh}\theta)\rangle = \langle(\lambda\gamma^m\theta)(\lambda\gamma^{nrs}\theta)(\lambda\gamma^{ptu}\theta)(\theta\gamma_{fgh}\theta) \tag{A.5}$$

$$+2\langle(\lambda\gamma^m\theta)\delta^{[r}_n(\lambda\gamma^{s]}\theta)(\lambda\gamma^{ptu}\theta)(\theta\gamma_{fgh}\theta)\rangle + 4\langle(\lambda\gamma^m\theta)\delta^{[r}_n(\lambda\gamma^{s]}\theta)\delta^{[t}_p(\lambda\gamma^{u]}\theta)(\theta\gamma_{fgh}\theta)\rangle.$$

And now we can easily use symmetry arguments to show that

$$\langle(\lambda\gamma^m\theta)(\lambda\gamma^s\theta)(\lambda\gamma^u\theta)(\theta\gamma_{fgh}\theta)\rangle = \frac{1}{120}\delta^{msu}_{fgh} \tag{A.6}$$

where δ^{msu}_{fgh} is the antisymmetrized combination of Kronecker deltas beginning with $\frac{1}{3!}\delta^m_f\delta^s_g\delta^u_h$. To see this note that the right hand side of (A.6) is the only tensor which is antisymmetric in $[msu]$ and $[fgh]$ and which is normalized to one (because $\delta^{msu}_{msu} = 120$), therefore respecting the normalization imposed by the rule (A.2). By the same token, using symmetry arguments alone one can show that

$$\langle(\lambda\gamma_m\theta)(\lambda\gamma_s\theta)(\lambda\gamma^{ptu}\theta)(\theta\gamma_{fgh}\theta)\rangle = \frac{1}{70}\delta^{[p}_{[m}\eta_{s][f}\delta^t_g\delta^{u]}_{h]} \tag{A.7}$$

$$\langle(\lambda\gamma_m\theta)(\lambda\gamma^{nrs}\theta)(\lambda\gamma^{ptu}\theta)(\theta\gamma_{fgh}\theta)\rangle = \frac{1}{8400}\epsilon^{fghmnprstu} +$$

$$+\frac{1}{140}\Big[\delta^{[n}_m\delta^r_{[f}\eta^{s][p}\delta^t_g\delta^{u]}_{h]} - \delta^{[p}_m\delta^t_{[f}\eta^{u][n}\delta^r_g\delta^{s]}_{h]}\Big] - \frac{1}{280}\Big[\eta_{m[f}\eta^{v[p}\delta^t_g\eta^{u][n}\delta^r_{h]}\delta^{s]}_v - \eta_{m[f}\eta^{v[n}\delta^r_g\eta^{s][p}\delta^t_{h]}\delta^{u]}_v\Big]. \tag{A.8}$$

In general, using several identities like $(\theta\gamma^{abc}\gamma^{mn}\theta) = (\theta\gamma^{r_1r_2r_3}\theta)K^{abcmn}_{r_1r_2r_3}$, where

$$K^{abcmn}_{r_1r_2r_3} = -\eta^{cn}\delta^{abm}_{r_1r_2r_3} + \eta^{cm}\delta^{abn}_{r_1r_2r_3} + \eta^{bn}\delta^{acm}_{r_1r_2r_3} - \eta^{bm}\delta^{acn}_{r_1r_2r_3} - \eta^{an}\delta^{bcm}_{r_1r_2r_3} + \eta^{am}\delta^{bcn}_{r_1r_2r_3}$$

or

$$(\lambda\gamma^{mnp}\theta)(\lambda\gamma^{qrs}\theta) = -\frac{1}{96}(\theta\gamma^{tuv}\theta)(\lambda\gamma^{mnp}\gamma_{tuv}\gamma^{qrs}\lambda)$$

$$\equiv -\frac{1}{96}(\lambda\gamma^{abcde}\lambda)(\theta\gamma^{tuv}\theta)f^{mnpqrs}_{abcdetuv}, \tag{A.9}$$

where

$$f^{mnpqrs}_{abcdetuv} = 18(\delta^{rs}_{uv}\delta^{abcde}_{mnpqt} - \delta^{np}_{uv}\delta^{abcde}_{qrsmt}) + 54(\delta^{ps}_{tv}\delta^{abcde}_{mnqru} + \delta^{mn}_{qr}\delta^{abcde}_{pstuv})$$
$$+54(\delta^{nv}_{rs}\delta^{abcde}_{mpqtu} - \delta^{rv}_{np}\delta^{abcde}_{qsmtu}) + [mnp] + [qrs] + [tuv]$$

and gamma matrix identities like (A.4) together with

$$(\lambda\gamma^{abc}\gamma^{de}\theta) = +(\lambda\gamma^{abcde}\theta) - 2\delta^{bc}_{de}(\lambda\gamma^{a}\theta) + 2\delta^{ac}_{de}(\lambda\gamma^{b}\theta) - 2\delta^{ab}_{de}(\lambda\gamma^{c}\theta)$$
$$-\delta^{c}_{e}(\lambda\gamma^{abd}\theta) + \delta^{c}_{d}(\lambda\gamma^{abe}\theta) + \delta^{b}_{e}(\lambda\gamma^{acd}\theta) - \delta^{b}_{d}(\lambda\gamma^{ace}\theta) - \delta^{a}_{e}(\lambda\gamma^{bcd}\theta) + \delta^{a}_{d}(\lambda\gamma^{bce}\theta)$$

(and many others) all pure spinor superspace expressions can be written in terms as a linear combination of the basic ones (A.6), (A.7), (A.8) and

$$\langle(\lambda\gamma^{mnpqr}\theta)(\lambda\gamma_{stu}\theta)(\lambda\gamma^{v}\theta)(\theta\gamma_{fgh}\theta) = \frac{1}{35}\eta^{v[m}\delta^{n}_{[s}\delta^{p}_{t}\eta_{u][f}\delta^{q}_{g}\delta^{r]}_{h]} - \frac{2}{35}\delta^{[m}_{[s}\delta^{n}_{t}\delta^{p}_{u]}\delta^{q}_{[f}\delta^{r]}_{g}\delta^{v}_{h]} \quad \text{(A.10)}$$
$$+\frac{1}{120}\epsilon^{mnpqr}{}_{abcde}\left(\frac{1}{35}\eta^{v[a}\delta^{b}_{[s}\delta^{c}_{t}\eta_{u][f}\delta^{d}_{g}\delta^{e]}_{h]} - \frac{2}{35}\delta^{[a}_{[s}\delta^{b}_{t}\delta^{c}_{u]}\delta^{d}_{[f}\delta^{e]}_{g}\delta^{v}_{h]}\right)$$

$$\langle(\lambda\gamma^{mnpqr}\theta)(\lambda\gamma_{d}\theta)(\lambda\gamma_{e}\theta)(\theta\gamma_{fgh}\theta) = -\frac{1}{42}\delta^{mnpqr}_{defgh} - \frac{1}{5040}\epsilon^{mnpqr}{}_{defgh} \quad \text{(A.11)}$$

$$\langle(\lambda\gamma^{mnp}\theta)(\lambda\gamma^{qrs}\theta)(\lambda\gamma_{tuv}\theta)(\theta\gamma_{ijk}\theta)\rangle = \quad \text{(A.12)}$$
$$-\frac{3}{175}\Big[-\delta^{[i}_{a}\delta^{j}_{[q}\delta^{k]}_{r}\delta^{[m}_{s]}\delta^{n}_{[t}\delta^{p]}_{u}\delta^{a}_{v]} + \delta^{[i}_{a}\delta^{j}_{[t}\delta^{k]}_{u}\delta^{[m}_{v]}\delta^{n}_{[q}\delta^{p]}_{r}\delta^{a}_{s]} + \delta^{[i}_{[q}\delta^{j}_{r}\eta^{k][m}\eta_{s][t}\delta^{n}_{u}\delta^{p]}_{v]}$$
$$+\delta^{a}_{[t}\eta^{b[i}\delta^{j}_{u}\eta^{k][m}\eta_{v][q}\delta^{n}_{r}\eta_{s]a}\delta^{p]}_{b} - \delta^{a}_{[q}\eta^{b[i}\delta^{j}_{r}\eta^{k][m}\eta_{s][t}\delta^{n}_{u}\eta_{v]a}\delta^{p]}_{b} - \delta^{[i}_{[t}\delta^{j}_{u}\eta^{k][m}\eta_{v][q}\delta^{n}_{r}\delta^{p]}_{s]}\Big]$$
$$+\frac{1}{33600}\epsilon^{abcde}{}_{a_1a_2a_3a_4a_5}f^{mnpqrs}_{abcdefgh}\Big[\delta^{[t}_{[f}\delta^{u}_{g}\eta^{v][a_1}\delta^{a_2}_{h]}\delta^{a_3}_{[i}\delta^{a_4}_{j}\delta^{a_5]}_{k]} + \delta^{[t}_{[i}\delta^{u}_{j}\eta^{v][a_1}\delta^{a_2}_{k]}\delta^{a_3}_{[f}\delta^{a_4}_{g}\delta^{a_5]}_{h]}$$
$$-\eta^{z[t}\delta^{u}_{[f}\eta^{v][a_1}\delta^{a_2}_{g}\eta_{h][i}\delta^{a_3}_{j}\delta^{a_4}_{k]}\delta^{a_5]}_{z} - \eta^{z[t}\delta^{u}_{[i}\eta^{v][a_1}\delta^{a_2}_{j}\eta_{k][f}\delta^{a_3}_{g}\delta^{a_4}_{h]}\delta^{a_5]}_{z}\Big].$$

$$\langle(\lambda\gamma^{mnpqr}\lambda)(\lambda\gamma^{u}\theta)(\theta\gamma_{fgh}\theta)(\theta\gamma_{jkl}\theta)\rangle = \quad \text{(A.13)}$$
$$-\frac{4}{35}\Big[\delta^{[m}_{[j}\delta^{n}_{k}\delta^{p}_{l]}\delta^{q}_{[f}\delta^{r]}_{g}\delta^{u}_{h]} + \delta^{[m}_{[f}\delta^{n}_{g}\delta^{p}_{h]}\delta^{q}_{[j}\delta^{r]}_{k}\delta^{u}_{l]} - \frac{1}{2}\delta^{[m}_{[j}\delta^{n}_{k}\eta_{l][f}\delta^{p}_{g}\delta^{q}_{h]}\eta^{r]u} - \frac{1}{2}\delta^{[m}_{[f}\delta^{n}_{g}\eta_{h][j}\delta^{p}_{k}\delta^{q}_{l]}\eta^{r]u}\Big]$$
$$-\frac{1}{1050}\epsilon^{mnpqr}{}_{abcde}\Big[\delta^{[a}_{[j}\delta^{b}_{k}\delta^{c}_{l]}\delta^{d}_{[f}\delta^{e]}_{g}\delta^{u}_{h]} + \delta^{[a}_{[f}\delta^{b}_{g}\delta^{c}_{h]}\delta^{d}_{[j}\delta^{e]}_{k}\delta^{u}_{l]}$$
$$-\frac{1}{2}\delta^{[a}_{[j}\delta^{b}_{k}\eta_{l][f}\delta^{c}_{g}\delta^{d}_{h]}\eta^{e]u} - \frac{1}{2}\delta^{[a}_{[f}\delta^{b}_{g}\eta_{h][j}\delta^{c}_{k}\delta^{d}_{l]}\eta^{e]u}\Big]$$

$$\langle(\lambda\gamma^{mnpqr}\lambda)(\lambda\gamma^{stu}\theta)(\theta\gamma_{fgh}\theta)(\theta\gamma_{jkl}\theta)\rangle = \quad \text{(A.14)}$$
$$-\frac{12}{35}\Big[\delta^{[s}_{[f}\delta^{t}_{g}\eta^{u][m}\delta^{n}_{h]}\delta^{p}_{[j}\delta^{q}_{k}\delta^{r]}_{l]} + \delta^{[s}_{[j}\delta^{t}_{k}\eta^{u][m}\delta^{n}_{l]}\delta^{p}_{[f}\delta^{q}_{g}\delta^{r]}_{h]}$$
$$-\eta^{v[s}\delta^{t}_{[f}\eta^{u][m}\delta^{n}_{g}\eta_{h][j}\delta^{p}_{k}\delta^{q}_{l]}\delta^{r]}_{v} - \eta^{v[s}\delta^{t}_{[j}\eta^{u][m}\delta^{n}_{k}\eta_{l][f}\delta^{p}_{g}\delta^{q}_{h]}\delta^{r]}_{v}\Big]$$

$$-\frac{1}{350}\epsilon^{mnpqr}{}_{abcde}\Big[\delta^{[s}_{[f}\delta^{t}_{g}\eta^{u][a}\delta^{b}_{h]}\delta^{c}_{[j}\delta^{d}_{k}\delta^{e]}_{l]} + \delta^{[s}_{[j}\delta^{t}_{k}\eta^{u][a}\delta^{b}_{l]}\delta^{c}_{[f}\delta^{d}_{g}\delta^{e]}_{h]}$$

$$-\eta^{v[s}\delta^{t}_{[f}\eta^{u][a}\delta^{b}_{g}\eta_{h][j}\delta^{c}_{k}\delta^{d}_{l]}\delta^{e]}_{v} - \eta^{v[s}\delta^{t}_{[j}\eta^{u][a}\delta^{b}_{k}\eta_{l][f}\delta^{c}_{g}\delta^{d}_{h]}\delta^{e]}_{v}\Big].$$

$$\langle(\lambda\gamma^{mnpqr}\lambda)(\lambda\gamma^{abcde}\theta)(\theta\gamma^{fgh}\theta)(\theta\gamma^{jkl}\theta)\rangle = \tag{A.15}$$

A.1 Obtaining epsilon terms

In the correlations above we obtained the epsilons terms by considering the duality of the gamma matrices

$$\left(\gamma^{m_1m_2m_3m_4m_5}\right)_{\alpha\beta} = +\frac{1}{5!}\epsilon^{m_1m_2m_3m_4m_5n_1n_2n_3n_4n_5}\left(\gamma_{n_1n_2n_3n_4n_5}\right)_{\alpha\beta}, \tag{A.16}$$

$$\left(\gamma^{m_1m_2m_3m_4m_5m_6}\right)_{\alpha}{}^{\beta} = +\frac{1}{4!}\epsilon^{m_1m_2m_3m_4m_5m_6n_1n_2n_3n_4}\left(\gamma_{n_1n_2n_3n_4}\right)_{\alpha}{}^{\beta},$$

$$\left(\gamma^{m_1m_2m_3m_4m_5m_6m_7}\right)_{\alpha\beta} = -\frac{1}{3!}\epsilon^{m_1m_2m_3m_4m_5m_6m_7n_1n_2n_3}\left(\gamma_{n_1n_2n_3}\right)_{\alpha\beta},$$

$$\left(\gamma^{m_1m_2m_3m_4m_5m_6m_7m_8}\right)_{\alpha}{}^{\beta} = -\frac{1}{2!}\epsilon^{m_1m_2m_3m_4m_5m_6m_7m_8n_1n_2}\left(\gamma_{n_1n_2}\right)_{\alpha}{}^{\beta}.$$

For example, to obtain the epsilon term of (A.8) we used the identity (A.9) to relate (A.8) with (A.13), whereas the epsilon terms in (A.13) were found by first computing its Kronecker delta terms and then using (A.16) to obtain the epsilon terms.

In fact, due to the pure spinor property of $(\lambda\gamma^m\lambda) = 0$ there are only three different correlations which need to be taken care of. That is because one can always use the properties of

$$\lambda^\alpha\lambda^\beta = \frac{1}{3840}(\lambda\gamma^{mnpqr}\lambda)\gamma^{\alpha\beta}_{mnpqr}$$

$$\theta^\alpha\theta^\beta = \frac{1}{96}(\theta\gamma^{mnp}\theta)\gamma^{\alpha\beta}_{mnp}$$

to write any correlation in terms of (A.13), (A.14) and (A.15). And as the epsilon terms of these three fundamental building blocks are easily found through the use of (A.16), all epsilon terms of an arbitrary pure spinor expression are easily[1] determined.

[1]If one has a computer to do the tedious calculations, of course.

Appendix B

$N = 1$ Super Yang-Mills Theory in $D = 10$

The basic reference is [37].

Definition 2. *The supercovariant derivatives are*

$$\nabla_m = \partial_m + A_m \tag{B.1}$$

$$\nabla_\alpha = D_\alpha + A_\alpha \tag{B.2}$$

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + \frac{1}{2}(\gamma^m\theta)_\alpha\partial_m, \tag{B.3}$$

and the field-strengths are defined by

$$F_{\alpha\beta} = \{\nabla_\alpha, \nabla_\beta\} - \gamma^m_{\alpha\beta}\nabla_m \tag{B.4}$$

$$F_{\alpha m} = [\nabla_\alpha, \nabla_m] \tag{B.5}$$

$$F_{mn} = [\nabla_m, \nabla_n]. \tag{B.6}$$

One can easily check the above field-strengths to be invariant under the gauge transformations of

$$\delta A_m = \partial_m\Omega, \quad \delta A_\alpha = D_\alpha\Omega. \tag{B.7}$$

Lemma 1. *The fermionic supercovariant derivative satisfies*

$$\{D_\alpha, D_\beta\} = \gamma^m_{\alpha\beta}\partial_m.$$

Proposition 1. $F_{\alpha\beta} = 0$ *if and only if*

$$\gamma^{\alpha\beta}_{mnpqr}\left(D_\alpha A_\beta + D_\beta A_\alpha + \{A_\alpha, A_\beta\}\right) = 0. \tag{B.8}$$

By straighforward computation using definition 2 we get

$$F_{\alpha\beta} = \{D_\alpha, D_\beta\} + \{D_\alpha, A_\beta\} + \{A_\alpha, D_\beta\} + \{A_\alpha, A_\beta\} - \gamma^m_{\alpha\beta}(\partial_m + A_m) = 0,$$

so

$$D_\alpha A_\beta + D_\beta A_\alpha + \{A_\alpha, A_\beta\} = \gamma^s_{\alpha\beta} A_s.$$

Multipling both sides by $\gamma^{\alpha\beta}_{mnpqr}$ and using the identity $\mathrm{tr}(\gamma_{mnpqr}\gamma_s) = 0$, equation (B.8) follows. By reversing the above steps, the converse can also be proved.

Proposition 2. *If* $\gamma^{\alpha\beta}_{mnpqr}\left(D_\alpha A_\beta + D_\beta A_\alpha + \{A_\alpha, A_\beta\}\right) = 0$ *(or equivalently* $\{\nabla_\alpha, \nabla_\beta\} = \gamma^m_{\alpha\beta}\nabla_m$*) then*

$$F_{\alpha m} \equiv (\gamma_m W)_\alpha \tag{B.9}$$

$$\nabla_\alpha W^\beta = -\frac{1}{4}(\gamma^{mn})_\alpha{}^\beta F_{mn}, \tag{B.10}$$

$$\nabla_\alpha F_{mn} = \nabla_m(\gamma_n W)_\alpha - \nabla_n(\gamma_m W)_\alpha. \tag{B.11}$$

The proof follows from the use of Bianchi and gamma matrix identities. The Bianchi identity

$$[\{\nabla_\alpha, \nabla_\beta\}, \nabla_\gamma] + [\{\nabla_\gamma, \nabla_\alpha\}, \nabla_\beta] + [\{\nabla_\beta, \nabla_\gamma\}, \nabla_\alpha] = 0,$$

implies that

$$\gamma^m_{\alpha\beta}[\nabla_m, \nabla_\gamma] + \gamma^m_{\gamma\alpha}[\nabla_m, \nabla_\beta] + \gamma^m_{\beta\gamma}[\nabla_m, \nabla_\alpha] = 0.$$

Consequentely,

$$\gamma^m_{\alpha\beta}F_{\gamma m} + \gamma^m_{\gamma\alpha}F_{\beta m} + \gamma^m_{\beta\gamma}F_{\alpha m} = 0. \tag{B.12}$$

The identity $\eta_{mn}\gamma^m_{\alpha(\beta}\gamma^n_{\gamma\delta)} = 0$ implies that (B.12) is trivially satisfied if $F_{\alpha m} = (\gamma_m W)_a$. Similarly, from

$$[[\nabla_m, \nabla_n], \nabla_\alpha] + [[\nabla_\alpha, \nabla_m], \nabla_n] + [[\nabla_n, \nabla_\alpha], \nabla_m] = 0$$

we obtain (B.11). From

$$[\{\nabla_\alpha, \nabla_\beta\}, \nabla_m] + \{[\nabla_m, \nabla_\alpha], \nabla_\beta\} - \{[\nabla_\beta, \nabla_m], \nabla_\alpha\} = 0,$$

we get

$$(\gamma_m)_{\beta\delta}\nabla_\alpha W^\delta + (\gamma_m)_{\alpha\delta}\nabla_\beta W^\delta = \gamma^n_{\alpha\beta}F_{nm}. \tag{B.13}$$

The identity $\nabla_\alpha W^\alpha = 0$ follows if we multiply (B.13) by $(g^m)^{\alpha\beta}$. Multiplication by $(\gamma^m)^{\beta\sigma}$ results in,

$$10\nabla_\alpha W^\sigma + \gamma^m_{\alpha\delta}\gamma^{\beta\sigma}_m\nabla_\beta W^\delta = -(\gamma^{mn})_\alpha{}^\sigma F_{mn}. \tag{B.14}$$

Multiplying (B.14) by $\gamma^p_{\sigma\kappa}\gamma^{\alpha\rho}_p$ and using the identities $(\gamma^p\gamma^{mn}\gamma_p)^\rho{}_\kappa = -6(\gamma^{mn})^\rho{}_\kappa$ and

$$(\gamma_m)_{\alpha\delta}(\gamma^m)^{\beta\sigma}(\gamma^p)_{\sigma\kappa}(\gamma_p)^{\alpha\rho} = -4\gamma_r^{\beta\rho}\gamma^r_{\delta\kappa} + 12\delta^\beta_\kappa\delta^\rho_\delta + 8\delta^\beta_\delta\delta^\rho_\kappa,$$

it follows that

$$12\nabla_\kappa W^\rho + 6\gamma^p_{\sigma\kappa}\gamma^{\alpha\rho}_p\nabla_\alpha W^\sigma = 6(\gamma^{mn})_\kappa{}^\rho F_{mn}. \tag{B.15}$$

Finally, from (B.14) and (B.15) we get $\nabla_\alpha W^\beta = -\frac{1}{4}(\gamma^{mn})_\kappa{}^\rho F_{mn}$.

Lemma 2. *The following identity holds true (written with the pure spinor λ^a for convenience)*

$$\lambda^\alpha D_\alpha A_m = (\lambda\gamma^m W) + \partial^m(\lambda A). \tag{B.16}$$

The proof follows trivially from (B.5) and (B.9).

Lemma 3. *The superfields A_m, W^α and F_{mn} can be written as*

$$A_m = \frac{1}{8}\gamma^{\alpha\beta}_m(D_\alpha A_\beta + A_\alpha A_\beta) \tag{B.17}$$

$$W^\alpha = \frac{1}{10}(\gamma^m)^{\alpha\beta}\left(D_\beta A_m - \partial_m A_\beta + [A_\beta, A_m]\right) \tag{B.18}$$

$$F_{mn} = \nabla_m A_n - \nabla_n A_m \tag{B.19}$$

Proposition 3. *The constraint equation*

$$\gamma^{\alpha\beta}_{mnpqr}\left(D_\alpha A_\beta + D_\beta A_\alpha + \{A_\alpha, A_\beta\}\right) = 0 \tag{B.20}$$

is equivalent to the super Yang-Mills equations

$$\gamma^m_{\alpha\beta}\nabla_m W^\beta = 0 \tag{B.21}$$

$$\nabla_m F^{mn} + \frac{1}{2}\gamma^n_{\alpha\beta}\{W^\alpha, W^\beta\} = 0. \tag{B.22}$$

From Proposition 2 we know that the constraint (B.20) implies the equations (B.10) and (B.11). Now we will show that (B.21) and (B.22) follow from those two equations, which proves the above proposition.

To prove (B.21) we act with the derivative ∇_γ over (B.10) and symmetrize over the spinor indices $(\gamma\alpha)$ to obtain

$$(\nabla_\alpha\nabla_\gamma + \nabla_\gamma\nabla_\alpha)W^\beta = -\frac{1}{4}(\gamma^{mn})_\gamma{}^\beta\nabla_\alpha F_{mn} - \frac{1}{4}(\gamma^{mn})_\alpha{}^\beta\nabla_\gamma F_{mn}$$

Using (B.4) in the left hand side and (B.11) on the right we get

$$\gamma^p_{\alpha\beta}\nabla_p W^\beta = -\frac{1}{2}(\gamma^{mn})_\gamma{}^\beta(\gamma_m\nabla_n W)_\alpha - \frac{1}{2}(\gamma^{mn})_\alpha{}^\beta(\gamma_m\nabla_n W)_\gamma,$$

from which we obtain upon multiplication by δ^γ_β on both sides and using $\gamma^{mn}_m = 7\gamma^n$ that

$$\gamma^p_{\alpha\beta}\nabla_p W^\beta = 0, \tag{B.23}$$

which proves (B.21). To obtain (B.22) we multiply (B.23) by $\gamma^{\alpha\delta}_n\nabla_\delta$ to get

$$\gamma^m_{\alpha\beta}\gamma^{\alpha\delta}_n\nabla_\delta\nabla_m W^\beta = 0.$$

Using $\nabla_\delta\nabla_m = [\nabla_\delta, \nabla_m] + \nabla_m\nabla_\delta$, $[\nabla_\delta, \nabla_m] = (\gamma_m W)_\delta$ and the equation of motion (B.10) we arrive at

$$\frac{1}{4}\mathrm{tr}(\gamma_n\gamma^{pq}\gamma_m)\nabla_m F_{pq} = (\gamma^m\gamma_n\gamma_m)_{\beta\kappa}W^\kappa W^\beta$$

which implies

$$\nabla^m F_{mn} = (\gamma_n)_{\alpha\beta}W^\beta W^\alpha = -\frac{1}{2}\{W^\alpha, W^\beta\}.$$

B.1 The θ-expansion of Super Yang-Mills superfields

In the sequence we use the following $\mathcal{N} = 1$ super-Yang-Mills θ expansions [38][39][40]

$$A_\alpha(x,\theta) = \frac{1}{2}a_m(\gamma^m\theta)_\alpha - \frac{1}{3}(\xi\gamma_m\theta)(\gamma^m\theta)_\alpha - \frac{1}{32}F_{mn}(\gamma_p\theta)_\alpha(\theta\gamma^{mnp}\theta)$$

$$+\frac{1}{60}(\gamma_m\theta)_\alpha(\theta\gamma^{mnp}\theta)(\partial_n\xi\gamma_p\theta) + \dots$$

$$A_m(x,\theta) = a_m - (\xi\gamma_m\theta) - \frac{1}{8}(\theta\gamma_m\gamma^{pq}\theta)F_{pq} + \frac{1}{12}(\theta\gamma_m\gamma^{pq}\theta)(\partial_p\xi\gamma_q\theta) + \dots$$

$$W^\alpha(x,\theta) = \xi^\alpha - \frac{1}{4}(\gamma^{mn}\theta)^\alpha F_{mn} + \frac{1}{4}(\gamma^{mn}\theta)^\alpha(\partial_m\xi\gamma_n\theta) + \frac{1}{48}(\gamma^{mn}\theta)^\alpha(\theta\gamma_n\gamma^{pq}\theta)\partial_m F_{pq} + \dots$$

$$\mathcal{F}_{mn}(x,\theta) = F_{mn} - 2(\partial_{[m}\xi\gamma_{n]}\theta) + \frac{1}{4}(\theta\gamma_{[m}\gamma^{pq}\theta)\partial_{n]}F_{pq} + \dots. \tag{B.24}$$

Here $\xi^\alpha(x) = \chi^\alpha e^{ik\cdot x}$ and $a_m(x) = e_m e^{ik\cdot x}$ describe the gluino and gluon respectively, while $F_{mn} = 2\partial_{[m}a_{n]}$ is the gluon field-strength.

B.2 Equations of motion in $U(5)$-covariant notation

If the equations of motion for the SYM superfields in $SO(10)$-covariant notation are given by yyy, then their $U(5)$-covariant form is the following

$$D_a W^+ = 0, \quad D_a W_{bc} = \epsilon_{abcde} F^{de} \quad D_a W^b = F_a^b,$$

$$D_+ W^a = 0, \quad D_+ W_{ab} = F_{ab}, \quad D_+ W^+ = F$$

Appendix C

The t_8 tensor

The famous t_8 tensor is defined by

$$t_8^{m_1n_1m_2n_2m_3n_3m_4n_4} = -\frac{1}{2}\Big[\,(\delta^{m_1m_2}\delta^{n_1n_2} - \delta^{m_1n_2}\delta^{n_1m_2})\,(\delta^{m_3m_4}\delta^{n_3n_4} - \delta^{m_3n_4}\delta^{n_3m_4}) \tag{C.1}$$

$$+\,(\delta^{m_2m_3}\delta^{n_2n_3} - \delta^{m_2n_3}\delta^{n_2m_3})\,(\delta^{m_4m_1}\delta^{n_4n_1} - \delta^{m_4n_1}\delta^{n_4m_1})$$

$$+\,(\delta^{m_1m_3}\delta^{n_1n_3} - \delta^{m_1n_3}\delta^{n_1m_3})\,(\delta^{m_2m_4}\delta^{n_2n_4} - \delta^{m_2n_4}\delta^{n_2m_4})\,\Big]$$

$$+\frac{1}{2}\Big[\delta^{n_1m_2}\delta^{n_2m_3}\delta^{n_3m_4}\delta^{n_4m_1} + \delta^{n_1m_3}\delta^{n_3m_2}\delta^{n_2m_4}\delta^{n_4m_1} + \delta^{n_1m_3}\delta^{n_3m_4}\delta^{n_4m_2}\delta^{n_2m_1}$$

$$+45 \text{ terms obtained by antisymmetrizing on each pair of indices}\Big].$$

One can check that its contraction with four field-strengths F_{mn} gives the following expression

$$t_8F^4 = 8(F^1F^2F^3F^4) + 8(F^1F^3F^2F^4) + 8(F^1F^3F^4F^2)$$

$$-2(F^1F^2)(F^3F^4) - 2(F^2F^3)(F^4F^1) - 2(F^1F^3)(F^2F^4),$$

which is a convenient way of summarizing the t_8 tensor. One can also check that in terms of components

$$t_8F^4 = \frac{1}{2}\left[-(k^2\cdot e^3)(k^2\cdot e^4)(e^1\cdot e^2)t - (k^2\cdot e^4)(k^4\cdot e^3)(e^1\cdot e^2)t\right.$$

$$+(k^2\cdot e^4)(k^3\cdot e^2)(e^1\cdot e^3)t - (k^3\cdot e^4)(k^4\cdot e^2)(e^1\cdot e^3)t + (k^2\cdot e^3)(k^4\cdot e^2)(e^1\cdot e^4)t$$

$$+(k^4\cdot e^2)(k^4\cdot e^3)(e^1\cdot e^4)t - (k^2\cdot e^4)(k^3\cdot e^1)(e^2\cdot e^3)t - (k^2\cdot e^3)(k^4\cdot e^1)(e^2\cdot e^4)t$$

$$-(k^3\cdot e^1)(k^4\cdot e^3)(e^2\cdot e^4)t - (k^4\cdot e^1)(k^4\cdot e^3)(e^2\cdot e^4)t + (k^3\cdot e^1)(k^4\cdot e^2)(e^3\cdot e^4)t$$

$$-(k^2\cdot e^3)(k^2\cdot e^4)(e^1\cdot e^2)u - (k^2\cdot e^3)(k^3\cdot e^4)(e^1\cdot e^2)u + (k^2\cdot e^4)(k^3\cdot e^2)(e^1\cdot e^3)u$$

$$+(k^3\cdot e^2)(k^3\cdot e^4)(e^1\cdot e^3)u+(k^2\cdot e^3)(k^4\cdot e^2)(e^1\cdot e^4)u-(k^3\cdot e^2)(k^4\cdot e^3)(e^1\cdot e^4)u$$
$$-(k^2\cdot e^4)(k^3\cdot e^1)(e^2\cdot e^3)u-(k^3\cdot e^1)(k^3\cdot e^4)(e^2\cdot e^3)u-(k^3\cdot e^4)(k^4\cdot e^1)(e^2\cdot e^3)u$$
$$-(k^2\cdot e^3)(k^4\cdot e^1)(e^2\cdot e^4)u+(k^3\cdot e^2)(k^4\cdot e^1)(e^3\cdot e^4)u+\frac{1}{2}(e^1\cdot e^3)(e^2\cdot e^4)t^2+$$
$$\frac{1}{2}(e^1\cdot e^4)(e^2\cdot e^3)u^2+\frac{1}{2}(e^1\cdot e^4)(e^2\cdot e^3)tu+\frac{1}{2}(e^1\cdot e^3)(e^2\cdot e^4)tu-\frac{1}{2}(e^1\cdot e^2)(e^3\cdot e^4)tu\Big],$$

which is a useful representation when comparing against scattering amplitude computations.

The t_8 tensor can also be represented in a $U(5)$-covariant fashion by taking (1.15) and going to the λ^+-frame

$$t_8F^4=\epsilon_{abcde}\langle(\lambda^+)^3\theta^aW^bW^cW^dW^e\rangle=\epsilon_{abcde}\langle(\lambda^+)^3F^b_fF^c_gF^d_hF^e_i\theta^a\theta^f\theta^g\theta^h\theta^i\rangle$$
$$=\delta^{fghi}_{bcde}F^b_fF^c_gF^d_hF^e_i.$$

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