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Filtro de Kalman Restrito: Teoria, Métodos e Aplicações

Tese de Doutorado

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Resumo

Pizzinga, Adrian Heringer; Fernandes, Cristiano Augusto Coelho (Orientador). **Filtro de Kalman Restrito: Teoria, Métodos e Aplicações**. Rio de Janeiro, 2008. 70p. Tese de Doutorado – Departamento de Engenharia Elétrica, Pontifícia Universidade Católica do Rio de Janeiro.

Nesta Tese, eu me concentro em desenvolvimentos sobre o filtro de Kalman sujeito a restrições lineares gerais. Há essencialmente três tipos de contribuições: (i) provas alternativas para resultados previamente estabelecidos na literatura sobre o filtro de Kalman com restrições; (ii) resultados que presumidamente destacam aspectos teóricos e metodológicos para modelagens em espaço de estado sob restrições; e (iii) aplicações que parecem ser inéditas até então em finanças (análise de investimentos) e em macroeconomia, nas quais os métodos propostos são ilustrados e avaliados. No final, eu sugiro algumas extensões adicionais sobre o tema, as quais, novamente, dividem-se em teoria, métodos e aplicações.

Palavras-chave

filtro de Kalman, modelos em espaço de estado, restrições lineares.

Abstract

Pizzinga, Adrian Heringer; Fernandes, Cristiano Augusto Coelho (Advisor). **Restricted Kalman Filtering: Theory, Methods and Applications.** Rio de Janeiro, 2008. 70p. PhD Thesis – Departamento de Engenharia Elétrica, Pontifícia Universidade Católica do Rio de Janeiro.

In this Thesis, I bring the attention to developments on Kalman filtering subject to general linear constraints. There are essentially three kinds of contributions: (i) new proofs for already established results within the restricted Kalman filtering literature; (ii) new results which are supposed to shed light on theoretical and methodological frameworks for linear state space modeling under linear restrictions; and (iii) applications that seem to be new in investment analysis and in macroeconomics, where the proposed methods are illustrated and evaluated. At the end, I suggest some further extensions in the subject, which, again, step into theory, methods and applications.

Keywords

linear restrictions, Kalman filtering, state space models.

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1
Introduction

1.1
Motivation

In some relevant practical situations from areas like economics, ¯nance, actuary, engineering etc, some state space models (Harvey, 1989; Brockwell and Davis, 1991; and Durbin and Koopman, 2001) would make more sense if estimated under some meaningful restrictions on the state vector.

Some examples:

{ One may consider state space models to conduct time-varying econometric models where well-established economic restrictions on the coe±cients should be at least attested.

{ Statistical models raised from physical considerations sometimes make sense only if considered under symmetry constraints on their parameters, whenever they are ¯xed or stochastically varying (cf. Pizzinga, Ruggeri and Guedes, 2005).

{ In the claims reserving issue, some dynamic models for *runo® data* (cf. de Jong and Zehnwirth, 1983) may have columns and/or nonnegativity restrictions in the development/delay e®ect.

{ Dynamic factor models for portfolio on-line recovering should be at least subject to accounting restrictions (eg. the portfolio allocations must add up to one, for every time period; cf. Pizzinga and Fernandes, 2006).

And, whenever one attempts to perform such constrained state estimation, some questions naturally arise. How should one implement this constrained estimation? When should this estimation be done? Which statistical properties do these methods of estimation share? Which theoretical and/or computational complications could emerge? Are all possible types of restrictions handleable? Are the imposed restrictions checkable for their plausibility under a speci¯c method? And what could be said about the initialization of the recursions from these restricted estimations?

This Doctorate Thesis concentrates on additional developments concerning the *restricted Kalman ¯ltering*, appropriate to problems that demand *linear* restrictions in the setting of *linear* state space models. By ranging from theoretical results and new methods to illustrative applications, the contributions address most - if not all - of the previous questions.

In the sequel, I will detail each one of the presumed novelties of the Thesis. But, ¯rst, let me review some literature on the subject.

1.2
A glimpse at the literature

There are essentially two directions the literature about linear state space models under restrictions has taken, one more \statistical-like"and another more \engineering-like". One very interesting point I have noticed was the absence of cross-references between these two ¯elds. The result: an unavoidable overlapping between contributions coming from both \worlds".

1.2.1
Statistical papers

From the statistical/econometric side, I understand that Doran (1992) is the most seminal paper on the subject, in which the restricted Kalman ¯ltering by augmentation was proved, under lots of matrix operations, for update and smoothing equations. At the end of his paper, Doran also made an attempt to further extend his approach for cases of nonlinear restrictions, but at least ¯rst-order di®erentiable. During the next ¯ve years, Doran have published two more papers. In Doran (1996), his previous approach was used in a problem of estimating Australian provincial populations according to annual national population. And, in Doran and Rambaldi (1997), the same approach was once more evoked to solve the problem of estimating time-varying econometric models (demand systems to be exact) also under time-varying and quite interpretable restrictions; in that same paper, the authors also discussed the relevant question concerning numerical optimization for the maximum likelihood estimation of unknown parameters. Some of those Doran's three important papers have been mentioned in the literature - mainly the very ¯rst one -, and his approach has been revisited as well. See for instance the book by Durbin and Koopman (2001), subsection 6.5.

Another more recent work on the subject is the paper due to Pandher (2002), which again cited Doran in his bibliographic review. In his paper, Pandher was fully concerned about forecasting multivariate time series under linear restrictions. His approach di®ered from the one proposed by Doran, although we can note again some augmentation strategy under a structural modeling framework. Pandher also gave some results on statistical e±ciency of the state prediction under restrictions and on observability of the state vector under his method. At last, Pandher recognized another previous and related work when he cites Leybourne (1993), who had tackled univariate state space models under time invariant restrictions over a random walk state vector.

1.2.2
Engineering papers

Now, I take a quick look at engineering articles on restricted Kalman ¯ltering. For instance, I can cite the papers by Wen and Durrant-Whyte (1991), Massicotte *et al.* (1995) and Geeter *et al.* (1997), none of which has referred to or has been referred to by the statisticians previously revisited in subsection 1.2.1. But two speci¯c and much more recent papers deserve a little bit of attention.

The ¯rst paper was written by Simon and Chia (2002), who derived at least two di®erent versions of the restricted Kalman ¯ltering by alternative perspectives, even though making use of the same type of solution, the Lagrange multipliers approach. Their ¯rst version of the Kalman ¯ltering is, in fact, almost the same as that originally obtained by Doran (1992) for the updating equations. So, this serves as an example of the aforementioned overlapping among the literature. Simon and Chia also presented ¯ve theorems that uncover good properties of their developed restricted Kalman ¯ltering; four of them relate to mean square error e±ciency. The methodological/theoretical part of the paper is closed by a discussion on how nonlinear restrictions could be encompassed by their restricted Kalman ¯ltering. And the outcome is another material previously released by Doran (1992), although apparently not perceived by Simon and Chia.

The second paper, by Simon and Simon (2004), clearly continues the ¯rst paper's *leitmotiv* by trying to incorporate inequality constraints in the Kalman ¯ltering. The authors solved the task by using quadratic programming and the interesting ¯nd was that the problem actually remains as a special case of imposing equality constraints.

1.3
This Thesis's contributions

The speci¯c contributions of my research consist of investigation and development of the following topics:

1. A more general and more elegant proof of the restricted Kalman ¯ltering (updating and smoothing equations) which uses *Hilbert space geometry*. This new proof is compared with those previous from Doran (1992).

2. An alternative proof based on Kalman recursions of the restricted Kalman ¯ltering. The importance of this alternative proof lies on its idea of *re-writing the augmented model* in a useful and equivalent way, which would be the building block for other methods and results to be also presented.

3. An alternative and \constructive"proof of the *statistical e±ciency* from the restricted Kalman ¯ltering, which is compared with the proof already presented in my Master dissertation (cf. Pizzinga, 2004).

4. An alternative approach for imposing time-invariant restrictions to the estimation of random-walk state vectors.

5. A comparison between the restricted Kalman ¯ltering and the restricted *recursive* least squares, and the establishment of the equivalence between both techniques under a particular albeit relevant case.

6. Development and implementation of a new restricted Kalman ¯ltering under a *reduced modeling approach*, followed by detailed confrontations with the usual restricted Kalman ¯ltering by augmentation.

7. Development of a *restricted Kalman predictor* which is applicable to general situations and which encompasses the method by Pandher (2002) as a particular case.

8. An alternative, \parametric", very short and quite general proof of the restricted Kalman ¯ltering under a *conditional expectation* framework, followed by comparisons with previous demonstrations.

9. The proof that the initial exact Kalman smoother (cf. Durbin and Koopman, 2001, ch. 5) still yields restricted smoothed state vectors within the \di®use" period, whenever applied to an appropriate augmented model.

10. A practical illustration in ¯nance, in which a dynamic factor model under a linear and interpretable restriction is used to understand the style of Brazilian exchange rate funds.

11. An application in macroeconomics, in which dynamic models for exchange rate past-through are proposed and estimated with Brazilian price indexes.

12. A practical illustration in macroeconomics, in which a univariate benchmarking model, recognized as a linear state space model under restrictions, is used to predict the Brazilian GNP.

If I have to classify the above contributions into three kinds, I might do as follows:

- Contributions 1, 2, 3, 5, 8 and 9 bear *theory*;
- Contributions 4, 6 and 7 bear *methods*;
- Contributions 10, 11 and 12 o®er three *applications*.

1.4 Thesis's organization

This Thesis is arranged as follows. Chapter 2 revisits the essentials of linear state space models and the Kalman ¯ltering. Chapter 3 is entirely dedicated to the theoretical issues concerning the imposition of linear restrictions to the Kalman equations, in which new proofs for already established results are given and, in addition, new results are derived. Chapter 4 focuses on some new methods that can be potentially useful in situations of linear state space modeling under linear restrictions on the state vector. Chapter 5 o®ers the aforementioned applications in ¯nance and macroeconomics, which illustrate the performance of some methods discussed in chapter 4. Finally, chapter 6 closes the Thesis by suggesting some additional research topics.

2
Linear state space models and the Kalman filtering

2.1
The model

A *linear wide sense* state space model for an observable *p*-variate stochastic process Y_t, de¯ned on an appropriate probability space (-*;* $\mathcal{F}$*;* $\mathcal{P}$), is described by the following set of equations:

$$
\begin{aligned}
Y_t &= Z_t ®_t + d_t + "_t \\
®_{t+1} &= T_t ®_t + c_t + R_t ´_t \\
®_1 &\sim (a_1; P_1):
\end{aligned}
\tag{2-1}
$$

The ¯rst equation is usually called the *measurement equation* and the second is known as the *state equation*. The unobservable *m*-variate process $®_t$ is termed *state*. The error terms $"_t$ and $´_t$ are respectively *p*-variate and *r*-variate second-order processes that are uncorrelated in time and from each other, with $var("_t) = H_t$ and $var(´_t) = Q_t$. The remaining *system matrices* $Z_t; d_t; H_t; T_t; c_t; R_t$ and Q_t evolve deterministically.

2.2
The Kalman equations

In this Thesis, I will adopt the following notation:

{ $a_{t|j}$ is a (an equivalence class of) random vector(s) with coordinates $a_{ti|j}$, $i = 1; : : : ; m$, representing the unique linear orthogonal projection (cf. Kubrusly, 2001, Theorem 5.52), evaluated on each (equivalence class of) coordinate(s) $®_{ti}$ of $®_t$, onto $S' \equiv span\{1; Y_{11}; : : : ; Y_{1p}; : : : ; Y_{j1}; : : : ; Y_{jp}\} \subseteq L_2 \equiv L_2(-; \mathcal{F}; \mathcal{P})$ - the subjacent topology is that induced by the usual inner product, which is given by $< X; Y > \equiv E(XY) = \int_\Omega X(!)Y(!)\mathcal{P}(d!); \forall X; Y \in L_2$.

{ $P_{t|j} \equiv E\left[(®_t - a_{t|j})(®_t - a_{t|j})'\right]$;

{ $À_t \equiv Y_t - Z_t a_{t|t-1} - d_t$ and $F_t \equiv E(À_t À_t') = Z_t P_{t|t-1} Z_t' + H_t$.

The *Kalman ¯ltering* (prediction, updating and smoothing) gives the above orthogonal projections evaluations and the corresponding mean square error matrices. The corresponding equations are given as follows:

{ Prediction equations

$$
\begin{aligned}
a_{t+1|t} &= T_t a_{t|t} + c_t \\
P_{t+1|t} &= T_t P_{t|t} T_t' + R_t Q_t R_t'
\end{aligned}
\tag{2-2}
$$

{ Updating or ¯ltering equations

$$
\begin{aligned}
a_{t|t} &= a_{t|t-1} + P_{t|t-1} Z_t' F_t^{-1} \text{À}_t \\
P_{t|t} &= P_{t|t-1} - P_{t|t-1} Z_t' F_t^{-1} Z_t P_{t|t-1}
\end{aligned}
\tag{2-3}
$$

{ Smoothing equations (for a given $n \geq t$)

$$
\begin{aligned}
a_{t|n} &= a_{t|t-1} + P_{t|t-1} r_{t-1} \\
r_{t-1} &= Z_t' F_t^{-1} \text{À}_t + (T_t - T_t P_{t|t-1} Z_t' F_t^{-1} Z_t)' r_t \\
P_{t|n} &= P_{t|t-1} - P_{t|t-1} N_{t-1} P_{t|t-1} \\
N_{t-1} &= Z_t' F_t^{-1} Z_t + (T_t - T_t P_{t|t-1} Z_t' F_t^{-1} Z_t)' N_t (T_t - T_t P_{t|t-1} Z_t' F_t^{-1} Z_t) \\
r_n &= 0 \text{ and } N_n = 0
\end{aligned}
\tag{2-4}
$$

Details concerning the derivations of these formulae are found in Harvey (1989), Brockwell and Davis (1991), Harvey (1993), de Jong (1989), Hamilton (1994), Tanizaki (1996), Durbin and Koopman (2001), Brockwell and Davis (2003) and Shumway and Sto®er (2006).

2.3 Introducing linear restrictions

From now on, it is assumed that the process ®_t in (2-1) satis¯es linear restrictions as follows:

Assumption 1 *The random vectors ®_t satisfy the following (possibly time varying) linear restrictions*

$$A_t \text{®}_t = q_t; \tag{2-5}$$

where, for each t, A_t is a $k \times m$ matrix and q_t is a $k \times 1$ (possibly random) vector.

Observe that the restrictions enunciated in eq.(2-5) are rather general. In fact, it encapsulates *a±ne* restrictions of the kind $A_t \text{®}_t + b_t = q_t$ by de¯ning $q_t' = q_t - b_t$ and allows the number of restrictions k to be time-varying. In practical situations, justi¯cation of such constraints in (2-5) arises naturally from the characteristics of the problem being modeled; see for instance the restrictions imposed on a demand system problem in Doran and Rambaldi (1997).

In the remaining of the Thesis, Assumption 1 will be considered in almost every topic to be discussed and, in due course, it may be added with some further structure on the linear restrictions.

3
Restricted Kalman filtering: theoretical issues

During all this chapter, I will discuss several topics concerning the theory of imposing the linear restrictions enunciated under a quite general form in (2-5) from Assumption 1. In section 3.1, I will present and compare three di®erent derivations of the restricted Kalman updating and smoothing equations under an augmented modeling approach. In section 3.2, I prove the statistical e±ciency due to the imposition of restrictions, and this shall be done using a geometrical framework. Stepping forward, I try in section 3.3 to establish the equivalence between the restricted Kalman ¯ltering and something that could be termed *recursive restricted least squares* estimator. Finally, in section 3.4, I investigate how initial di®use state vectors a®ect the use of the Kalman smoother under linear restrictions.

3.1
Augmented restricted Kalman filtering: alternative proofs

3.1.1
Geometrical proof

When estimating state space models under linear restrictions as given in eq. (2-5), the natural task is to impose these very restrictions on the state estimators given by the Kalman equations in order to obtain a more meaningful result. The following theorem guarantees that such task is possible for the updating and smoothing equations, whenever one adopts an augmented measurement equation:

Theorem 1 *If the measurement vectors Y_t are replaced by $Y_t^* = (Y_t'; q_t')'$, the matrices Z_t are replaced by $Z_t^* = [Z_t' \; A_t']'$, the vectors d_t are replaced by $d_t^* = (d_t'; 0')'$, and the measurement equation error vectors ε_t are replaced by $\varepsilon_t^* = (\varepsilon_t'; 0')'$, then the Kalman updating and smoothing equations applied to this new linear state space models satisfy the same linear restrictions given in (2-5), that is,*

$$A_t a_{t|t} = q_t \tag{3-1}$$

$$A_t a_{t|n} = q_t \tag{3-2}$$

First proof of Theorem 1: Denote the subspace generated by the augmented measurements up to time j, where $j \in \{t; t+1; :::; n\}$, by $S'' = span\{1; Y_{11}; :::; Y_{1p}; q_{11}; :::; q_{1k}; :::; Y_{j1}; :::; Y_{jp}; q_{j1}; :::; q_{jk}\}$, the unique linear orthogonal projection onto S'' by $¼_{S''}$ and the i^{th} row from A_t by $A_{ti} = [c_{ti1} ::: c_{tim}]$. Then, by making use of linearity of $¼_{S''}$ and of the linear restrictions established in Assumption 1, it follows that

$$\begin{aligned} A_{ti}a_{t|j} &= c_{ti1}a_{t1|j} + \cdots + c_{tim}a_{tm|j} = c_{ti1}¼_{S''}(®_{t1}) + \cdots + c_{tim}¼_{S''}(®_{tm}) \\ &= ¼_{S''}(c_{ti1}®_{t1} + \cdots + c_{tim}®_{tm}) = ¼_{S''}(A_{ti}®_t) = ¼_{S''}(q_{ti}) \\ &= q_{ti}\, ; \end{aligned}$$

where the last equality comes from the fact that q_{ti} belongs to $\mathcal{R}(¼_{S''}) = S''$. Since i is arbitrary, the theorem is proved. □

Theorem 1 was originally due to Doran (1992, p.570, 571), but our proof also reveals some gains:

1. It does not presume that F_t is invertible for all t.
2. It uni¯es in a single argument both updating and (*any type of*) smoothing equations.
3. It does not make any explicit use of Kalman updating or smoothing equations.
4. It is a shorter and consequently more elegant proof.

The ¯rst methodological contribution of this proof, namely the guarantee that the augmented measurement procedure is able to deal with *any* type of linear restriction, is directly related to item 1. Many examples of restrictions that would decrease the rank of F_t are of deterministic nature, whether they originate from economic theories or not (to be even more speci¯c: consider for instance the *portfolio* accounting restriction in time-varying extensions of the asset class factor models due to Sharpe, 1992). The second contribution, related to item 2, is that *any* set of state smoothing - from which we mention the traditional ¯*xed-interval*, ¯*xed-point* and ¯*xed-lag* estimators (cf. Anderson and Moore, 1979) - must yield restricted estimated states.

The following consequence from Theorem 1 has already proved to be useful, once it was conveniently used by Doran (1996) in a state space estimation of population totals.

Corollary 1 *If some univariate equations of the measurement vector Y_t have errors with zero variance, then*

$$Z_{t2}a_{t|t} = Y_{t2} \text{ and } Z_{t2}a_{t|n} = Y_{t2} \tag{3-3}$$

where Z_{t2} is the block from Z_t that corresponds to the block Y_{t2} from Y_t whose coordinates have null variance errors.

Proof: It is enough to see that Y_t can be written as $[Y'_{t1}\ Y'_{t2}]'$ and that Z_t, in turn, can be written as $[Z'_{t1}\ Z'_{t2}]'$. Establishing that $A_t = Z_{t2}$ and $q_t = Y_{t2}$, Theorem 1 guarantees the desired result. □

3.1.2
Computational proof

During this subsection, I shall consider the following additional structure to the restrictions in (2-5):

Assumption 2 *The linear restrictions in (2-5) are such that the coordinates of q_t are linearly independent in L_2 and from $1; Y_{11}; : : : ; Y_{1p}; : : : ; Y_{t1}; : : : ; Y_{tp}$. Also, suppose that $F_t > 0$ for all t.*

For the Kalman updating and smoothing equations, it is in fact an attainable task, as Theorem 1 says, to carry out Kalman ¯ltering estimations under the above these linear restrictions. Here, this is now proved by explicitly using updating and smoothing equations, even though under strategies somewhat di®erent from those tackled by Doran (1992).

Second proof of Theorem 1: Uncouple the augmented model by recognizing that, for all t, q_t is a \new" measurement vector that is observed \after" Y_t and \before" Y_{t+1}. This recognition leads to a new linear state space representation entirely equivalent to the augmented model. The measurement equation for this representation is de¯ned by

$$Y_{t,j} = Z_{t,j}\text{®}_{t,j} + d_{t,j} + "_{t,j}\ ;\ "_{t,j} \sim (0; H_{t,j}) \tag{3-4}$$

When $j = 1$, nothing is changed from the measurement equation from (2-1) of section 2.1. But for $j = 2$ we must have

$$Y_{t,2} = q_t; \quad Z_{t,2} = A_t; \quad d_{t,2} = 0 \text{ and } H_{t,2} = 0: \tag{3-5}$$

Regarding the state equation, just notice that, for all t, $\text{®}_{t,2} = \text{®}_{t,1}$ and $\text{®}_{t+1,1} = T_t\text{®}_{t,2} + c_t + R_t\text{´}_t$, $\text{´}_t \sim (0; Q_t)$. Within this equivalent framework, it becomes possible to treat the imposing of the linear restriction in time t as a new update of the state vector. Implementing: consider the state updating equation given in (2-3), already applied to the above equivalent model for t ¯xed and $j = 2$:

$$\begin{aligned} a_{t,2|t,2} &= a_{t|t-1,2} + P_{t|t-1,2}Z'_{t,2}F_{t,2}^{-1}(Y_{t,2} - Z_{t,2}a_{t|t-1,2}) \\ &= a_{t|t-1,2} + P_{t|t-1,2}Z'_{t,2}(Z_{t,2}P_{t|t-1,2}Z'_{t,2} + H_{t,2})^{-1}(Y_{t,2} - Z_{t,2}a_{t|t-1,2}) \\ &= a_{t|t-1,2} + P_{t|t-1,2}A'_t(A_tP_{t|t-1,2}A'_t)^{-1}(q_t - A_ta_{t|t-1,2}); \end{aligned}$$

where the second equality comes from the very expression of F_t (cf. the established notation in section 2.2) and the third comes from (3-5). Now, since $(A_tP_{t|t-1,2}A'_t)^{-1}$ is a genuine inverse (cf. Assumption 2) and $a_{t,2|t,2} = a_{t|t}$, this last updated state vector being the one associated with the augmented model, pre-multiply both sides of the last identity by A_t in order to get (3-1).
Now, rephrase the state smoothing equations in (2-4) for the augmented model as follows:

$$\begin{aligned} a_{t|n} &= a_{t|t-1} + P_{t|t-1}r_{t-1} \\ r_{t-1} &= Z_t^{*\prime}F_t^{-1}\grave{A}_t + \left(T_t - T_tP_{t|t-1}Z_t^{*\prime}F_t^{-1}Z_t^*\right)' r_t; \quad \text{where} \quad Z_t^* = \begin{bmatrix} Z_t \\ A_t \end{bmatrix}: \end{aligned}$$

Of course, other quantities would also have deserved asterisks, but they are suppressed for ease of notation. Placing the expression of r_t in $a_{t|n}$, it follows that

$$\begin{aligned} a_{t|n} &= a_{t|t-1} + P_{t|t-1}(Z_t^{*\prime}F_t^{-1}\grave{A}_t + (T_t - T_tP_{t|t-1}Z_t^{*\prime}F_t^{-1}Z_t^*)'r_t) \\ &= a_{t|t-1} + P_{t|t-1}Z_t^{*\prime}F_t^{-1}\grave{A}_t + P_{t|t-1}(T_t - T_tP_{t|t-1}Z_t^{*\prime}F_t^{-1}Z_t^*)'r_t \\ &= a_{t|t} + (P_{t|t-1}T_t' - P_{t|t-1}Z_t^{*\prime}F_t^{-1}Z_t^*P_{t|t-1}T_t')r_t; \end{aligned}$$

where the last equality follows from the Kalman updating equation in (2-3). Pre-multiplying both sides by A_t, it follows that

$$A_ta_{t|n} = A_ta_{t|t} + (A_tP_{t|t-1}T_t' - A_tP_{t|t-1}Z_t^{*\prime}F_t^{-1}Z_t^*P_{t|t-1}T_t')r_t:$$

According to Doran (1992), eq. (22) (from Assumption 2, F_t from the augmented model is invertible), it follows that $A_tP_{t|t-1}Z_t^{*\prime}F_t^{-1} = \begin{bmatrix} \underset{k\text{x}p}{0} & \underset{k\text{x}k}{I} \end{bmatrix}$: Use this together with (3-1) already proved to obtain

$$\begin{aligned} A_ta_{t|n} &= q_t + \left(A_tP_{t|t-1}T_t' - [0 \quad I]\begin{bmatrix} Z_t \\ A_t \end{bmatrix}P_{t|t-1}T_t'\right)r_t \\ &= q_t + \left(A_tP_{t|t-1}T_t' - A_tP_{t|t-1}T_t'\right)r_t = q_t; \end{aligned}$$

which gives identity (3-2) □

It is worth to be noticed that there is no methodological novelty here. In turn, the contribution o®ered here comes from this second proof, which certainly deserves some quali¯cation. Even though not encompassing signi¯cant generalizations like those veri¯ed in the ¯rst proof, and besides being considerably longer, this second proof makes use of simple matrix operations, which illustrate potentially useful strategies that can be evoked more times in future research. Indeed, one should recall that quite the same decomposition used in the part of the proof related to the updating equations has been the great responsible for the well-known treatment of multivariate state space models under an univariate framework; cf. Durbin and Koopman (2001), section 6.4. On the other hand, the part related to the smoothing equation is entirely based on de Jong (1989)'s smoothing recursions, which are mathematically transparent and computationally e±cient.

3.1.3 Conditional expectation proof

The main goal of this subsection is to give a third and last proof for the augmented restricted Kalman ¯ltering. For such, I must add other structure (quite \traditional", we would say) to the linear state space model in (2-1).

Assumption 3 *$"_t$ and $´_t$ are independent (in time, between each other and of $®_1$) Gaussian stochastic processes. Also, $®_1$ is a Gaussian random vector.*

Besides considering this new \parametric"framework, denote by $\mathcal{F}_j$ the ¾-¯eld generated by the measurement vectors up to time j; that is $\mathcal{F}_j \equiv ¾(Y_1; : : : ; Y_j)$. Also set $\hat{a}_{t|j} \equiv E(®_t|\mathcal{F}_j)$ and $\hat{P}_{t|j} \equiv V(®_t|\mathcal{F}_j)$. Under Assumption 3, the Kalman recursions are versions of these conditional moments when $j = t - 1$, $j = t$ and $j = n$; see Anderson and Moore (1979), Harvey (1989), Harvey (1993), Tanizaki (1996) and Durbin and Koopman (2001). Consequently, all the properties of the conditional expectation can be conveniently used in order to allow a very quick proof for Theorem 1:

Third proof of Theorem 1: Let t be an arbitrary time instant. De¯ne $\mathcal{F}^*_j \equiv ¾(Y_1; q_1; : : : ; Y_j; q_j)$. Fixing j in $\{t; t + 1; : : : ; n\}$, it follows with probability 1 that

$$A_t\hat{a}_{t|j} = A_t E\left(®_t|\mathcal{F}^*_j\right) = E\left(A_t®_t|\mathcal{F}^*_j\right) = E\left(q_t|\mathcal{F}^*_j\right) = q_t; \qquad (3\text{-}6)$$

where the third equality is due to the restrictions in (2-5) and the fourth equality naturally comes from the very $\mathcal{F}^*_j$-measurability of q_t. Finally make $j = t$ and $j = n$. □

The most evident comparison between this third proof and the previous proofs is concerned with length and elegance. Besides, it maintains the same generality in terms of linear restrictions and state smoothing, which has been guaranteed already by the ¯rst proof given in subsection 3.1.1.

Now, on the potentially usefulness from this third proof:

1. The additional normality and independence assumptions, although restricting a little the scope of Theorem 1, can be considered an asset because these are straightforwardly generalizable to other types of state space models - the non-Gaussian and/or nonlinear state space models. The only drawback is that most of statistical techniques designed to handle more general state space models do require the existence of an expression for the conditional laws $p(y_t|®_t)$, which are obscured by the \singularity" incurred in the augmenting procedure.

2. Finally, the third proof plainly reveals that the *Bayesian approach* for state space modeling (cf. West and Harisson, 1997; Durbin and Koopman, 2001; and Shumway and Sto®er, 2006) can also deal with linear restrictions by adopting augmented measurement equations as well. Yet, one in such case shall be aware of some unavoidable singularities.

3.2 Statistical efficiency

In this section I demonstrate the statistical e±ciency - in terms of mean square estimation error - of the restricted Kalman ¯ltering discussed so far. For this, I shall make use of a geometrical perspective, something that might be general enough, while still grasping at intuition and simplicity. For what follows, it is important to bear in mind that the Kalman recursions, in addition to being recursive computational formulae from an operational standpoint, give linear orthogonal projections evaluations onto some speci¯c subspaces spanned by the model measurements.

I begin by quoting the well-established and useful fact:

Lemma 1 *Take a Hilbert space $\mathcal{H}$, two subspaces $\mathcal{M};\mathcal{N}$ of $\mathcal{H}$ and the linear orthogonal projections $¼_{\mathcal{M}}$ and $¼_{\mathcal{N}}$. If $\mathcal{M} \subseteq \mathcal{N}$, then, for each $x \in \mathcal{H}$, $¼_{\mathcal{M}}(¼_{\mathcal{N}}(x)) = ¼_{\mathcal{M}}(x)$.*

Proof: $\mathcal{N}$ is, by its own, a Hilbert space (because it is closed) and $\mathcal{M}$ is a closed subspace of $\mathcal{N}$. Then, using the Orthogonal Projection Theorem (cf. Theorem 5.20 of Kubrusly, 2001), we get $\mathcal{N} = \mathcal{M} + \left(\mathcal{M}^{\perp} \cap \mathcal{N}\right)$. So, from Proposition

5.58 of Kubrusly (2001), it follows that $¼_{\mathcal{N}} = ¼_{\mathcal{M}} + ¼_{\mathcal{M}^\perp \cap \mathcal{N}}$; tautologically, for each $x \in \mathcal{H}$,

$$¼_{\mathcal{N}}(x) = ¼_{\mathcal{M}}(x) + ¼_{\mathcal{M}^\perp \cap \mathcal{N}}(x): \qquad \text{(3-7)}$$

Now, apply $¼_{\mathcal{M}}$ on both sides of (3-7). □

Some additional notation must also be set:

{ $a_{t|j}$, $P_{t|j}$ and S' are de¯ned as previously and relate to the *standard* state space model;

{ $a^*_{t|j}$, $P^*_{t|j}$ and S'' are obtained with the *augmented* state space model, associated with Theroem 1

Now, everything needed for formally guaranteeing the statistical e±-ciency has been gathered. Two demonstrations are given. Both are based on a strong geometrical appeal and have an inductive style, in the sense that, ¯rstly, individual coordinates of the state vector are tackled and then, in a second moment, the strategy is generalized for arbitrary linear combinations of these coordinates. But they do di®er in some aspects. The ¯rst proof concentrates on the optimality of the linear orthogonal projection that comes directly from ¯rst principles, while the second proof is rather \constructive", uses Lemma 1 and focuses on a standard decomposition.

Theorem 2 $P^*_{t|j} \leq P_{t|j}$ *in the usual ordering of symmetric matrices.*

First proof of Theorem 2: Let $i = 1; : : : ; m$. Since the set containing the original model measurements until time j is contained in the correspondent set the from augmented model, it follows that $S' \subseteq S'' \equiv span\{1; Y_{11}; : : : ; Y_{1p}; q_{11}; : : : ; q_{1k}; : : : ; Y_{j1}; : : : ; Y_{jp}; q_{j1}; : : : ; q_{jk}\}$. Therefore, from Theorem 5.53 of Kubrusly (2001),

$$E\left[\left(®_{ti} - a^*_{ti|j}\right)^2\right] = \inf_{Y \in S''} E\left[(®_{ti} - Y)^2\right] \leq E\left[\left(®_{ti} - a_{ti|j}\right)^2\right]:$$

Generalizing: take $x = (x_1; : : : ; x_m)' \in R^m$. Using linearity, the linear orthogonal projections onto S' and onto S'', both evaluated in $x'®_t = x_1®_{t1} + \cdots + x_m®_{tm}$, are given by

$$x'a_{t|j} = x_1 a_{t1|j} + \cdots + x_m a_{tm|j}$$

and

$$x'a^*_{t|j} = x_1 a^*_{t1|j} + \cdots + x_m a^*_{tm|j}:$$

Observing that $x'a_{t|j} \in S''$ (because $a_{ti|j} \in S' \subseteq S'' \forall i = 1;:::;m$ and S'' is a linear manifold), it follows that

$$\begin{aligned} x'P^*_{t|j}x &= x'E\left[(®_t - a^*_{t|j})(®_t - a^*_{t|j})'\right]x = E\left[x'(®_t - a^*_{t|j})(®_t - a^*_{t|j})'x\right] \\ &= E\left[(x'®_t - x'a^*_{t|j})(x'®_t - x'a^*_{t|j})'\right] = E\left[\left(x'®_t - x'a^*_{t|j}\right)^2\right] \\ &= \inf_{Y \in S''} E\left[(x'®_t - Y)^2\right] \leq E\left[\left(x'®_t - x'a_{t|j}\right)^2\right] \\ &= E\left[(x'®_t - x'a_{t|j})(x'®_t - x'a_{t|j})'\right] = E\left[x'(®_t - a_{t|j})(®_t - a_{t|j})'x\right] \\ &= x'E\left[(®_t - a_{t|j})(®_t - a_{t|j})'\right]x = x'P_{t|j}x: \end{aligned}$$

Since x is arbitrary, the conclusion is that $P^*_{t|j}$ is, in fact, \less than or equal to" $P_{t|j}$. □

Second proof of Theorem 2: Consider again an arbitrary $i = 1;:::;m$. Recall once more that S' and S'' already de¯ned are subspaces (closed linear manifolds) of L_2 and that $S' \subseteq S''$. Theorem 5.20 of Kubrusly (2001) asserts the existence of $» \in S''^{\perp}$ such that

$$®_{ti} = a^*_{ti|j} + »: \tag{3-8}$$

Also, Theorem 5.20 of Kubrusly (2001) and Lemma 1 (make $\mathcal{H} = L_2$, $\mathcal{M} = S'$, $\mathcal{N} = S''$ and $x = ®_{ti}$) assure the existence of $º \in S'' \cap S'^{\perp}$ such that

$$a^*_{ti|j} = a_{ti|j} + º: \tag{3-9}$$

From the decomposition (3-8), obtain

$$E\left[\left(®_{ti} - a^*_{ti|j}\right)^2\right] = E\left(»^2\right): \tag{3-10}$$

Now, add both decompositions (3-8) and (3-9) in order to get $®_{ti} - a_{ti|j} = » + º$, and evoke Pythagorean theorem - which is licit since $»$ and $º$ are orthogonal - to have

$$E\left[\left(®_{ti} - a_{ti|j}\right)^2\right] = E\left(»^2\right) + E\left(º^2\right): \tag{3-11}$$

Identities (3-10) and (3-11) assert the claimed e±ciency for each coordinate estimation of the state vector. The case of an arbitrary linear combinations $x'®_t$ is dealt with in a similar fashion. □

Looking at cases in which $j \geq t$, the last theorem shows that Kalman updating and smoothing equations, when used with the augmented model, besides respecting the linear restrictions from equation (3-1), give more accurate estimators.

3.3 Restricted Kalman filtering versus restricted recursive least squares

Consider the following univariate special case of model (2-1) where the state vector is time invariant and $Z_t \equiv x_t'$ is a row vector of exogenous explanatory variables:

$$\begin{aligned} Y_t &= x_t'\beta_t + \varepsilon_t \ ; \quad \varepsilon_t \sim (0, \sigma^2) \\ \beta_{t+1} &= \beta_t. \end{aligned} \tag{3-12}$$

This model can and should be viewed as a linear regression model written in a linear state space representation. It is known (see Harvey, 1981 and 1993) that application of the Kalman state updating equation to (3-12) numerically coincides with the method of recursive least squares. Back in the days when matrix inversion was a computational burden, this equivalence proved useful since it turned out to be possible to estimate a regression model with no need to invert a \big" $X'X$ matrix. In addition, this made attainable the updating of ordinary least squares (OLS) estimates whenever new observations added to the data set. Nowadays this equivalence still deserves its merits in statistics and econometrics. Firstly, depending on the ill-conditioning of the regressors, it still may be di±cult to invert \big" $X'X$ matrices, a problem that justi¯es recursive estimation. Secondly, this equivalence is in full connection with the traditional coe±cients stability test by Brown *et al.* (1975).

The purpose of this section is to generalize the above parallel in the context of linear restrictions. I shall admit that the coe±cient vector of a regression model is supposed to obey certain linear restrictions which are enunciated as

$$A\beta = q, \tag{3-13}$$

where A is a known $k \times m$ matrix, $k \leq m$, and $q = (q_1, \ldots, q_k)'$ is a known $k \times 1$ vector. Since the main objective is to bridge the restricted recursive estimation to the restricted Kalman ¯ltering, structures (3-12) and (3-13) are now taken together to generate the following augmented measurement equation:

$$\begin{pmatrix} Y_t \\ q \end{pmatrix} = \begin{pmatrix} x_t' \\ A \end{pmatrix} \beta_t + \begin{pmatrix} \varepsilon_t \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \varepsilon_t \\ 0 \end{pmatrix} \sim \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma^2 & 0 \\ 0 & 0 \end{pmatrix} \right). \tag{3-14}$$

From Theorem 1, the application of the Kalman updating equation to the model in (3-14) produces updated state vectors which satisfy $Ab_{t|t} = q$. But, in fact, there is more: the terms of the sequence $\left(b_{t|t}\right)$ are the output from *on line* successive applications of restricted least squares. In order to establish this link, the restricted least square (RLS) estimator and its covariance matrix for a linear regression model $Y = X\beta + \varepsilon$, $\varepsilon \sim (0, \sigma^2 I)$, where β obeys (3-13), is recalled below:

$$\hat{\beta}_{RLS} = \hat{\beta}_{OLS} + (X'X)^{-1}A'[A(X'X)^{-1}A']^{-1}(q - A\hat{\beta}_{OLS}) \tag{3-15}$$
$$Var\left(\hat{\beta}_{RLS}\right) = \sigma^2(X'X)^{-1} - (X'X)^{-1}A'[A(X'X)^{-1}A']^{-1}A(X'X)^{-1}.$$

The derivation of the expression in (3-15) is presented in almost any book on econometrics. See for instance Johnston and Dinardo (1997) or Greene (2003).

This section's result:

Theorem 3 *Under the state space model in (3-14), the Kalman state updating equation is identical to a recursive application of (3-15).*

Proof: The model in (3-14) can be decomposed in a way that recognizes q as a "new" measurement vector obtained/observed just "after" Y_t and right "before" Y_{t+1}. Thus, the measurement equation is recast as

$$Y_{t,j} = Z_{t,j}\beta_{t,j} + \varepsilon_{t,j}\ ;\ \varepsilon_{t,j} \sim (0, H_{t,j}). \tag{3-16}$$

Notice that $Y_{t,1} = Y_t$, $Z_{t,1} = x'_t$ and $H_{t,1} = \sigma^2$; on the other hand $Y_{t,2} = q$, $Z_{t,2} = A$ and $H_{t,2} = 0$. The new state equation is written in the same way as before.

It now becomes possible to regard the imposing of the linear restrictions as a new updating of the state vector. In fact, for an arbitrary t, denote the output of the Kalman updating using all the measurements from eq.(3-16) up to $Y_{t,1}$ by $\hat{\beta}_{t,1|t,1}$, which, as already discussed, equals the output of the recursive least squares - and consequently the OLS estimator - applied to the "observations" $\{Y_1, q_1, ..., q_k, ..., Y_{t-1}, q_1, ..., q_k, Y_t\}$. The state equation implies that $\hat{\beta}_{t,2|t,1} = \hat{\beta}_{t,1|t,1}$. Then, as $Y_{t,2} = q$ arrives, and by noticing that $P_{t,2|t,1} = P_{t,1|t,1} = \sigma^2\left(X'_tX_t\right)^{-1}$, $Z_{t,2} = A$ and $\nu_{t,2} = q - A\hat{\beta}_{t,2|t,1}$ and $F_{t,2} = A\sigma^2\left(X'_tX_t\right)^{-1}A'$, the Kalman state updating equation in (2-3) becomes

$$\hat{\beta}_{t,2|t,2} = \hat{\beta}_{t,1|t,1} + \left(X'_tX_t\right)^{-1}A'\left(A\left(X'_tX_t\right)^{-1}A'\right)^{-1}\left(q - A\hat{\beta}_{t,1|t,1}\right). \tag{3-17}$$

But, as just mentioned, $\hat{\beta}_{t,1|t,1} = \hat{\beta}_{MQO}$. Therefore, the conclusion is that the Kalman updating in (3-17) is indeed an application of RLS estimator of (3-15). The equivalence between covariance matrices can be also established analogously. □

Some conceptual and practical consequences follow. First of all, it now becomes clear that the restricted Kalman filtering is indeed a generalization of the RLS estimator, a statement that, albeit intuitive, was lacking a proper formalization. In addition, Theorem 3 also shows that a regression model with random walk time-varying coefficients *under restrictions* (set $\beta_{t+1} = \beta_t + \eta_t$, $\eta_t \sim (0; Q)$), as the state equation for model (3-14)) does encapsulate the regression model with static coefficients, still under the same restrictions. Then, whenever the restricted Kalman filtering is applied to the time-varying version, both models can be compared as usual - to estimate the static model, just set $Q \equiv 0$. Finally, note that the recursive residuals obtained from recursive application of (3-15) are automatically uncorrelated - indeed, Theorem 3 says they are innovations. This is a desirable property in paving the way towards the development of a generalization of the stability test by Brown *et al.* (1975).

3.4 Initialization

3.4.1 Motivation

Besides considering the linear restrictions in equation (2-5), in this section I will also admit that some coordinates of the initial state vector α_1 have infinity variances. This is the basic set-up of the so-called *diffuse initialization* of the Kalman recursions, a subject extensively studied in Ansley and Kohn (1985), de Jong (1988), Harvey (1989), de Jong (1991), Koopman (1997), Durbin and Koopman (2001), Koopman and Durbin (2003), and de Jong and Chu-Chun-Lin (2003). Under this at least partially unspecified initial conditions, a question that comes is whether the methods of imposing linear restrictions can be derived from the very beginning. Observe that, once some elements of P_1 explode, there shall be no L_2 theory available anymore, nor could even the traditional Kalman equations be tackled, at least in the period when the effect of diffuseness - which lasts for an initial portion of the data - has not vanished yet. So, the strategies used in proofs by Doran (1992) and by Pizzinga (2008) to achieve the augmented restricted Kalman filtering (cf. Theorem 1) unfortunately become useless here. The purpose of this section is to address this theoretical issue precisely, by exploring the conditions which allow one to extend the restricted estimation to diffuse initializations, and by working out appropriately the modified versions of the Kalman equations; that is, the proof shall be "computational" instead of "geometrical".

3.4.2 Reviewing the initial exact Kalman smoother

From now on, the initial state vector is modeled as

$$\alpha_1 = a + B\delta + R_0\eta_0,$$

in which a is ¯xed and known, $\pm \sim (0; \cdot I_q)$, $´_0 \sim (0; Q_0)$, and B and R_0 are $m \times q$ and $m \times (m - q)$ selection matrices respectively, such that $B' R_0 = 0$ and $B' ®_1 = \pm$. In general, ± consists of initial conditions for the non-stationary terms of the m-variate process $®_t$. Under this ¯x, the exact initial Kalman smoother, obtained when $\cdot \longrightarrow +\infty$, is, in Durbin and Koopman (2001)'s notation,

$$\begin{aligned} À_t^{(0)} &= Y_t - Z_t a_t^{(0)} - d_t; \qquad F_{*,t} = Z_t P_{*,t} Z_t' + H_t; \qquad F_{\infty,t} = Z_t P_{\infty,t} Z_t' \\ L_t^{(0)} &= T_t - T_t P_{\infty,t} Z_t' F_{\infty,t}^{-1} Z_t; \quad L_t^{(1)} = -T_t P_{*,t} Z_t' F_{\infty,t}^{-1} Z_t + T_t P_{\infty,t} Z_t' F_{\infty,t}^{-1} F_{*,t} F_{\infty,t}^{-1} Z_t; \end{aligned} \tag{3-18}$$

$$\begin{aligned} a_{t+1}^{(0)} &= T_t a_t^{(0)} + c_t + T_t P_{\infty,t} Z_t' F_{\infty,t}^{-1} À_t^{(0)}; \quad P_{*,t+1} = T_t P_{\infty,t} L_t^{(1)'} + T_t P_{*,t} L_t^{(0)'} + R_t Q_t R_t'; \\ P_{\infty,t+1} &= T_t P_{\infty,t} L_t^{(0)'}; \quad t = 1; : : : ; n; \end{aligned}$$

$$\begin{aligned} r_{t-1}^{(0)} &= L_t^{(0)'} r_t^{(0)}; \quad r_{t-1}^{(1)} = Z_t F_{\infty,t}^{-1} À_t^{(0)} + L_t^{(0)'} r_t^{(1)} + L_t^{(1)'} r_t^{(0)}; \\ a_{t|n} &= a_t^{(0)} + P_{*,t} r_{t-1}^{(0)} + P_{\infty,t} r_{t-1}^{(1)}; \quad r_n^{(0)} = 0; \quad r_n^{(1)} = 0; \quad t = n; : : : ; 1; \end{aligned} \tag{3-19}$$

whenever $F_{\infty,t}$ just de¯ned above is nonsingular. Otherwise, changes must take place in the recursions (3-18) and (3-19) (cf. Koopman and Durbin, 2003). According to Koopman (1997), there exists a time instant d after which the above recursions collapse to the traditional Kalman smoother; therefore, $P_{\infty,t} = 0$ for $t > d$ necessarily.

The presented recursions constitute the paradigm proposed in Koopman (1997), Durbin and Koopman (2001, sec. 5.3), and Koopman and Durbin (2003) for the treatment of state smoothing di®use initialization. An alternative approach, based on the augmentation of the measurement equation, is proposed in de Jong and Chu-Chun-Lin (2003)

3.4.3 Combining exact initialization with linear restrictions

Before going to the main result of the paper, some preliminary steps must me addressed. The ¯rst is to list and to discuss the conditions under which it will be possible to combine di®use initial conditions for the Kalman smoothing equations with the imposition of linear restrictions. Let them be enunciated and, without any loss of generality, consider them valid for all $t = 1; : : : ; n$.

Assumption 4 $\{q_{ti}; : : : ; q_{tk}\}$ *is a linearly independent subset of* $L_2(-; \mathcal{F}; \mathcal{P})$.

Assumption 5 $\forall i = 1; : : : ; k : q_{ti} \in$ *span*$\{1; Y_{11}; : : : ; Y_{1p}; q_{11}; : : : ; q_{1k}; : : : ; Y_{t-1,1}; : : : ; Y_{t-1,p}; q_{t-1,1}; : : : ; q_{t-1,k}; Y_{t,1}; : : : ; Y_{t,p}\}$.

Assumption 6 *The matrices Z_t and A_t are such that the rank of $[Z_t' \; A_t']'$ is $p + k$.*

Each Assumption should be examined in terms of generality and plausibility. Assumption 4 guarantees non-redundance of the linear restrictions. Assumption 5, in turn, de¯nitively moves the focus towards the augmented restricted Kalman ¯ltering, since this is actually the appropriate way of handling \stochastic" restrictions; indeed, if one faces some q_{ti} ¯xed or perfectly predictable given the past, the corresponding linear restrictions would be much better dealt with by the reduced restricted Kalman ¯ltering (section 4.2), under which the exact initial Kalman smoother in (3-18) and (3-19) are straightforwardly applicable. Notice also that, under any state equation chosen, Assumptions 4 and 5 necessarily imply $A_t P_{t|t-1} A_t' > 0$. Finally, Assumption 6, besides reinforcing that the linear restrictions are distinct, should be understood as an impossibility of repeated signals along the lines of the augmented measurement equation proposed in Theorem 1, something quite natural from a practical perspective.

The second preliminary step is to quote the following auxiliary result:

Lemma 2 *Consider a state space model with the augmented measurement proposed in Theorem 1. If $F_t > 0$, then $A_t P_{t|t-1} [Z_t' \; A_t'] F_t^{-1} = [0_{k\times p} \; I_{k\times k}]$.*

Notice that Lemma 2 is actually a rephrasing of eq.(22) from Doran (1992), which has already proved to be key in the second proof of Theorem 1.

Finally, here is the main result concerning initialization:

Theorem 4 *(The initial exact restricted Kalman smoother) Suppose the augmented state space model associated to Theorem 1 satis¯es Assumptions 4, 5 and 6. Then the initial exact Kalman smoother in (3-18) and (3-19) yields*

$$A_t a_{t|n} = q_t: \qquad (3\text{-}20)$$

Proof: Fix an arbitrary $t \in \{1; : : : ; d\}$, where d is the length of the di®use period associated with the augmented model. De¯ne $\bar{Z}_t = [Z_t' \; A_t']'$. Other quantities would also have deserved tildes, but they are suppressed for conserving notation. From the Assumptions 4 and 5, $F_{\infty,t}$ cannot be a zero matrix. Supposing ¯rst that $F_{\infty,t}$ is nonsingular, take the recursive formulae of $r_{t-1}^{(0)}$ and $r_{t-1}^{(1)}$ in (3-19) and place them in the expression of $a_{t|n}$, which gives

$$a_{t|n} = a_t^{(0)} + P_{\infty,t} Z_t' F_{\infty,t}^{-1} \grave{A}_t^{(0)} + P_{\infty,t} L_t^{(0)\prime} r_t^{(1)} + P_{\infty,t} L_t^{(1)\prime} r_t^{(0)} + P_{*,t} L_t^{(0)\prime} r_t^{(0)}: \qquad (3\text{-}21)$$

From (3-21), identity (3-20) will be proved whenever the following three claims are established.

Claim 1. $A_t\left(a_t^{(0)} + P_{\infty,t}Z_t'F_{\infty,t}^{-1}\grave{A}_t^{(0)}\right) = q_t$.
Proof: De¯ne $a_{t|t}^{(0)} \equiv a_t^{(0)} + P_{\infty,t}Z_t'F_{\infty,t}^{-1}\grave{A}_t^{(0)}$. Looking at the recursion in (3-18), it follows that $a_{t|t}^{(0)}$ is the Kalman updating equation given in (2-3) applied to an augmented state space model with system matrices given by $Z_t^{\dagger} = Z_t$, $d_t^{\dagger} = (d_t'\ 0')'$, $H_t^{\dagger} = 0_{(p+k)\times(p+k)}$, $T_t^{\dagger} = T_t$, $c_t^{\dagger} = c_t$, and $Q_t^{\dagger} = 0$; and also $a_1^{\dagger} = 0$ and $P_1^{\dagger} = P_{\infty,1} = BB'$. □

Claim 2: $A_tP_{\infty,t}L_t^{(0)'} = 0$.
Proof: Still considering the auxiliary state space model from the previous claim, use the expression of $L_t^{(0)'}$ in (3-18) and Lemma 2 to get

$$\begin{aligned} A_tP_{\infty,t}L_t^{(0)'} &= A_tP_{\infty,t}T_t' - A_tP_{\infty,t}Z_t'F_{\infty,t}^{-1}Z_tP_{\infty,t}T_t' \\ &= A_tP_{\infty,t}T_t' - [0_{k\times p}\ I_{k\times k}]\,Z_tP_{\infty,t}T_t' \\ &= 0: \ \square \end{aligned}$$

Claim 3: $A_t\left(P_{\infty,t}L_t^{(1)'} + P_{*,t}L_t^{(0)'}\right) = 0$.
Proof: From the expression of $L_t^{(1)'}$ in (3-18), it follows that

$$\begin{aligned} A_tP_{\infty,t}L_t^{(1)'} &= -A_tP_{\infty,t}Z_t'F_{\infty,t}^{-1}Z_tP_{*,t}T_t' + A_tP_{\infty,t}Z_t'F_{\infty,t}^{-1}F_{*,t}F_{\infty,t}^{-1}Z_tP_{\infty,t}T_t' \\ &= -[0_{k\times p}\ I_{k\times k}]\,Z_tP_{*,t}T_t' + [0_{k\times p}\ I_{k\times k}]\left[Z_tP_{*,t}Z_t' + diag\,(H_t; 0_{k\times k})\right]F_{\infty,t}^{-1}Z_tP_{\infty,t}T_t' \\ &= -A_tP_{*,t}T_t' + A_tP_{*,t}Z_t'F_{\infty,t}^{-1}Z_tP_{\infty,t}T_t'; \end{aligned} \tag{3-22}$$

where the second equality comes from Lemma 2 combined with the auxiliary state space model ¯rstly evoked in Claim 1, and from the expression of $F_{*,t}$ in (3-18) associated with the the augmented model.
On the other hand, the expression of $L_t^{(0)}$ implies

$$A_tP_{*,t}L_t^{(0)'} = A_tP_{*,t}T_t' - A_tP_{*,t}Z_t'F_{\infty,t}^{-1}Z_tP_{\infty,t}T_t': \tag{3-23}$$

Add (3-22) and (3-23). □

In case of $F_{\infty,t}$ being singular, uncouple the augmented measurement $(Y_t'; q_t')'$ in such a way that

$$Y_{t,1}; \colon\colon\colon; Y_{t,p-1}; (Y_{t,p}; q_t')' :$$

Without losing generality, assume that $Z_{t,p}P_{\infty,t}Z_{t,p}' > 0$, where $Z_{t,p}$ is the p^{th}-row of Z_t (recall: $P_{\infty,t} > 0$ for $t \le d$). Since $F_{\infty,t}$ associated with $(Y_{t,p}; q_t')'$ is nonsingular (cf. Assumption 6), proceed exactly as before in order to attain (3-20). This completes the proof. □

From a practical perspective, a point coming from this last result, which must be reinforced, is that, under quite general conditions, it is *always possible* to yield restricted smoothed state vectors, even when the estimation lies in the \di®use" period (that is, for $t = 1; : : : d$, whatever d may be). Said in other words: the beginning of the series is not critical anyhow to get more interpretable results (which is certainly the case whenever estimated state vectors under meaningful restrictions are achieved).

4
Restricted Kalman filtering: methodological issues

This chapter is concerned with some presumed new methods about imposing linear restrictions in state space modeling. The plan I will follow is this. In section 4.1, I propose an alternative restricted Kalman filtering that is indicated to situations in which the linear restrictions are time-invariant and the state vector follows a general random walk. In section 4.2, I present another alternative restricted Kalman filtering under a reduced linear state space model, which will be confronted with the previous augmented restricted Kalman filtering from several standpoints. At last, section 4.3 deals with the imposition of linear restrictions in the prediction of the state vector.

4.1
Random walk state vectors under time-invariant restrictions

In this section, the paradigm of augmenting the measurement equation, in order to accomplish linear restrictions in state vector estimation, changes. Actually, this brief change in course deserves some attention because it may highlight a potential framework in restricted Kalman filtering.

The result of this section, the proof of which is still carried out by elementary Hilbert space theory, is the following:

Theorem 5 *If the linear state space model in (2-1) is such that $c_t = 0$ and $T_t = R_t = I$, then (i)$A\alpha_1 = q$ (with q deterministic) and (ii)$AQ_tA' = 0$ for all t=1,2,... are sufficient to*

$$Aa_{t|j} = q \quad for\ all \quad t; j = 1; 2; ::: \tag{4-1}$$

Proof: Fix t and j. Once again, denote by $\pi_{S'}$ the linear orthogonal projection onto S'. Now observe that, from a trivial recursion on the state equation,

$$\alpha_t = \alpha_1 + \sum_{j=1}^{t-1} \eta_{t-j}: \tag{4-2}$$

Pre-multiplying both sides of (4-2) by A implies

$$A\alpha_t = A\alpha_1 + \sum_{j=1}^{t-1} A\eta_{t-j} = q + 0 = q; \tag{4-3}$$

where the second equality comes from hypotheses (i) and (ii). Denoting the i^{th} row from A by $A_i = [c_{i1} \ldots c_{im}]$, it follows that

$$\begin{aligned} A_i a_{t|j} &= c_{i1} a_{t1|j} + \cdots + c_{im} a_{tm|j} = c_{i1} \pi_{S'}(\alpha_{t1}) + \cdots + c_{im} \pi_{S'}(\alpha_{tm}) \\ &= \pi_{S'}(c_{i1}\alpha_{t1} + \cdots + c_{im}\alpha_{tm}) = \pi_{S'}(A_i \alpha_t) = \pi_{S'}(q_i) \\ &= q_i; \end{aligned}$$

where the third, fifth and sixth equalities come respectively from the linearity of $\pi_{S'}$, from (4-3), and from the fact that $q_i \in \mathcal{R}(\pi_{S'}) = S'$. Since t, j and i were taken arbitrarily, the theorem is proved. □

I should make explicit some practical gains from this last proposition, applicable to models in which the state vector evolves as (possibly heteroscedastic) random walks. The first bonus is that there is no need to increase the dimension of the measurement equation any longer. The second is that, by imposing the enunciated restrictions on the initial state vector and on the covariance matrices of the error terms from the state equation, maximum likelihood estimation can be sharply enhanced whenever some of the unknown parameters belong to those matrices. The third advantage is that the restrictions are satisfied by any type of state estimation, whether it is a prediction, updating or smoothing.

4.2 Reduced restricted Kalman filtering

4.2.1 Motivation

In dealing with a linear regression model under linear restrictions, there are two ways of estimation. Actually, both prove to be numerically equivalent and are known by the name of restricted least squares. The first way was already revisited in section 3.3 (cf. expressions in (3-15)), while the second is implemented by rewriting a reduced model with transformed data and then applying usual *OLS* estimation to the transformed data (cf. Davidson and MacKinnon, 1993).

My aim in this section is to propose a restricted Kalman filtering under a reduced modeling framework. While the usual restricted Kalman filtering by augmentation discussed so far can be viewed as a generalization of the first way to impose linear restrictions in a static linear regression model, the new approach to be now developed, in turn, resembles the second. One feature to be listed among others is that, even though both approaches of restricted least squares produce exactly the same result, the two restricted Kalman ¯ltering (the augmented and the reduced) *do not* always result in the same estimated state vectors.

4.2.2 The method

In the remaining of this section 4.2, consider the measurement equation in (2-1), the restrictions in (2-5) and the following

Assumption 7 *The (possibly random) vector $q_t = (q_{t1}; \dots; q_{tk})'$ is linearly predeterminate to $1; Y_1; \dots; Y_t$, for all $t = 1; 2; \dots$; that is, $q_{ti} \in$ span$\{1; Y_{11}; \dots; Y_{1p}; \dots; Y_{t1}; \dots; Y_{tp}\}$, for all $i = 1; \dots; k$ and $t = 1; 2; \dots$.*

The basic perspective behind the alternative restricted Kalman ¯ltering is much the same as that of the reduced modeling in linear regression under linear restrictions: some state coordinates are rewritten as an a±ne function of the others and the result is appropriately placed in the measurement equation.

The *method*:

Let t be an arbitrary time index.

1. Without any loss of generality write the linear restrictions in (2-5) as
$$A_{t,1}®_{t,1} + A_{t,2}®_{t,2} = [A_{t,1}\ A_{t,2}]\left(®'_{t,1}; ®'_{t,2}\right)' = q_t; \qquad (4\text{-}4)$$
where $A_{t,1}$ is a $k \times k$ full rank matrix.

2. Solve (4-4) for $®_{t,1}$ which should result in
$$®_{t,1} = A_{t,1}^{-1}q_t - A_{t,1}^{-1}A_{t,2}®_{t,2}: \qquad (4\text{-}5)$$

3. Take (4-5) and put it in the measurement equation of the model in (2-1) - from which we drop d_t without loosing generality at all - aiming to obtain
$$\begin{aligned} Y_t &= Z_{t,1}®_{t,1} + Z_{t,2}®_{t,2} + ″_t \\ &= Z_{t,1}\left(A_{t,1}^{-1}q_t - A_{t,1}^{-1}A_{t,2}®_{t,2}\right) + Z_{t,2}®_{t,2} + ″_t \\ &= Z_{t,1}A_{t,1}^{-1}q_t - Z_{t,1}A_{t,1}^{-1}A_{t,2}®_{t,2} + Z_{t,2}®_{t,2} + ″_t \\ \Rightarrow Y_t^* &\equiv Y_t - Z_{t,1}A_{t,1}^{-1}q_t = \left(Z_{t,2} - Z_{t,1}A_{t,1}^{-1}A_{t,2}\right)®_{t,2} + ″_t \\ &\equiv Z_{t,1}^*®_{t,2} + ″_t: \end{aligned}$$

4. Now, postulate a transition equation for the unrestricted state vector $®_{t,2}$. This equation leads to the following *reduced* linear state space model

when it is put together with the measurement equation derived in the last step:

$$
\begin{aligned}
Y_t^* &= Z_{t,2}^* \alpha_{t,2} + \varepsilon_t \;\; ; \;\; \varepsilon_t \sim (0, H_t) \\
\alpha_{t+1,2} &= T_{t,2}\alpha_{t,2} + c_{t,2} + R_{t,2}\eta_{t,2} \;\; ; \;\; \eta_{t,2} \sim (0, Q_{t,2}) \\
\alpha_{1,2} &\sim (a_{1,2}, P_{1,2}).
\end{aligned}
\tag{4-6}
$$

5. For the reduced model in (4-6), apply the usual Kalman filtering to obtain $a_{t,2|j}$ and $P_{t,2|j}$, for all $j \geq t$.

6. Reconstitute the estimate $a_{t,1|j}$ and its mean square error matrix $P_{t,1|j}$ by means of the affine relation given in (4-5):

$$
\begin{aligned}
a_{t,1|j} &= A_{t,1}^{-1} q_t - A_{t,1}^{-1} A_{t,2} a_{t,2|j} \\
P_{t,1|j} &= (A_{t,1}^{-1} A_{t,2}) P_{t,2|j} (A_{t,1}^{-1} A_{t,2})'.
\end{aligned}
\tag{4-7}
$$

The above algorithm deserves some qualification. First, I must say that the approach is not completely new, since a particular case was conveniently used in Doran and Rambaldi (1997); what I am doing here is to put it in a more general framework. In addition, observe that *j does have to be greater than or equal to t* due to steps 5 and 6 (cf. Assumption 7). Another aspect is that the specification for the state equation in step 4 could be extracted from the *complete* state equation in (2-1), but if one does not want to think or worry about a full transition system, then one could concentrate only in modeling the block $\alpha_{t,2}$.

4.2.3 Reducing versus augmenting

Among the advantages of the reduced model approach over the augmented model, I cite:

- *Mathematical consistency*: Once the state equation is chosen after the reducing task, the method avoids any risk of obtaining measurement and state equations theoretically inconsistent with each other.
- *Computational efficiency*: While the augmenting approach increases the dimension of the practical problem (indeed, the length of the measurement vectors increases from p to $p + k$!), the reduced model approach goes in an opposite direction by not altering the size of the measurement equation and shortening the size of the state equation (from m to $m - k$). In other words, the augmenting approach "augments" the dimensions of the practical problem while the reduced model approach "reduces" them.

{ *Model selection*: The reduced model approach enables one to investigate the plausibility of the assumed linear restrictions by using information criteria (*e.g.* AIC and BIC). The competing model would be the unrestricted one as given by (2-1), the (*quasi*) likelihood function of which is surely comparable with that one from the restricted model in equations (4-6).

Stepping further towards the comparison between the reducing and the augmenting approaches, I present two results. Both are related to the augmented model suggested in Theorem 1, and reveal that, for certain types of state restrictions, it is much less °exible. The ¯rst proposition concerns limitations on the state equation. Note that the ¯rst three conditions listed below are quite general, since they are veri¯ed for several state space speci¯cations (*e.g.* zero-mean initial state vectors, whatever di®use or non-di®use) and for many types of linear restrictions (*e.g.*, all the deterministic ones):

Proposition 1 *Suppose the partition in (4-4) is such that $A_{t,1} \equiv A_1$. Also, admit the following conditions:*

(i) $T_t = diag\,(T_{t,1}; T_{t,2})$, *where* $T_{t,1}$ *is* $k \times k$.

(ii) $\left(A_1^{-1}A_{t,2}T_{t,2} - T_{t,1}A_1^{-1}A_{t,2}\right) E\,(®_{t,2}) = 0$.

(iii) $E\,(q_t) = E\,(q_{t+1}) = ¹$.

Then, (i), (ii) *and* (iii) *are su±cient for* $T_{t,1} = I_{k\times k}$. *Now, suppose* (i), (ii) *and* (iii) *valid for all* $t \geq 1$ *and consider the additional conditions:*

(iv) $\forall t \geq 1 : A_{t,2} \equiv A_2$, *such that* A_2 *has null kernel.*

(v) $\forall t \geq 1 : q_t \equiv q$ *(possibly random).*

Now, (i) *to* (v) *are su±cient for* $T_t = I_{m\times m}$.

Proof: For ease of notation, set $c_t = 0$ and $R_t = I$ in the augmented version of model (2-1). From (4-5) and from condition (i), I have

$$A_1^{-1}q_{t+1} - A_1^{-1}A_{t,2}T_{t,2}®_{t,2} - A_1^{-1}A_{t,2}´_{t,2} = T_{t,1}A_1^{-1}q_t - T_{t,1}A_1^{-1}A_{t,2}®_{t,2} + ´_{t,1},$$

which is equivalent to

$$A_1^{-1}q_{t+1} - T_{t,1}A_1^{-1}q_t = \left(A_1^{-1}A_{t,2}T_{t,2} - T_{t,1}A_1^{-1}A_{t,2}\right)®_{t,2} + ´_{t,1} + A_1^{-1}A_{t,2}´_{t,2}. \quad (4\text{-}8)$$

Taking expectations on both sides of (4-8) and using conditions (ii) and (iii), I arrive at

$$(I - T_{t,1})\,A_1^{-1}¹ = 0. \quad (4\text{-}9)$$

From (4-9), I necessarily have $T_{t,1} = I_{k\times k}$. Finally, under (i), (ii) and (iii) valid for all $t \geq 1$, the conditions (iv) and (v) imply $T_{t,2} = I_{(m-k)\times(m-k)}$ (indeed: get $A_2®_{t,2} = q - A_1®_{t,1}$ from (4-4), pre-multiply this latter identity by a left inverse of A_2, and recall that a±ne functions of random walks are also random walks). □

It becomes clear from Proposition 1 that, if one chooses the augmenting approach for dealing with important types of restrictions, there would be no possibility left but a random walk evolution for at least a block of the state vector.

The second proposition is stated below. Its condition (vii), as one can directly see, is a quite natural set-up, since this avoids some pathological behaviors from the measurement equation, such as non-ergodic stationarity:

Proposition 2 *Suppose conditions* (i), (ii) *and* (iii) *of Proposition 1 are valid for all* $t \geq 1$, *as well as* (iv) *and* (v), *with* q *degenerated. Also assume that:*

(vi) $\forall t \geq 1 : Q_t \equiv diag\,(¾^2_{t1}; : : : ; ¾^2_{tm})$.

(vii)$\forall t \geq 1$ *and* $\forall i = 1; : : : ; m : ¾^2_{1i} = \cdots = ¾^2_{ti} = 0 \Rightarrow Var\,(®_{1i}) = 0$.

Then, $Q_t = O_{m\times m}$ *for all* $t \geq 1$.

Proof: Take an arbitrary $t \geq 1$. From (i) to (v), the decomposition in (4-4) collapses to $A_1®_{t+1,1} + A_2®_{t+1,2} = q$. This implies $rank\,(V\,(®_{t+1})) \leq m - k$. But, as $T_s = I$ for all $s = 1 : : : t$ (cf. Proposition 1), I must have $max\,\{rank\,(P_1)\,;rank\,(Q_1)\,; : : : ; rank\,(Q_t)\} \leq m - k$. Then, using (vi), there exist $i_1; : : : ; i_k \in \{1; : : : ; m\}$ such $¾^2_{s\,i_j} = 0$ for all $s = 1; : : : ; t$ and $j = 1; : : : ; k$. Conveniently rearranging $®_{t+1}$, I get a partition $\left(®^{*\prime}_{t+1,1}; ®^{*\prime}_{t+1,2}\right)'$ such that $Var\left(®^*_{t+1,1}\right) = O_{k\times k}$ (cf. Proposition 1 again and condition (vii)). Than,

$$O_{k\times k} = Var\,(q^*) = Var\left(A^*_1®^*_{t+1,1} + A^*_2®^*_{t+1,2}\right) = A^*_2 Var\left(®^*_{t+1,2}\right) A^{*\prime}_2 : \quad (4\text{-}10)$$

From (4-10) I ¯nally obtain $Q^*_{s,2} = O_{(m-k)\times(m-k)}$ for all $s = 1; : : : t$. □

This last result rules out any possibility of non-degenerated state vectors under contemporaneously uncorrelated errors $´_{t1}; : : : ; ´_{tm}$. This limitation, as the previously raised from Proposition 1, surely does not arise under the reducing approach.

4.2.4 Geometrical considerations

Let me now grasp some intuitive insight from the described method and therefore geometrically understand what this alternative restricted Kalman ¯ltering, as well as the previous one by augmentation, is in fact \doing" to the state vector.

I in the ¯rst place defend that one could work in an equivalent way with the model

$$
\begin{aligned}
Y_t &= Z^*_{t,2}\alpha_{t,2} + d^*_t + \varepsilon_t \;;\quad \varepsilon_t \sim (0; H_t) \\
\alpha_{t+1,2} &= T_{t,2}\alpha_{t,2} + c_{t,2} + R_{t,2}\eta_{t,2} \;;\quad \eta_{t,2} \sim (0; Q_{t,2}) \\
\alpha_{1,2} &\sim (a_{1,2}; P_{1,2});
\end{aligned}
\tag{4-11}
$$

where $d^*_t \equiv Z_{t,1}A^{-1}_{t,1}q_t$, since (4-6) and (4-11) are equivalent linear state space representations. Indeed, by Assumption 7, d^*_t just defined is necessarily deterministic - or linearly predeterminate to $Y_1;\dots;Y_t$. So the sets $\{1; Y_{11};\dots;Y_{1p}\ \dots; Y_{j1};\dots;Y_{jp}\}$ and $\{1; Y^*_{11};\dots;Y^*_{1p}\ \dots; Y^*_{j1};\dots;Y^*_{jp}\}$ produce the same univariate innovations in $L^2(\Omega;\mathcal{F};\mathcal{P})$, which implies they span the same subspace. Therefore, the Kalman filtering (updating or smoothing equations) applied to (4-6) - or, as already argued, to (4-11) - is projecting each α_{tj}, $j = k+1;\dots;m$ onto $span\{1; Y_{11};\dots;Y_{1p};\dots;Y_{j1};\dots;Y_{jp}\}$, $j \geq t$, as it is done in a regular state space estimation. *But (4-6) - or equivalently (4-11) - together with (4-5) make explicit the fact that* $\alpha_{t,1} = (\alpha_{t1};\dots;\alpha_{tk})'$ *can be affinely extracted from* $\alpha_{t,2} = (\alpha_{t,k+1};\dots;\alpha_{tm})'$. Then, it becomes possible to project each coordinate of $\alpha_{t,2}$ first and subsequently obtain the projections of each coordinate of $\alpha_{t,1}$ using (4-7). In light of such considerations, one could consider the reduced model approach as some kind of a "two-stage" state estimation.

Within the previous augmenting procedure, one in turn has to project directly (by means of the Kalman equations applied to an augmented model) each coordinate of the entire $\alpha_t = (\alpha_{t1};\dots;\alpha_{tm})'$ onto the *bigger* subspace *span*$\{1; Y_{11};\dots;Y_{1p}; q_{11};\dots;q_{1k};\dots;Y_{j1};\dots;Y_{jp}; q_{j1};\dots;q_{jk}\}$ - strictly bigger if at least one of the $q_1;\dots;q_t$ is not linearly predeterminate to the measurements. Figures 4.1 and 4.2 illustrate these highlighted geometrical differences.

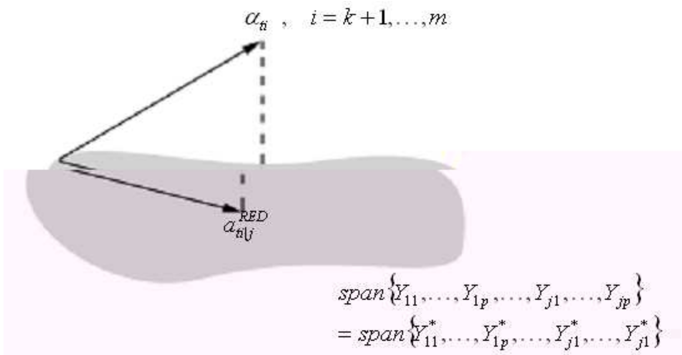

Figura 4.1: Geometrical meaning of this section's reduced model approach: Here only part of the coordinates of the state vector are directly projected onto the original spanned subspace. The other coordinates projections are obtained by formula (4-7).

4.3

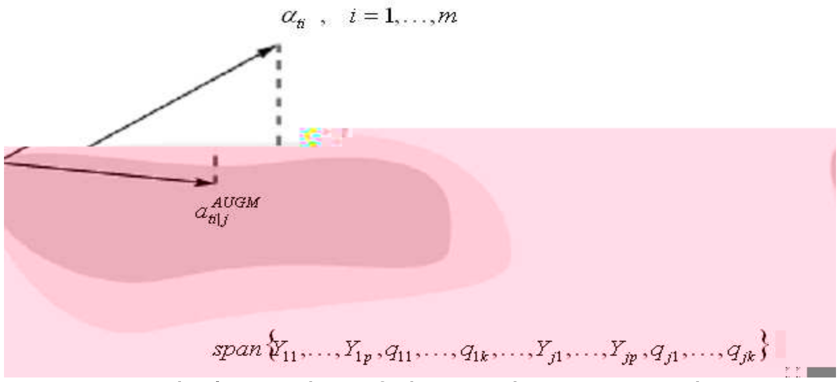

Figura 4.2: Geometrical meaning of the previous augmenting approach: Here each coordinate of the state vector is directly projected onto an augmented subspace.

Predictions from a restricted state space model

The original proposal for adopting an augmented model, which I re-evoke again in this section, does not, in general, guarantee that linear restrictions on the state vector are carried over to the Kalman prediction equations (*immediate example*: except for the local level model, there is no extension of Corollary 1 for the prediction equations when one is dealing with any of the structural models - cf. Harvey, 1989 - put in their respective state space forms). However, there is one exception. This particular case is described by a state vector that follows a possibly heteroscedastic random walk, and is considered in the following

Corollary 2 *Under the conditions presented in Theorem 1, in addition to* (*i*) $c_t = 0$ *and* (*ii*) $T_t = R_t = I$, *it follows that*

$$A_t a_{t+1|t} = q_t:$$

In this section I propose a simple strategy to further extend (namely, for *any* type of linear state space model) the restricted Kalman ¯ltering and smoothing up to schemes aimed at prediction. As a matter of fact, what I seek certainly di®ers from the method by Pandher (2002). In turn, the grounds of my proposal are built up on the ideas of missing values state space treatment and of the decomposition used in the second proof of Theorem 1 and in the proof of Theorem 3.

Consider that one is willing to extrapolate the state vector and/or the measurements up to h steps ahead in the future; that is, one wants to obtain $a_{n+1|n};:::;a_{n+h|n}$ and/or $\hat{Y}_{n+1|n};:::;\hat{Y}_{n+h|n}$. But, similarly to everything that has been done so far in this Thesis, it is known *a priori* that, for all $j = 1;:::;h$, A_{n+j}®$_{n+j} = q_{n+j}$, where A_{n+j} is a $k \times m$ matrix and q_{t+j} is a $k \times 1$ (possibly random) vector; this knowledge is nothing more than the confirmation that Assumption 1 is not confined to a particular time series of size n. So the question is how to make $a_{n+1|n}, \ldots, a_{n+h|n}$ satisfy those same theoretical constraints.

The proposed answer starts again from adopting an augmented model. Then, the augmented version of (

the original Kalman equations due to missing observations are discussed in Durbin and Koopman (2001), section 4.8. If the researcher wants to treat the series in (4-14) under a univariate framework, he or she shall refer to section 6.4 of that same book - this univariate choice brings some gains exclusively on the computational side since the whole vectors $Y_{n+1}, Y_{n+2}, \ldots, Y_{n+k}$ are lacking. Finally, notice that Theorem 1 is doing its job by guaranteeing that, for all $j = 1, \ldots, h$, $A_{n+j} a_{n+j|n} = q_{n+j}$. For the reasons just explained in this paragraph, "$n + j|n$" is an abuse of notation.

It is now time to harvest the resuming computational algorithm:

1. Decompose the model in (4-12) striving to get the series in (4-14).

2. Store the "new" observations while respecting the missing values positions.

3. Apply the Kalman smoothing equation to the stored observations, appropriately modified to account for the missing values.

4. Take the smoothed states corresponding to the missing values positions as the *predicted state vectors under linear restrictions*.

5
Applications

After presenting and discussing several theoretical and methodological issues, I entirely dedicate some practical examples to this chapter. In those, two methods from chapter 4 - namely: the reduced restricted Kalman ¯ltering and the restricted Kalman prediction equations - are considered, implemented and evaluated. The remaining of this chapter is structured as follows. Section 5.1[1] presents an application in time-varying factor modeling for dynamic style analysis, in which an accounting restriction on the coe±cients is tackled by the reduced restricted Kalman ¯ltering. In section 5.2[2], time-varying econometric models are considered for the estimation and interpretation of the dynamic exchange rate pass-through over Brazilian price indexes; again, the reduced restricted Kalman ¯ltering is key for testing some economic hypothesis imposed under two speci¯c restrictions. And, in section 5.3, the material concerning restricted predictions from section 4.3 is conveniently implemented for obtaining predictions of quarterly GNP that must be somehow consistent with the annual GNP (that is: for each year, the sum of quarterly GNP data is *restricted* to equal the annual GNP).

The models to be discussed in the sequel have been implemented using the Ox 3.0 language (cf. Doornick, 2001) with occasional use of the Ssfpack 3.0 library for linear state space modeling (Koopman *et al.*, 2002). My computer was an Athlon XP 2200 Mhz, with 378 Mb-RAM. The computational e±ciency of the estimations are separately analyzed and discussed in the appropriate sections. All the estimations were carried under the *exact maximum likelihood estimation* and the *exact initial Kalman ¯lter* (cf. Durbin and Koopman, 2001, chapters 5 and 7).

[1] I am in great debt with Luciano Vereda, who was the sole responsible for the economic interpretations of the proposed models and of the corresponding empirical results.

[2] Most of the conception, theory, and results and their interpretations have been carried out by Rafael Martins de Souza and Luiz Felipe Pires Maciel, to whom I am grateful.

5.1
Case I: Semi-strong dynamic style analysis

5.1.1
Motivation

Depending on the type of an investment fund under investigation, detailed information on the actual portfolio composition is not usually available. *Return-based style analysis*, or simply *style analysis*, is a statistical method for the estimation/approximation of the unknown composition of an investment fund portfolio. Standard practice of style analysis only uses the so-called *external information*, which is represented by the fund returns and some market indexes returns, and is implemented by the *asset class factor model* (cf. Sharpe, 1988 and 1992). Later, this has been modified by the add of an intercept term (cf. de Roon, Nijman and ter Horst, 2004), as follows:

$$R_t^P = \alpha + \beta_1 R_{t1} + \beta_2 R_{t2} + \cdots + \beta_m R_{tm} + \varepsilon_t. \tag{5-1}$$

Assumptions: R_t^P is the portfolio return; $R_t = (R_{t1}, R_{t2}, \ldots, R_{tm})'$ represents some asset class indexes returns, which should satisfy the assumptions of *exhaustiveness*, of *mutual exclusiveness* and of *different behavior* (cf. Sharpe, 1988 and 1992); $\beta_1, \beta_2, \ldots, \beta_m$ are the unknown allocations/exposures which are sometimes supposed to satisfy an accounting constraint, known as the *portfolio restriction*[3]: $\sum_{i=1}^{m} \beta_i = 1$; α is the *Jensen's measure* or *Jensen's alpha* (cf. de Roon, Nijman and ter Horst, 2004), and represents the idiosyncratic fund return, i.e., it measures how much the fund aggregates - or looses - by means of its selectivity strategies[4]; and ε_t is a typical random error process with finite second moments.

Even though being a much used tool in investment analysis, model (5-1) has a drawback: it ignores the fact that asset class exposures and selectivity *do change* over time, reflecting the very plausible and possible reallocations of the assets by the portfolio manager - an idea that was also evoked in Pizzinga and Fernandes (2006) and in Swinkels and Van der Sluis (2006). Later, in Pizzinga *et al.* (2008) a class of *semi-strong*[5] style analysis models was proposed, the

[3] There is also a *short-sale restriction*, which is sometimes considered and is implemented by forcing non-negativeness of $\beta_1, \beta_2, \ldots, \beta_m$. But, as this restriction is not always meaningful (*e.g.*, most of hedge funds take positions in derivative markets), this is not adopted here.

[4] In fact, the *actual* Jensen's measure emerges in the context of equilibrium models, such as the CAPM or the APT model - cf. Elton *et al.* (2006) - or the multi-factor model of Carhart (1997). However, I shall retain the intercept term of (5-1) as the selectivity measure, since this and the former are used to achieve the same ends of the measure proposed in Jensen (1968).

[5] This means that only the portfolio restriction is imposed; cf. the style analysis taxonomy proposed by de Roon, Nijman and ter Horst (2004).

exposures and Jensen's measure of which were both made stochastically time-varying as a (vector) random walk. This represented a direct generalization of the static model (5-1), and its estimation was carried out by an appropriate restricted linear state space model. Empirical illustrations were presented using return series of Brazilian US Dollar/Real exchange rate funds. Among several points, there was clearly a visual evidence that the time-varying exposures onto US Dollar-Real exchange rate markets behaved under di®erent autoregressive regimes, one of those directly associated to the 2002 Brazilian presidential election, a period of some political turbulence and high volatility.

This sections's exercise aims at, ¯rstly, uncovering evidence on switching regimes for the time-varying exposures of Brazilian US Dollar-Real exchange rate funds under an econometrically more compelling way and, secondly, interpreting the estimated exposures from the \more appropriate" model. In it, I empirically evaluate several sounding dynamics: (1)random walk; (2)simply autoregressive; (3)autoregressive with abrupt switching regimes; and (4)nonlinear under a general smoothing transition function. The elected regime switching variable for the 3rd and the 4th models is the AR(1)-GARCH(1,1) volatility of US Dollar/Real exchange rate. As it will be seen, the adoption of time-varying portfolio-restricted exposures following such processes makes it necessary to use the reduced restricted Kalman ¯ltering of section 4.2.

5.1.2
Competing models

I now present the analytical expressions of several *time-varying asset class factor models* for *semi-strong dynamic style analysis*. In what follows, the reducing method from the subsection 4.2.2 has been evoked in order to make the portfolio restriction attainable.

Let me ¯rst obtain the expression corresponding to the portfolio restriction on the state vector, which I shall denote in this section by γ_t and whose coordinates represent the exposures and the Jensen's measure. To do this, we use steps 1 and 2 of the algorithm of subsection 4.2.2:

$$
\begin{aligned}
1 &= [1\ 1 \dots 1\ 0]\,(\beta_{t1}, \beta_{t2}, \dots, \beta_{tm}, \alpha_t)' \\
\Rightarrow \quad 1 &= \beta_{t1} + [1 \dots 1\ 0]\,(\beta_{t2}, \dots, \beta_{tm}, \alpha_t)' \\
\Rightarrow \quad \beta_{t1} &= 1 - [1 \dots 1\ 0]\,(\beta_{t2}, \dots, \beta_{tm}, \alpha_t)' \\
\Rightarrow \quad \gamma_{t,1} &= 1 - [1 \dots 1\ 0]\,\gamma_{t,2}.
\end{aligned}
$$

I now move on to the measurement equation of the reduced model by making use of step 3 of the algorithm in conjunction with the last equality obtained above:

$$
\begin{aligned}
R_t^c &= R_{t1}\beta_{t,1} + [R_{t2} \ldots R_{tm}\ 1]\,(\beta_{t2}, \ldots, \beta_{tm}, \alpha_t)' + \varepsilon_t \\
&= R_{t1} - R_{t1}\,[1 \ldots 1\ 0]\,(\beta_{t2}, \ldots, \beta_{tm}, \alpha_t)' + [R_{t2} \ldots R_{tm}\ 1]\,(\beta_{t2}, \ldots, \beta_{tm}, \alpha_t)' + \varepsilon_t \\
\Rightarrow R_t^c - R_{t1} &= [R_{t2} - R_{t1} \ldots R_{tm} - R_{t1}\ 1]\,(\beta_{t2}, \ldots, \beta_{tm}, \alpha_t)' + \varepsilon_t \\
\Rightarrow R_t^c - R_{t1} &= [R_{t2} - R_{t1} \ldots R_{tm} - R_{t1}\ 1]\,\gamma_{t,2} + \varepsilon_t.
\end{aligned}
$$

Finally, combining a rather encompassing state equation with the expression above, I arrive at the following general structure:

$$
\begin{gathered}
R_t^P - R_{t,1} = [R_{t2} - R_{t1} \ldots R_{tm} - R_{t1}\ \ 1]\,\gamma_{t,2} + \varepsilon_t;\ \ \varepsilon_t \sim NID(0, \sigma^2 X_t) \\
\gamma_{t+1,2} = diag\left(T_t^{\beta}, 1\right)\gamma_{t,2} + \eta_t\ \ ;\ \ \eta_t \sim NID(0, Q) \\
\gamma_{t,1} = 1 - [1 \ldots 1\ 0]\,\gamma_{t,2}.
\end{gathered}
\tag{5-2}
$$

I enumerate some features of model (5-2). Firstly, it should be reinforced that the last coordinate of γ_t is the time-varying Jensen's measure α_t and the remaining coordinates are the time-varying exposures $\beta_{t1}, \beta_{t2}, \ldots, \beta_{tm}$. Secondly, as can be directly seen from the second line of (5-2), the Jensen's measure follows a random walk And thirdly, X_t in the measurement error's variance is some nonnegative variable that must respond for occasional heteroscedastic behavior, and Q can be set full[6].

The decision towards the random walk for the evolution of the Jensen's measure deserves some justi¯cation. Although such choice seems too simple and perhaps \unrealistic" in ¯rst glance, three reasons support such recognition. The ¯rst is that of *parsimony* and *simplicity*, as there is no additional clue to guide one in choosing a more complex transition equation. The second is the allowance for the possibility of fundamental selectivity changes along time due to non-stationarity. The third comes from the next result:

Proposition 3 *For the model in (5-2), set $\mathcal{F}_\infty \equiv \sigma\left\{R_s^P : s \geq 1\right\}$ and $\hat{\alpha}_{t|\infty} \equiv E\left(\alpha_t|\mathcal{F}_\infty\right)$. Then, $\limsup_{t \longrightarrow +\infty} \hat{\alpha}_{t|\infty} = +\infty$ and $\liminf_{t \longrightarrow +\infty} \hat{\alpha}_{t|\infty} = -\infty$ $\mathcal{P} - a.s.$*

Proof: Without loss of generality, suppose the underlying probability space is *complete*. Take an arbitrary $i \in \{1, \ldots, m\}$. According to Chung (2001), ch.8, I have $\limsup_{t \longrightarrow +\infty} \alpha_t = +\infty$ $\mathcal{P} - a.s.$ It means that there is a subsequence

[6] The impacts on the exposures, represented by the components of η_t, \communicate" among themselves. Note that it would be unreasonable to assume that investment decisions (and hence the exposures) are related only by the portfolio restriction (which is an accounting constraint), since they re°ect the same underlying shocks. In other words, shocks that lead investors to augment their exposures onto some asset classes can also make them decide to reduce their positions on others.

®$_{t_j}$ such that $\lim_{j \longrightarrow +\infty}$®$_{t_j}$ = $+\infty$ $\mathcal{P} - a{:}s$. Taking such subsequence as nondecreasing, I obtain

$$\lim\sup_{t \longrightarrow +\infty} \hat{a}_{t|\infty} \geq \lim_{j \longrightarrow +\infty} \hat{a}_{t_j|\infty} = +\infty \quad \mathcal{P} - a{:}s,$$

where the equality follows from the Monotone Convergence Theorem for conditional expectations (cf. Chung, 2001, ch.9). The liminf case is dealt with under the same fashion. □

The interpretation: although non-stationary, the choice of a random walk has the advantage that, for\large" series, the smoothed Jensen's measure must intercept the time x-line in¯nitely often $\mathcal{P} - a{:}s{:}$ (in other simple words: with probability 1, the estimated Jensen's measures *never explode*).

The remaining part of model (5-2)'s speci¯cation lies on the transition sub-matrix $T_t^{\beta} \equiv diag\,(Á_{t2}; : : : ; Á_{tm})$, which drives the evolution of the unrestricted block of time-varying exposures in °$_{t,2}$. Let me ¯rst enumerate the possibilities I am going to investigate empirically and, in the sequel, give appropriate rationalities to each of them:

1. *Random walk* (*RW*): $T_t^{\beta} \equiv I_{(m-1)\times(m-1)}$.

2. *Purely autoregressive* (*AR*): $T_t^{\beta} \equiv diag\,(Á_2; : : : ; Á_m)$, where $|Á_i| < 1$ for all i.

3. *Autoregressive with abrupt switching regimes*: some diagonal entries of T_t^{β} take the form $Á_{i1} + Á_{i2}d_{ti}$, where $d_{ti} = 1$ if some exogenous variable z_t assumes certain values and $d_{ti} = 0$ otherwise.

4. *Nonlinear under a general smoothing transition function*: some diagonal entries of T_t^{β} take the form $Á_{i,1} + Á_{i,2}z_t + Á_{i,3}z_t^2$, where z_t is some exogenous variable.

The ¯rst model is clearly the most parsimonious and had already been used by Pizzinga *et al.* (2008) within this same style analysis framework[7]. In the occasion, two Brazilian US Dollar/Real exchange rate funds had their time-varying Jensen's measures and exposures analyzed. As already told, exposures estimated under this framework seem to follow two di®erent patterns, one during the months near the 2002 Brazilian presidential election

[7] According to Swinkels and van der Sluis (2006), this speci¯cation should be used if one believes that exposures can increase or decrease over time when responding to shocks (that turn out to exert a permanent e®ect). In contrast, if one believes that exposures can deviate for some time from normal (or \steady-state") levels but will forcefully come back to them (which means that shocks exert a transitory e®ect), then a variant like the second model should be used.

(when exposures onto US Dollar/Real markets appeared to be more erratic and less persistent), and the other during the remaining months (in which exposures were much more stable). This \stylized fact" was interpreted as suggesting that linear models should be abandoned in favor of more sophisticated ones (specially regime switching models), in which the consequences from changes in the decision making process (which possibly varies with the state of the economy, perhaps with market volatility) could be better captured.

The second model captures situations in which managers try to target \steady-state" exposures. When compared with the ¯rst model, the number of parameters grows by $m-1$ autoregressive coe±cients. Since the eigenvalues of T_t^{β} have absolute values strictly smaller than 1 (one), non-stationarity and/or \explosive" behaviors for the exposures are ruled out, something that brings some inferential attractiveness. Besides, this second model can be understood as a bridge to the nonlinear third and fourth versions.

The third and fourth models undoubtedly add complexity to the process of parameter estimation but are justi¯ed by their ability of capturing the state-dependent behavior of managers and investors (which generates the aforementioned possibility of multiple regimes in exposures' dynamics). One might recall that the nonlinear processes[8] used here are respectively the threshold autoregressive (TAR) model and a general smoothing transition autoregressive (STAR) model, in which the second-order polynomial on z_t is an attempt to approximate a more general \smooth" transition function. For a comprehensive treatment of these types of switching-regime proposals outside the state space framework, see Enders (2004). Once these dynamics are postulated to the state equation, parameter estimation can be accomplished under the usual paradigm of maximizing the prediction error decomposition form of the likelihood (see, for example, Harvey, 1989, ch.3; and Durbin and Koopman, 2001, ch.7).

5.1.3
Model Selection

As there are four alternatives to describe the time-varying exposures, I must discuss how to decide in practice which model seems to be the most appropriate. I actually adopt the following selection mechanism:

{ Likelihood-ratio (LR) tests to validate or to refuse the nonlinear proposals (3) and (4).

[8]Even though being *nonlinear processes* (*i.e.*, there is no corresponding Gaussian nor *i.i.d* Wold decomposition - cf. Brockwell and Davis, 1991 and 2003), these choices for the state equation still provide us with a Gaussian \linear" space model.

{ Information criteria, such as AIC and BIC.

{ Predictive power by comparing Pseudo R^2 and MSE measures.

{ Diagnostic tests over the standardized innovations.

The listed strategies had been fully discussed in Harvey (1989), ch. 5. Here, the null for the LR test shall be H_0: *\The parameters associated to the switching regimes are all zero"*. Consequently, our test aims at comparing the \reduced"model (2) to the \complete"model (3) or (4). There are strong theoretical evidences that, asymptotically, $LR = 2\left[logL_{Max,Comp} - logL_{Max,Red}\right] \sim \hat{A}^2_k$, where k is the number of parameters set to zero under the null, since at least the reduced model maintains the standards for good properties of maximum likelihood estimation[9] (cf. Pagan, 1980).

5.1.4
Reducing versus Augmenting

This subsection is dedicated to some discussion concerning the type of restricted Kalman ¯ltering that would be the most appropriate to obtain portfolio-restricted estimated exposures. As discussed in previous chapters, two major possibilities could *a priori* be evoked for such task, and so far we have concentrated only in the details of the *reduced* restricted Kalman ¯ltering. But, as we will show, the use of the *augmented* restricted Kalman ¯ltering considerably limits the choices of models for the exposures' evolution. Observe that, even though Propositions 1 and 2 from section 4.2 could be evoked here, I shall make use of the rather special structure of the portfolio restriction in order to uncover more drawbacks of the augmented restricted Kalman ¯ltering.

The time-varying asset class factor model corresponding to the augmenting approach shall have its measurement equation displayed as

$$\begin{pmatrix} R_t^P \\ 1 \end{pmatrix} = \begin{pmatrix} R_t' & 1 \\ 1\cdots 1 & 0 \end{pmatrix} °_t + \begin{pmatrix} ''_t \\ 0 \end{pmatrix}; \begin{pmatrix} ''_t \\ 0 \end{pmatrix} \sim NID\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}; \begin{pmatrix} ¾^2 X_t & 0 \\ 0 & 0 \end{pmatrix} \right): \tag{5-3}$$

The state equation retains its general form as presented in the second line of (5-2), but with the dimension increased by an unit. In addition, I use the following presupposition for the remaining of this section:

Assumption 8 *The state equation associated to model (5-3) is such that:*

1. $\forall t \geq 1$ *and* $\forall i = 1;:::;m : \sum_{j \neq i} (Á_{ti} - Á_{tj})\, E\,(¯_{tj}) = 0;$

[9] Analytical and/or Monte Carlo investigations for the LR test about its asymptotic properties would deserve some special attention, but I rather leave this important issue for future research.

2. $\forall i = 1, \ldots, m+1 : \sigma_i^2 \equiv Var(\eta_{ti}) = 0 \Rightarrow Var(\gamma_{1i}) = 0$.

Note that, even though the first statement of Assumption 8 looks artificial in first glance, this includes standard zero-mean setups for the initial state vector as particular cases, whatever diffuse or non-diffuse (in other words: it is more stringent to suppose the initial state vector γ_1 is at least integrable with unconditional expectation given by $a_1 \equiv E\left[(\beta_{11}, \ldots, \beta_{1m}, \alpha_1)'\right] = (0_{1\times m}, a_{1,m+1})'$).

The next two propositions, which are related to the augmented model (5-3) reveal that the augmented restricted Kalman filtering looses much flexibility in terms of the possible choices for the state equation.

Proposition 4 *For all t, each diagonal entry of the matrix T_t^{β} must be equal to one. That is, $T_t^{\beta} \equiv T^{\beta} = I_{m\times m}$.*

Proof: Take an arbitrary $t \geq 1$. From the second line of the measurement equation (5-3), I obtain the portfolio restriction

$$\beta_{t1} = 1 - \beta_{t2} - \cdots - \beta_{tm}. \tag{5-4}$$

From the state equation given in (5-2), I have

$$\beta_{t+1,1} = \phi_{t1}\beta_{t1} + \eta_{t1}. \tag{5-5}$$

Now, put (5-4) into (5-5) and make use of the state equation again to get

$$1 - \phi_{t2}\beta_{t2} - \cdots - \phi_{tm}\beta_{tm} - \sum_{j=2}^{m}\eta_{tj} = \phi_{t1} - \phi_{t1}\beta_{t2} - \cdots - \phi_{t1}\beta_{tm} + \eta_{t1}, \tag{5-6}$$

which is equivalent to

$$(1-\phi_{t1}) + (\phi_{t1} - \phi_{t2})\beta_{t2} + \cdots + (\phi_{t1} - \phi_{tm})\beta_{tm} = \sum_{j=1}^{m}\eta_{tj}. \tag{5-7}$$

Taking unconditional expectations on both sides of (5-7) and evoking the first item of Assumption 8, I finally get $1 - \phi_{t1} = 0$. The other exposures are dealt with analogously. □

The message is clear. If one chooses the augmented Kalman filtering for estimating the time-varying asset class factor model under the portfolio restriction, there is no possibility left but a random walk evolution for *every coordinate* the state vector. Therefore, proposals 2, 3 and 4 listed in subsection 5.1.2, or any other non-random walk specification, whether time-varying or fixed, would not be even checkable for a given data set. Note that Proposition 1 from section 4.2 would at the most reveal that only one coordinate of the state vector is a random walk.

The second proposition, stated below, rules out any possibility of time-varying exposures under contemporaneously independent errors $\varepsilon_{t,1}, \ldots, \varepsilon_{t,m}$.

Proposition 5 *Let Q^{β} be the covariance matrix associated to the exposures' random error. If $Q^{\beta} \equiv diag\left(\sigma_1^2, \ldots, \sigma_m^2\right)$, then $Q^{\beta} = O$.*

Proof: From the portfolio restriction imposed in the second line of (5-2), β_t is in fact a singular random vector. This is, by Proposition 4, equivalent to $max\left\{rank\left(Q^{\beta}\right), rank\left(P_1^{\beta}\right)\right\} < m$, where P_1^{β} is the block of P_1 associated to the initial exposures $\left(\beta_{11}, \ldots, \beta_{1m}\right)'$. Then, as Q^{β} is diagonal, it must follow that there exists $i \in \{1, \ldots, m\}$ such that $\sigma_i^2 = 0$, which in turn implies in $Var\left(\beta_{1i}\right) = 0$ (cf. the second item of Assumption 8). But, as $\beta_{ti} = 1 - \sum_{j \neq i} \beta_{tj}$, I must have

$$0 = Var\left(\beta_{ti}\right) = \sum_{j \neq i} Var\left(\beta_{tj}\right), \tag{5-8}$$

where both equalities follow from the state equation being a random walk (cf. Proposition 4) and from Q^{β} being diagonal. The conclusion from (5-8) is that $\sigma_j^2 = 0$ for all $j = 1, \ldots, m$. □

From this last result, one should learn that the adoption of the augmented restricted Kalman filtering also forces one to always consider exposures whose impacts, which are represented by the components of ε_t, are correlated. Such limitation, like the former, certainly does not arise under the reduced restricted Kalman filtering.

5.1.5
Empirical results

The asset class indexes were the CDI (the average rate charged in overnight transactions between depository institutions), the US Dollar/Real exchange rate (in percentage points) and observed variations in two financial indicators, Quantum Cambial and Quantum Fixed Income[10]. The data comprise 209 observations on weekly returns from 2001 to 2004 and were obtained from *Quantum Axis* (www.quantumfundos.com.br). Two US Dollar/Real exchange rate funds inside the Brazilian industry were considered[11]: HSBC Cambial FIF and Itau Matrix US Hedge FIF.

[10] The Quantum Fixed Income indicator approximately tracks variations in the market price of a certain fund's share, whose objectives are such that its market value increases whenever the six-month swap rate decreases. The Quantum Cambial indicator, in turn, follows variations in the market price of another fund's share, whose value increases whenever the premium from a swap contract - the so called DI-Dollar - decreases. For additional discussion on these two indicators, see Varga (1999).

[11] Since the fund Itau Matrix US Hedge FIF has been bought by another fund - namely Itau B Cambial FI - at the end of 2004, I made the corresponding estimation with the data

The covariance matrix Q (cf. the state equation in (5-2)) was considered full in all the estimations[12]. The X_t heteroscedastic variable (cf. the measurement equation of (5-2)) was chosen to be the US Dollar/Real AR(1)-GARCH(1,1)[13] volatility in the analyzed period, and its standardized version was used as the z_t switching regime variable for the exposures onto the US Dollar/Real exchange rate and the Quantum Cambial. The dummy variable d_t from the TAR speci¯cation takes 1 whenever $z_t \geq 1:3$, and 0 otherwise. This calibration was chosen to capture the period of high volatility, which took place from the last week of September 2002 (which is located around the 90th observation) to the third week of February 2003 (which is located around the 110th observation)[14]. Figure 5.1, which helps recognizing these patterns, also illustrates what happened throughout the period. Note that the 2nd half of 2002 was marked by a con¯dence crisis that surged on the eve of the Brazilian presidential elections. This crisis found a very fertile ground to grow due to fears about the macroeconomic policies that could be followed by the candidate who was leading the polls, Luis Inacio Lula da Silva. When agents perceived that Lula administration would not change economic fundamentals like the °oating exchange rate and in°ation targeting regimes, expectations about future economic developments became favorable, ¯nancial market indicators turned positive and volatility dropped.

If one makes careful inspection on the information depicted in tables 5.1 and 5.2, several points emerge. Looking ¯rst at computational e±ciency, it is clear that the computational times, even though larger for the nonlinear proposals, remain essentially negligible. This could be of great value, should one try to use/implement these dynamic style-analysis proposals in practice.

Stepping further, one should note that the predictive power from the competing proposals gives us no clue about which model is the most adequate - it seems that, for these particular estimations, all models can reproduce the data almost under similar capabilities (cf. Pseudo R^2 and MSE measures). Also, the use of AIC and BIC criteria is of no help in deciding which model should be considered.

The diagnostic tests in the last three lines of these tables[15], though, uncover important aspects. They actually tell that, in terms of model basic

until the ¯rst week of 2003 November (149 observations). Additional information can be obtained at the National Association of Investment Banks (ANBID) (www.anbid.com).

[12] But it must be remarked that, from what was discussed in subsection 5.1.3, there would be no problem in attempting diagonal speci¯cations under the reduced restricted Kalman ¯ltering, on which the estimations are based.

[13] The corresponding implementation has been made using EViews (www.eviews.com)

[14] I could have tried to estimate the value of the limiar, but it would demand more periods of high volatility in the data.

[15] The three tests were applied to the standardized innovations.

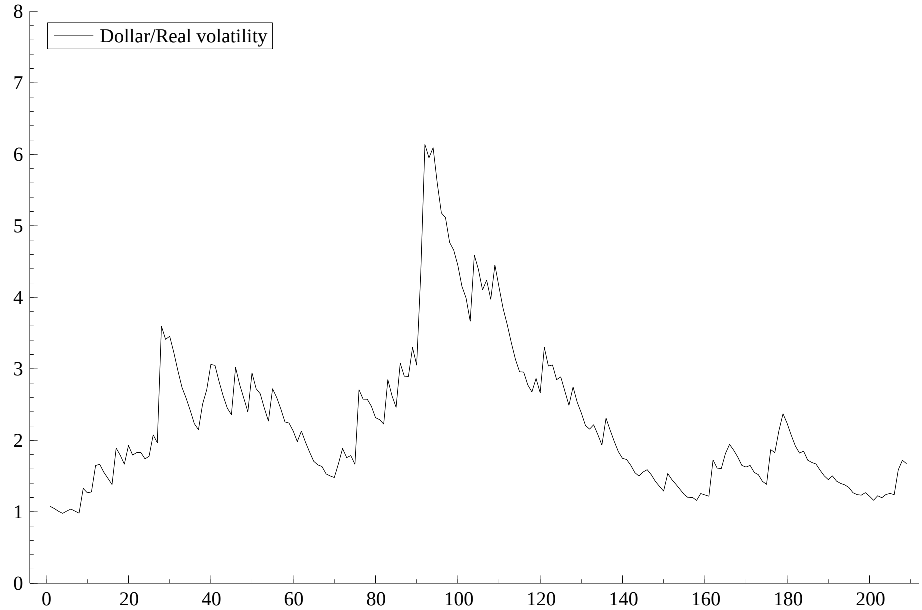

Figura 5.1: US Dollar/Real volatility obtained under the AR(1)-GARCH(1,1) model.

Tabela 5.1: Results from the estimations with HSBC FIF Cambial.

Attribute	RW	AR	TAR	$STAR$
Log-likelihood	-149.462	-148.124	-143.421	-143.657
Computational time	0.61	3.85	6.76	10.66
Pseudo R^2	0.905	0.907	0.905	0.905
MSE	0.549	0.534	0.545	0.550
AIC	1.516	1.561	1.554	1.576
BIC	1.756	1.801	1.858	1.912
Linearity LR test	-	-	9.406 (0.009)	8.935 (0.063)
Ljung-Box test (30 lags)	54.984 (0.004)	52.910 (0.006)	61.172 (0.001)	51.236 (0.009)
Homoscedasticity F test	0.435 (0.010)	0.481 (0.02)	0.384 (0.003)	0.442 (0.011)
Jarque-Bera test	18.694 (0.000)	18.478 (0.000)	60.247 (0.000)	11.182 (0.004)

assumptions, the *STAR* proposal systematically behaves better than the others. This is an indication that, in the analyzed period, exchange rate exposures were driven by some switching regime nonlinear process.

Finally, looking at the results from the *LR* linearity test, there is evidence, at least under a 10% signi¯cance level, that the *STAR* speci¯cation is supported by the data.

Taking into account these ¯ndings, there is no option left but to accept, amongst the four considered proposals, the *STAR* model as the best description for the exposures onto the exchange rate markets.

Figures 5.2 and 5.3 depict time plots for the restricted Kalman smoothing estimates of Jensen's measure and of the exposures onto U.S. Dollar/Real exchange rate spot markets and Quantum Cambial. Visual inspection suggests that the investment strategy followed by the managers of HSBC FIF Cambial

Tabela 5.2: Results from the estimations with Itau Matrix US Hedge FIF.

Attribute	RW	AR	TAR	$STAR$
Log-likelihood	-243.201	-242.417	-234.786	-233.996
Computational time	0.33	0.77	11.1	5.44
Pseudo R^2	0.697	0.679	0.761	0.684
MSE	3.008	3.029	2.366	3.009
AIC	3.489	3.479	3.430	3.446
BIC	3.793	3.782	3.814	3.871
Linearity LR test	-	-	15.262 (0.000)	16.841 (0.002)
Ljung-Box test (30 lags)	32.144 (0.361)	37.576 (0.161)	31.757 (0.379)	37.376 (0.166)
Homoscedasticity F test	0.670 (0.210)	0.899 (0.738)	0.944 (0.857)	0.779 (0.434)
Jarque-Bera test	6.286 (0.043)	4.242 (0.120)	7.412 (0.024)	2.618 (0.270)

is such that exposures onto U.S. Dollar/Real exchange rate spot markets were negligible throughout the sample, except during the period of higher volatility, when a signi¯cant long position was taken. Furthermore, exposures onto Quantum Cambial were always signi¯cant, wandering around a share of approximately 75% of the portfolio throughout the period. This outcome probably re°ects preventive measures taken by fund managers during the crisis, which protected the portfolio against the losses caused by the decrease in the market value of dollar-indexed bonds issued by the Brazilian government. Managers of Itau Matrix US Hedge FIF, in turn, followed an investment strategy in which the exposures onto U.S. Dollar/Real exchange rate spot markets wandered around 75%-80% of the portfolio throughout the sample (even though the large con¯dence intervals observed during the period avoid ascertaining this); on the other hand, exposures onto Quantum Cambial were negative and signi¯cant at several occasions.

I now look at information extracted by the model on selectivity skills by analyzing the time path of Jensen's measure. The graphs on the top of Figures 5.2 and 5.3 suggest that managers of HSBC FIF Cambial and Itau Matrix US Hedge FIF revealed a slight tendency of generating gains during the period marked by the con¯dence crisis. One can understand these facts concerning HSBC and Itau funds by recalling that it is precisely during periods of increased volatility that managers have signi¯cant pro¯t opportunities by engaging in high-frequency operations (e.g., day-trade transactions).

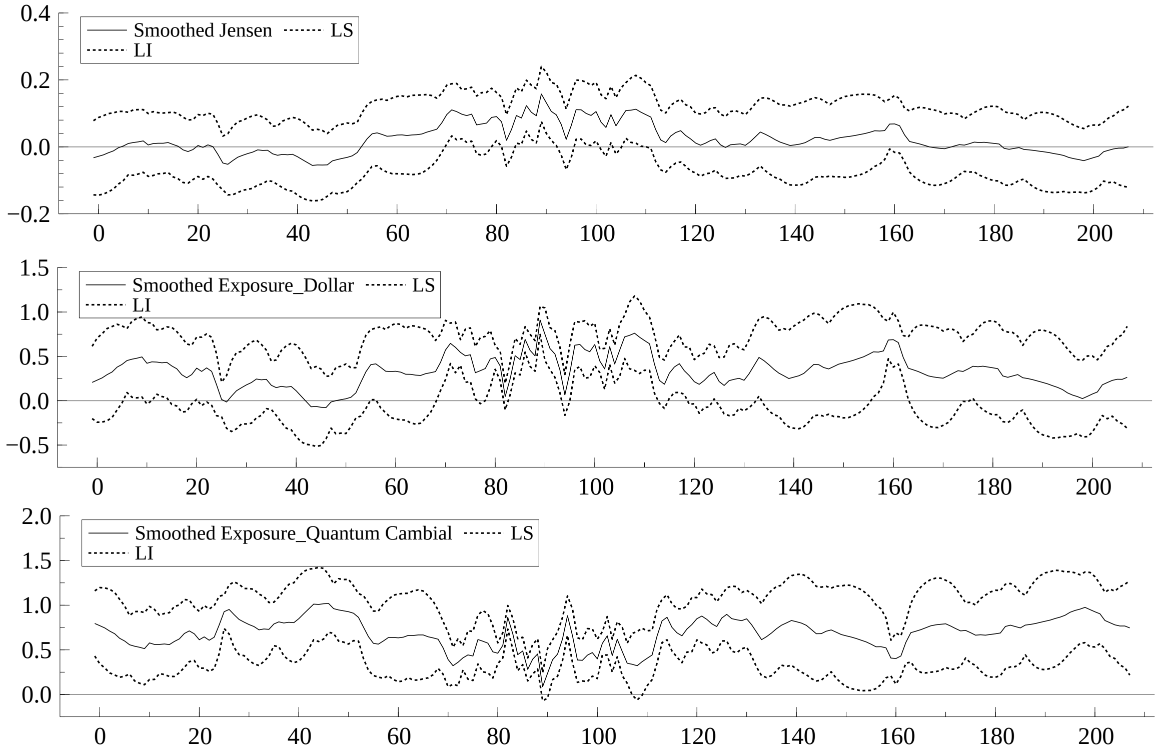

Figura 5.2: Smoothed *STAR* exposures and Jensen's measure for the HSBC FIF Cambial with respective 95% con¯dence intervals.

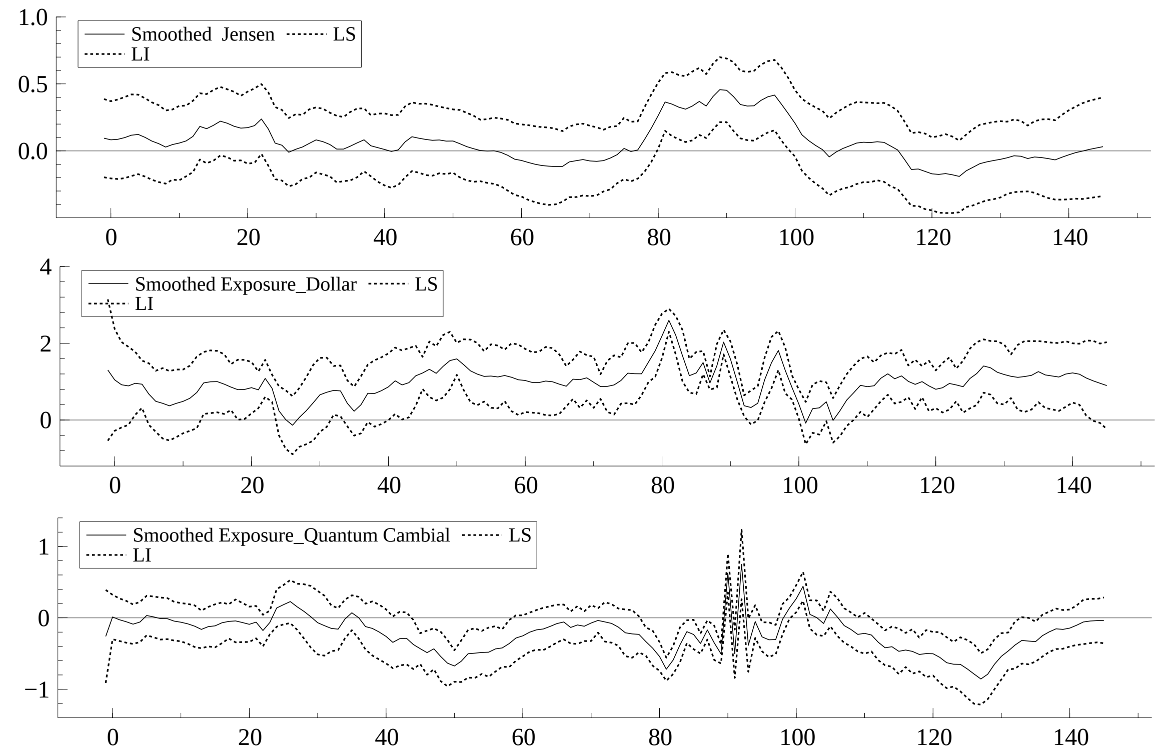

Figura 5.3: Smoothed *STAR* exposures and Jensen's measure for the Itau Matrix US Hedge FIF with respective 95% con¯dence intervals.

Tabela 5.3: Pairwise correlation summary for the smoothed exposures onto exchange rate markets.

Fund	Correlation
HSBC FIF Cambial	-0.9592
Itau Matrix US Hedge FIF	-0.6230

Table 5.3, which shows pairwise relations for smoothed exposures onto U.S. Dollar/Real exchange rate and Quantum Cambial, allows a better understanding of the funds' behavior. Each series was tested individually for unit root (cf. Enders, 2004, ch.4) and all results indicated stationarity. In the sequel, pairwise correlations were estimated. The information in Table 5.3 suggests that the positions in the U.S. Dollar/Real exchange rate spot market and Quantum Cambial were negatively related, re°ecting the fact that managers engaged in hedge operations to avoid a devaluation in their funds' shares. These results can even reveal a tendency to incur in some degree of leverage. One can see this by looking at what would happen if the U.S. Dollar/Real exchange rate has increased during the period (event that actually happened). Note that Itau Matrix US Hedge FIF would pro¯t from this movement from two sources: (i) the long position in the U.S. Dollar/Real exchange rate market and; (ii) the short position in the Quantum Cambial. This outcome can be understood by recalling that there is a positive relationship between the U.S. Dollar/Real exchange rate and the premium in the DI-Dollar swap contract.

5.2 Case II: Estimation of dynamic exchange rate past-through

5.2.1 Motivation

In this section, linear state space models are proposed to estimate the pass-through of Brazilian price indexes against the US Dollar/Real exchange rate from 1996 to 2005. The methodological framework encompasses the reduced restricted Kalman ¯ltering from section 4.2, which permits verifying the plausibility of some economic hypothesis.

There are three main targets. The ¯rst is to decide whether models of null (or of full) pass-through are acceptable to the price indexes investigated here. The second is to carry out likelihood ratio tests for the signi¯cance of some economic exogenous variables, which shall be termed determinants in this paper and are theoretically associated with the pass-through. The third is to analyze the behavior of the estimated pass-through from the best models.

Basic concepts and some review of the literature

In an open economy, domestic prices can be a®ected by external shocks, whether from currency relative prices adjustment or from movements of international supply and demand. The exchange rate is a quite volatile economic variables in macroeconomic policy. How much the exchange rate a®ects the economy? One of the faster channels is into prices. This channel is called *(exchange rate) pass-through*. There are few studies for this e®ect in Brazil in which the response of the prices to a change in exchange rate is suitably tackled.

The importance of past-through estimation has increased since the adoption of in°ation targeting regime (cf. Fraga, Goldfajn and Minella, 2003), and the recognition that it is crucial for in°ation forecasting. In addition to these motivations above, there is some evidence of a time-varying pass-through, even though only few studies have considered this assumption. Indeed, as Parsley (1995) points out, the stability of exchange rate pass-through is not well tested in common econometric speci¯cations of pass-through equations.

Pass-Through Determinants

According to Menon (1996), Taylor (2000), and Campa and Goldberg (2002), the main drivers of price sensibility to exchange rate changes can be inferred. In face of literature with macroeconomic approach, the pass-through depends on: *in°ation persistence*, *openness degree of the economy*, the *output gap*, and *real exchange rate disalignments*. From the standpoint of disaggregated analysis, the exchange rate pass-through is also associated with *the competition degree of each industry* and with *¯rm's market power* (with the elasticity price-demand).

5.2.2
The model setting and inference

I now present the state space model for the exchange rate pass-through for a given index price as follows:

$$\Delta \log p_t = \sum_{k=1}^{m} \beta_{kt} \Delta \log e_{t-k} + \psi_0 + \psi_1 \Delta \log(ap_t) + \varepsilon_t; \quad \varepsilon_t \sim NID(0; \sigma^2) \quad (5\text{-}9)$$

$$¯_{t+1} = ¯_t + °_1 ¢\, log(IPA_t)\mathbf{1}_{q\times 1} + °_2 ¢\, log(ip_t)\mathbf{1}_{q\times 1} + °_3 ¢\, log(re_t)\mathbf{1}_{q\times 1}$$
$$+ °_4 ¢\, log(o_t)\mathbf{1}_{q\times 1} + »_t; \quad »_t \sim NID(0; Q) \tag{5-10}$$

The former equation linearly relates the observed monthly log-variation of price to the log-variation of exchange rate until time $t - m$ and to an exogenous variable, the American price index, ap_t. The coe±cients of $¢ log\ e_{t-k}$ in equation (5-9) are the state coordinates, which represent the *components* of the past-through (the sum of them is termed *long run past-through*) and whose dynamics are given in equation (5-10), which also sets the impact from the following determinants: IPA series that represents the in°ationary environment; ip_t is the industrial production index, re_t is the exchange rate disalignment, and o_t is the openness of the economy. The matrix $Q_{m\times m}$ is set diagonal, even though the components from the past-through (*i.e.* the state coordinates) do maintain degrees of dependency due to the presence of common determinants in the state equation.

The reduced restricted Kalman ¯ltering has to be evoked in order to make the restrictions of *full* past-through ($\sum_{i=1}^{m} ¯_{it} = 1$) and of *null* past-through ($\sum_{i=1}^{m} ¯_{it} = 0$) attainable. The completeness of the exchange rate passing-through (the ¯rst restriction) means that all the variation of the exchange rate is passed to the domestic prices. This is key for Economic Theory standpoint, since it means that the PPP hypothesis is acceptable. On the other hand, the acceptance that null exchange rate passing-through model is the most adequate scenario implies the exchange rate movements do not have any e®ect in the domestic prices, and so, the monetary authority needs not be concerned with exchange rate movements to make monetary policy with such price indexes.

Besides checking the hypotheses of completeness (or absence) of exchange rate passing-through, another purpose of this application is to identify the most adequate number of lags of the exchange rate, that is the value of m. For such, quite the same steps listed in subsection 5.1.3 shall be used.

Finally, the signi¯cance of the parameters $Ã_0$, $Ã_1$, $°_1$, $°_2$, $°_3$ and $°_4$ will be tested under a likelihood ratio (LR) testing approach. Since both the reduced and the complete model maintain the standards for good properties of maximum likelihood estimation (cf. Pagan, 1980), it follows that, asymptotically, $LR \equiv 2\,[logL_{Max,Comp} - logL_{Max,Red}] \sim Â_1^2$, in which $logL_{Max,Red}$ represents the maximum of the log-likelihood for a model with a particular explanatory variable dropped from the speci¯cation.

5.2.3

Empirical results

The analyzed data contain monthly observations from August of 1999 to January of 2007 of the Brazilian wholesale price index (IPA), the Brazilian consumer price index (IPC), the American price index, the exchange rate between the Brazilian Real and the American Dollar, the Brazilian industrial production index and a measure of openness, which is the sum of imports and exports as a proportion of GNP. The decision of using data since August of 1999 is justi¯ed by the in°ation target system adopted by the Banco Central (institution corresponding to the American Federal Reserve in Brazil) in June of 1999. The data has been obtained from IPEA Data (www.ipeadata.gov.br), and each estimation has taken less than 2 seconds, something that highlights the computational e±ciency of the adopted state space framework.

Overall IPA

The most adequate model for the IPA series is the model with 7 lags on the exchange rate. Even though only the 4 ¯rst states have a con¯dence interval that does not contain zero, this decision has been based on the lack of serial correlation for the residuals. Figure 5.4 shows the evolution of the coe±cients along time. The $PseudoR^2 = 0{:}64$ suggests that the model provides a reasonable adjustment for the IPA. The long run pass-through given in Figure 5.5 has some variation when we compare the beginning of the sample to the end with a edge at the 2002, the year of elections preceding the Lula's administration in Brazil, a period of great volatility in the exchange rate.

The restricted models were estimated to verify whether the hypothesis of null and full exchange rate pass-through have some support from the data. The information criteria shown in table 5.4 do not provide any evidence that these extremes allow a better ¯t. The *LR* signi¯cance tests are given in table 5.5. The p-values reveal no evidence that the proposed determinants help to explain the behavior of the pass-through.

Criterium	unrestricted	$\sum_{i=1}^{7} ¯_{i,t} = 0$	$\sum_{i=1}^{7} ¯_{i,t} = 1$
AIC	3.000		

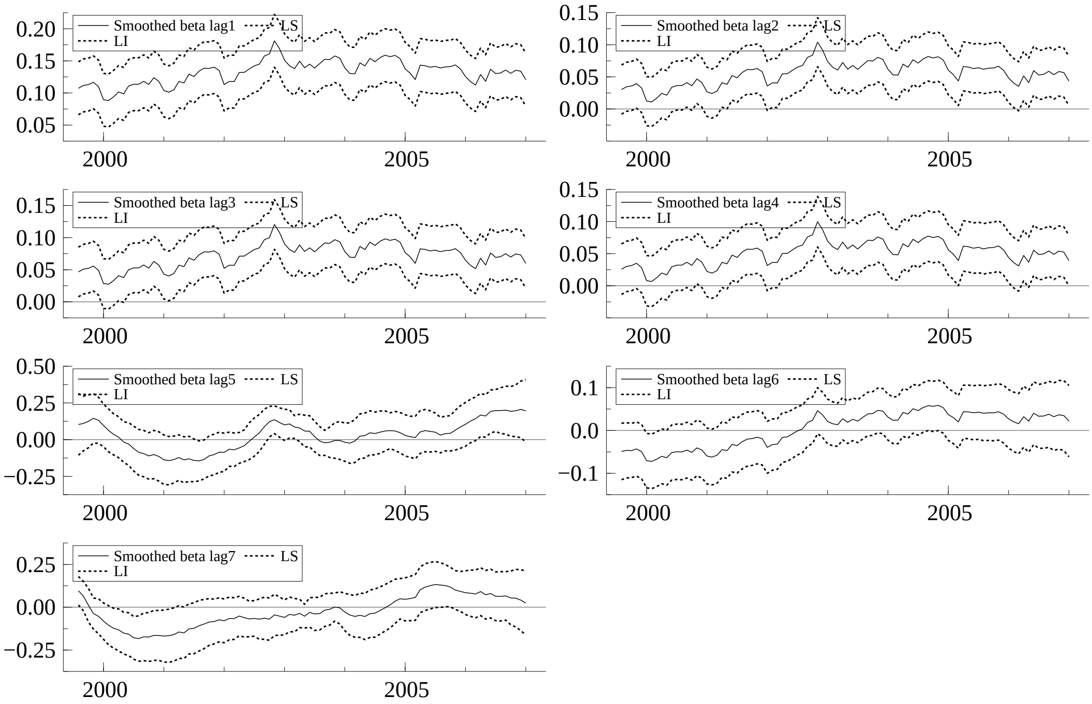

Figura 5.4: IPA smoothed betas.

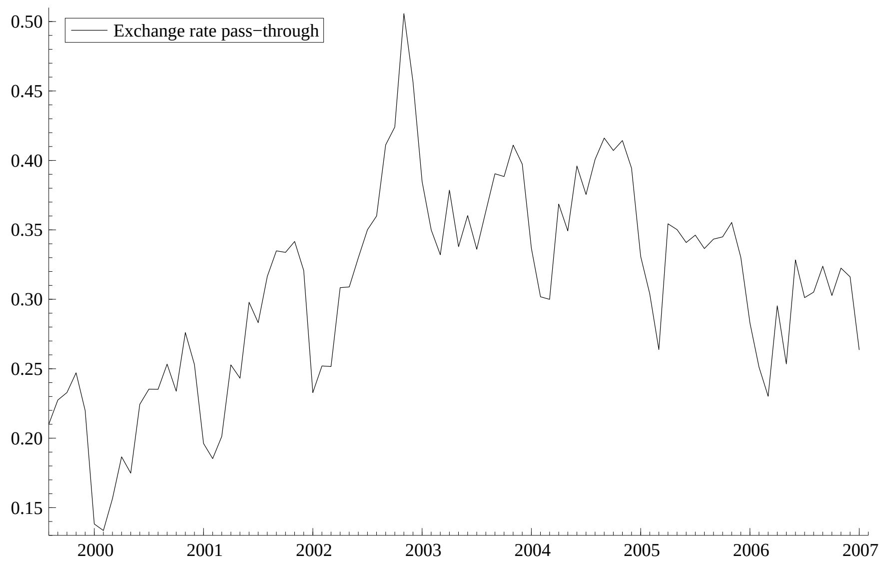

Figura 5.5: IPA long run exchange rate pass-through.

$\tilde{A}_1$	$°_1$	$°_2$	$°_3$	$°_4$
0.001	0.000	0.001	0.001	0.000
(1.000)	(1.000)	(1.000)	(0.505)	(1.000)

Tabela 5.5: IPA estimated parameters and corresponding p-values in parenthesis.

level of disaggregation splits the overall IPA into two main groups: consumption and production goods.

The more adequate model for the IPA consumption series has only a lag of the exchange rate, since it has the lower information criteria values and its residuals shows no serial correlation. The $PseudoR^2 = 0{:}615434$ provides evidence in favor of goodness-of-¯t. Since the decision of having only one lag for the exchange rate, the short and long run exchange rate pass-through are the same. Its variation over time can be seen at ¯gure 5.6. During the year of 2002, the exchange rate pass-through presented higher values compared to the rest of the sample period, probably due to the same explanations already given. Also, there is some indication of seasonal patterns, since the pass-through seems to be close to zero in the very begging of each year.

As shown in table 5.6, the LR signi¯cance tests reveal that three proposed determinants are supported by the data. One might also observe that the inertial parameter $\tilde{A}_1$ is statistically signi¯cant for the measurement equation.

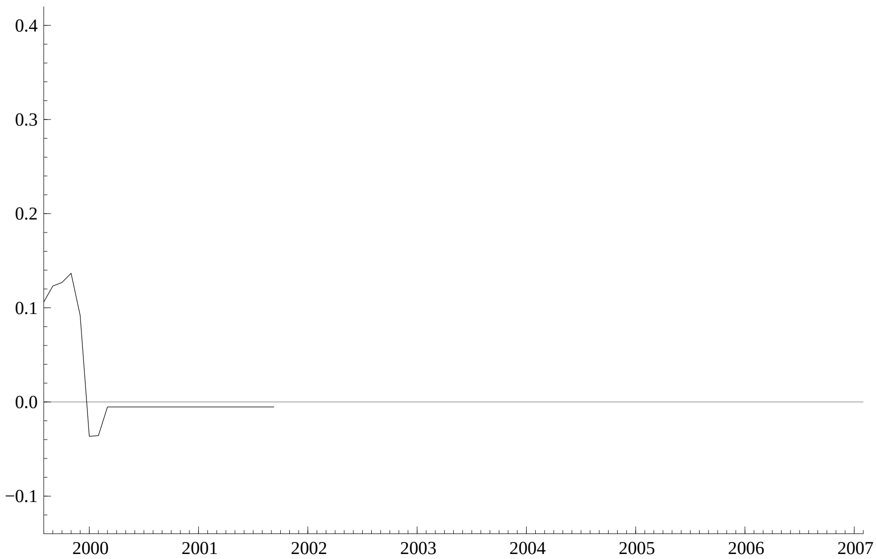

Figura 5.6: IPA consumption smoothed betas.

As it happened to the IPA consumption series, the most adequate model to the IPA production series was the model with only one lag of exchange rate pass-through. Again, the high value of the $PseudoR^2 = 0{:}729$ provides

$\tilde{A}_1$	$°_1$	$°_2$	$°_3$	$°_4$
0.561	-0.0004	0.012	0.005	-0.002
(0.000)	(0.000)	(0.061)	(0.006)	(0.347)

Tabela 5.6: IPA consumption estimated parameters and corresponding p-values in parenthesis.

us some con¯dence that the model ¯ts the data in a proper way. The pass-through variation over time can be seen in ¯gure 5.7. This remarks some aspects similar to those found in the previous analysis, except for the lack of evidence on seasonality.

The *LR* signi¯cance tests shown in table 5.7 provide us with two statistically signi¯cant determinants. Still, the inertial parameter $\tilde{A}_1$ is again statistically signi¯cant at the measurement equation.

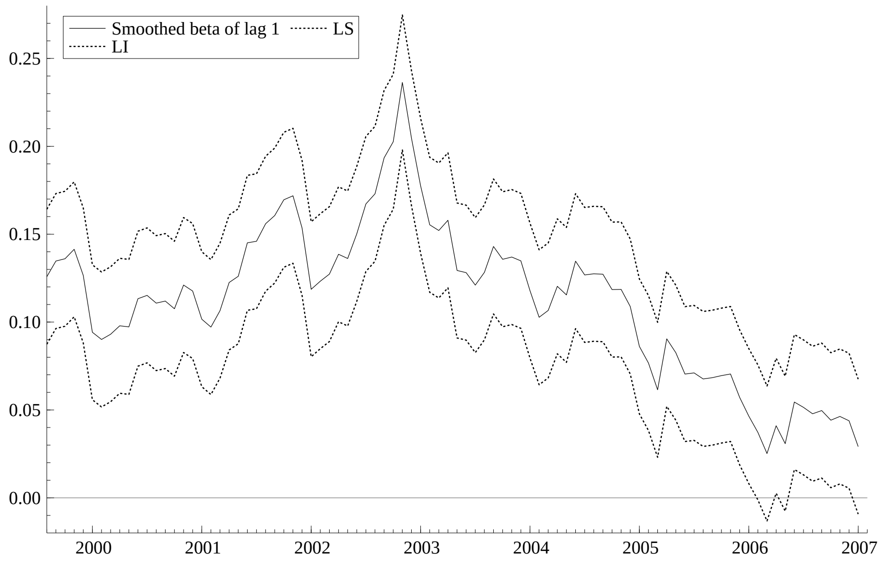

Figura 5.7: IPA production smoothed betas.

$\tilde{A}_1$	$°_1$	$°_2$	$°_3$	$°_4$
0.590	-0.001	0.002	0.002	-0.001
(0.000)	(0.699)	(0.000)	(0.000)	(0.823)

Tabela 5.7: IPA production estimated parameters and corresponding p-values in parenthesis.

IPC

The model adjusted with 2 lags of the exchange rate shows that the IPC seems to be not responding to the exchange rate movements. As can be seen in ¯gure 5.8, the states corresponding to all lags are varying around zero within the whole sample period. The long-run pass-through presented in ¯gure 5.9 is also oscillating around zero. This shall be taken as the ¯rst symptom of absence of passing-through, and this is reinforced by the application of the restricted Kalman ¯ltering, since the model which has the null pass-through restriction has the best information criteria; see table 5.8.

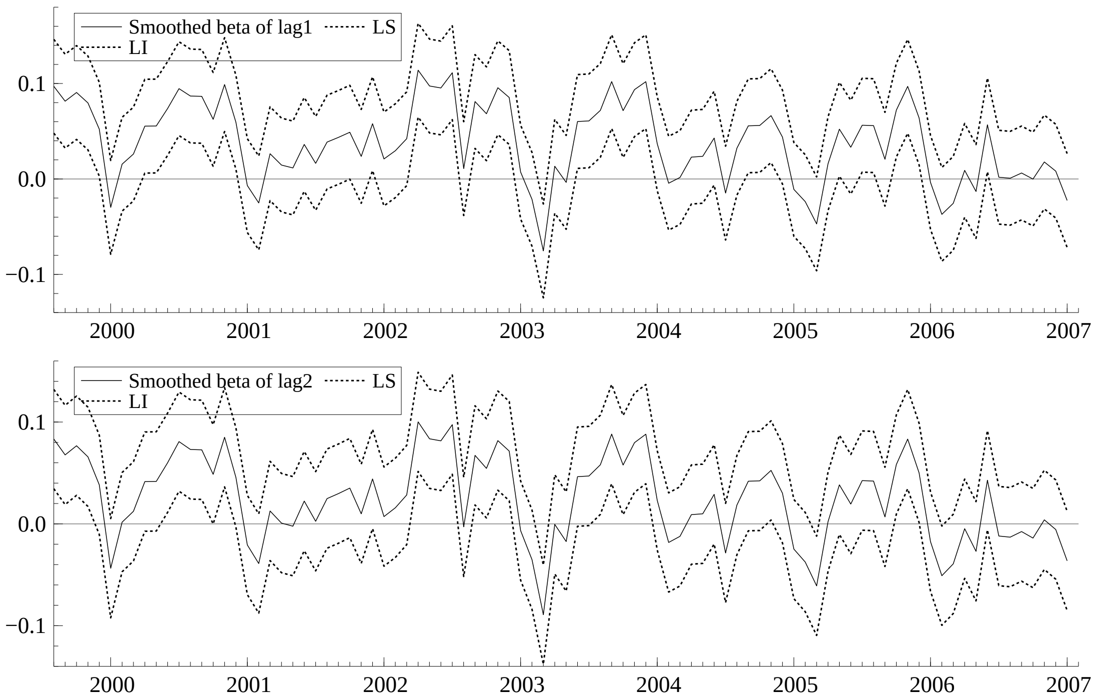

Figura 5.8: IPC smoothed betas.

Criterium	unrestricted	$\sum_{i=1}^{2}$ ¯$_{i,t}$ = 0	$\sum_{i=1}^{2}$ ¯$_{i,t}$ = 1
AIC	2.838	2.801	5.291
BIC	3.144	3.051	5.541

Tabela 5.8: IPC information criteria of the unrestricted and the restricted models.

5.3 Case III: GNP benchmarking estimation and prediction

5.3.1 Motivation

I close the applications of this Thesis by facing GNP quarterly prediction. Here is the setting. There are two series, a quarterly series of GNP which is subject to measurement error and an annual total series of the same economic variable that is \accurately" recorded. The goal is to produce a quarterly GNP

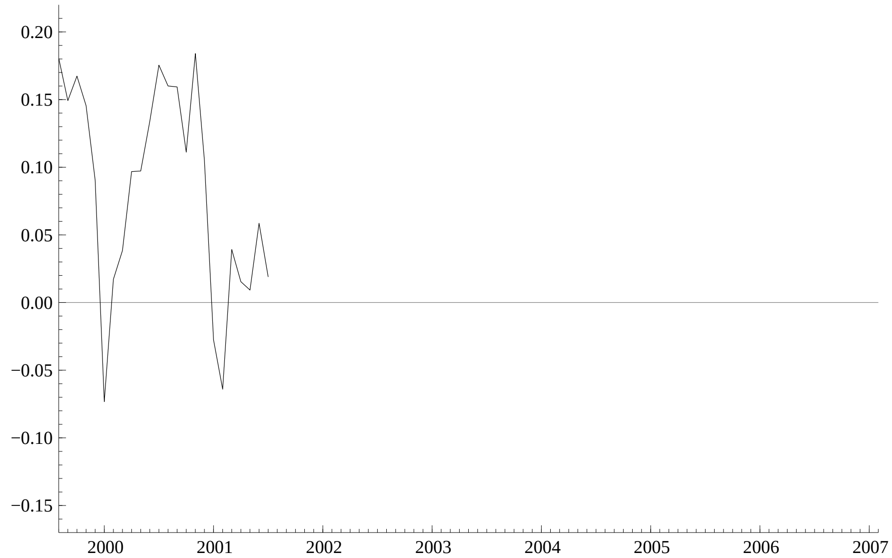

Figura 5.9: IPC long run pass-through.

free from those measurement errors, and this is supposed to be accomplished by conveniently using the information from the annual totals. This is in fact a *benchmarking* problem and its general formulation was examined in a rather comprehensive state space fashion by Durbin and Quenneville (1997). Later, Durbin and Koopman (2001), ch.3, quickly revisited the corresponding state space forms.

5.3.2
Model setup

Here the focus is to make predictions under this benchmarking framework, which shall generate quarterly predictions from the GNP free from measurement error and under *consistency* (that is, the estimated quarterly GNP must sum up to the annual totals GNP). For this purpose, I use the restricted Kalman predictor of section 4.3 with an alternative state space form that evinces the consistency restriction. This representation is an augmented state space model, the augmentations of which only appear in time periods multiple of 4 (four): that is, in these time periods the information from the annually totals GNP is attached to the measurement equation (this is making use of the time- and size-varying °exibility of the augmenting restricted Kalman ¯ltering!). In this sense, the measurements would be Y_t if we \are not" in $4i$, and would be $(Y_t; X_t)'$ if we \are" in $4i$, where Y_t represents some quarterly GNP, X_t represents some totally GNP of some year and $i = 1; 2; :::$. The state vector would be ®$_t \equiv$ (¹$_t$; ¹$_{t-1}$; ¹$_{t-2}$; ¹$_{t-3}$; °$_t$; °$_{t-1}$; °$_{t-2}$; °$_{t-3}$; "$_t$; "$_{t-1}$; "$_{t-2}$; "$_{t-3}$; »$_t$)$'$, where ¹$_t$ is a local level, °$_t$ is a dummy seasonal e®ect, "$_t$ is a Gaussian white noise irregular component and »_t represents the $AR(1)$ measurement error. In addition, one must set $H_t \equiv 0$, $d_t \equiv 0$ and $c_t \equiv 0$. Finally, check below the Z_t matrices for this alternative restricted state space form:

$$Z_t = \begin{cases} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}; \ if \ t \neq 4i; \ i = 1; 2; ::: \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}; \ if \ t = 4i; \ i = 1; 2; ::: \end{cases}$$

The matrices $T_t \equiv T$, $R_t \equiv R$ and $Q_t \equiv Q$ are obvious and are omitted to save space. Observe that the speci¯cation is purely based on the structural modeling framework (see Harvey, 1989, chapter 2) for the quarterly GNP series. Also see that the *theoretical* consistency correction acts in the time indexes multiple of four. From Theorem 1 and from the computational algorithm described in the end of section 4.3, the *empirical* consistency correction is achieved in-sample and out-of-sample periods (the latter would be the prediction) whenever the Kalman updating and smoothing equations actuate on a series extended in the way proposed in 4-14 (notice that $q_t = X_t$ for every t multiple of 4).

5.3.3 Empirical results

To illustrate the proposed benchmarking prediction model, I applied it to the Brazilian GNP series constructed by the methodology proposed in Cerqueira *et al.* (2007). The very original series had been obtained from IBGE (*www.ibge.gov.br*) and IPEADATA (*www.ipeadata.gov.br*). I estimated the model with 140 observations ranging from the ¯rst quarter of 1960 to the fourth quarter of 1994, a 21.7-second task. In the sequel, I used the restricted Kalman predictor to the next two years using the annual totals of 1995 and 1996 to get the consistency restrictions satis¯ed. Table 5.9 presents the prediction results. The reader can easily con¯rm that the predicted quarterly GNP is consistent with the annual totals.

Tabela 5.9: Results of the benchmarking prediction.

Year/Quarter	1st	2nd	3rd	4th	Annual total
1995	1,066	1,153	1,129	1,096	4,444
1996	1,030	1,165	1,167	1,139	4,502

6
Further extensions

At the end of this Thesis, I list some additional points in my research about restricted Kalman filtering. Some of them are already under investigation.

Firstly, I cite additional *theoretical* points that sound interesting within the theme. Here they are:

- A study about state observability and parameters identification, which are two important issues to firmly establish the inferential grounds for state space models under linear restrictions.
- A formal investigation about possible connections between the results in Simon and Chia (2002) and the new proofs of restricted Kalman filtering presented in this Thesis.
- Analytical and/or Monte Carlo formal investigations into how the presumed additional information due to the use of the augmented restricted Kalman filtering translates to improvements for (*quasi*) maximum likelihood estimators.
- Derivation of results about combining diffuse initialization and linear restrictions under the approach by de Jong and Chun-Chun-Lin (2003), and, consequently, the analysis of how the new assumptions needed are more or less stringent than those considered in section 3.4.

Now, I concentrate on additional *methods*, so far not explored:

- Implementation of an *extended* restricted Kalman filtering in order to accomplish not only nonlinear equality constraints but also *inequality* constraints. In this respect, specific topics of interest would be the investigation of convergence of this extended approach and how this should be combined with *quasi* maximum likelihood estimation for the fixed parameters.
- Derivation of new tests for coefficients stability under linear restrictions, to which the material from section 3.3 could be of some value.

Lastly, I believe the following *applications* would be relevant to further illustrate and value already developed methodologies.

{ Estimation of dynamic factor model with an exact *smoothing transition* coe±cients under the same linear and interpretable portfolio restriction and also under some linear restriction about *leverage/hedge*.

{ Formulation and estimation of *multivariate* benchmarking models aimed at prediction.

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